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A Log-Dagum Weibull Distribution: Properties and Characterization

Aneeqa Khadim[1], Aamir Saghir[1], Tassadaq Hussain[1], Mohammad Shakil[2], Mohammad Ahsanullah[3]

[1]Department of Mathematics, Faculty of Sciences, Mirpur University of Science and Technology (MUST), Mirpur, Pakistan

[2]Department of Liberal Arts and Sciences, Faculty of Mathematics, Miami Dade College, Hialeah, USA

[3]Department of Management Sciences, Rider University, Lawrenceville, USA

Email address:
Aneeqa89@gmail.com (A. Khadim)

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Abstract: This article proposes a new family of continuous distributions generated from a log dagum random variable (named Log-Dagum Weibull Distribution) on the basis of T-X family technique. mathematical and statistical properties including survival function, hazard and reverse hazard function, Rth moments, L-moments, incomplete rth moments, quantile points, Order Statistics, Bonferroni and Lorenz curves as well as entropy measures for this class of distributions are presented also LDW distribution characterized by truncated moments order statistics and upper record values. Simulation study of the proposed family of distribution has been derived. The model parameters are obtained by the method of maximum likelihood estimation. We illustrate the performance of the proposed new family of distributions by means of four real data sets and the data sets show the new family of distributions is more appropriate as compared to Exponentiated exponential distribution (EED), Weibull distribution (WD), Gamma distribution (GD), NEED Nadarajah Exponentiated exponential distribution and Lomax distribution (LD). Moreover, perfection of competing models is also tested via the Kolmogrov-Simnorov (K S), the Anderson Darling (A^*) and the Cramer-von Misses (W^*). The measures of goodness of fit including the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) are computed to compare the fitted models.

Keywords: Probability Distributions, Log-Dagum Distribution, Parameter Estimation, Weibull Distribution

1. Introduction

Statistical distributions are extensively used in the literature for the modeling and the forecasting of real life phenomena. The recent literature has suggested numerous ways of extending well-known distributions. There has been an increased interest in defining new classes of the univariate continuous distributions by introducing one or more additional shape parameter(s) to the baseline distribution. This induction of parameter(s) has been proven useful in exploring tail properties and also for improving the goodness-of-fit of the generator family. The well-known families are: The beta-G of Eugene et al. [19] the Gamma-G (type-1) proposed by Zografos and Balakrishnan [35] the Kumaraswamy-G derived by Cordeiro and de Castro [14] the Mc-G family considered by Alexander et al. [5] the Weibull-X family of distributions developed by Alzaatreh et al. [3] the exponentiated generalized class derived by Cordeiro et al. [15] the Exponentiated T-X family developed by Alzaghal et al. [1] the weibull-G family proposed by Bourguignon et al. [13] the exponentiated half-logistic family considered by Cordeiro et al. [17] the gamma-G (type 3) family introduced by Torabi and Montazari [33] the log-gamma-G family developed by Amini et al. [7] the gamma-x family introduced by Alzaatreh et al. [4] the logistic-G family introduced by Torabi and Montazari [32]. The Kumaraswamy weibull-generated family developed by Hassan and Elgarhy [23] the new weibull-G family discussed by Tahir et al.[34] Exponentiated generalized exponential dagum distribution extended by Nasiru et al [25] the new generalized family of distributions discussed by Ahmad [2] Some new members of the T-X family of distributions of Farrukh Jamal and Muhammad Nasir [20]. A modified T-X family of distributions discussed by Muhammad Aslam et al. [6] Handique et al. [22] discussed properties and applications of a new member of the T-X family of distributions, Shakil et al. [31] derived properties of Burr (4P) distribution and Shakil et al. [30] discussed some inferences on the Dagum (4P) distribution. Hamed et al. [21] established new class of Lindley distributions properties and application etc.

The current work presents a new distribution called the Log-Dagum Weibull (LDW) distribution with three parameters. The proposed distribution due to its flexibility in accommodating all the forms of the hazard rate function can be used in a variety of problems for modeling lifetime data. The LDW distribution is not only suitable for modeling comfortable bathtub shaped failure rates data but is also suitable for testing goodness-of-fit of some models.

The rest of the article is organized as follows. Section 2 presents the T-X family of distributions. In section 3, the proposed Log-Dagum Weibull distribution is derived and studied graphical behavior of its probability density functions (pdf), survival function hazard functions shape of hazard function and concavity. In section 4, some statistical properties including rth moments, L- moment's, incomplete rth moments, quantile function and order statistics are presented. Section 5, contains the Shannon entropy and Renyi entropy. Section 6 presents Bonferroni and Lorenz curves. The characterization via hazard function, reverse hazard function and truncated moments and ordered statistics of distribution is derived in section 7. Estimation of model parameters is presented in section 8. Evaluation measures and practical data examples of the proposed model to real data are given in section 9, followed by concluding remarks.

2. T-X Family

Let $r(t)$ be the probability density function (pdf) of a random variable $T \in [a;\ b]$ for $-\infty \leq a < b < \infty$ and let $W[G(x)]$ be a function of the cumulative distribution function (cdf) of a randomvariable X such that $W[G(x)]$ satisfies the following conditions

$$\begin{cases} \text{(i) } W[G(x)] \in [\mathrm{a}, \mathrm{b}] \\ (ii) W[G(x)] \text{ Is differentiable and monotonically non} - \text{decreasing} \\ \text{and } (iii) \mathrm{W}[\mathrm{G}(\mathrm{x})] \rightarrow \mathrm{a} \text{ as } \mathrm{x} \rightarrow -\infty \text{ and } \mathrm{W}\,[\mathrm{G}(\mathrm{x})] \in \text{ b as } \mathrm{x} \rightarrow \infty \end{cases} \quad (1)$$

Recently, Alzaatreh et al. [3] defined the T-X family of distributions by

$$F(x) = \int_0^{W[G(x)]} r(t)dt \quad (2)$$

Where W[G(x)] satisfies the condition (1). The pdf corresponding to (2) is given by

$$f(x) = \left\{\frac{d}{dx} W[G(x)]\right\} r\{W[\mathrm{G(x)}]\} \quad (3)$$

In Table 1, we provide the W [G(x)] functions for some members of the T-X family of distributions.

Table 1. *Different $W[G(x)]$ functions for special models of the T-X family.*

S.No.	$W[G(x)]$	Range of T	Members of T-X family
1	$G(x)$	$[0, 1]$	Beta-G 19] KW-G type 1 [14] Mc-G [5]
2	$-\log [G(x)]$	$[0, \infty]$	Exp-G (KW-G type 2) [15] Gamma-G Type-2 [35] Log-Gamma-G Type-2 [7]
3	$-\log [1 - G(x)]$	$[0, \infty]$	Gamma-G Type-1 [26] Log-Gamma-G [7] Weibull-X [3] Gamma-X [4]
4	$- \log [1 - G^{\alpha}(x)]$	$[0, \infty]$	Exponentiated T-X [1]
5	$\log [- \log [1 - G(x)]$	$[-\infty, \infty]$	Logistic-X
6	$\log[\frac{G(x)}{1-G(x)}]$	$[-\infty, \infty]$	Logistic-G [32] proposed
7	$\frac{G(x)}{1-G(x)}$	$[0, \infty]$	Gamma-G Type-3 [35]

3. New Model

A random variable T has the log dagum distribution with shape parameter $\beta > 0$ and $\lambda > 0$ if it's cumulative distribution function (cdf) is given by

$$\pi(t) = (1 + \mathrm{e}^{-\lambda x})^{-\beta} t \in R, \beta > 0, \lambda > 0 \quad (4)$$

Its corresponding probability density function (pdf) can be expressed as

$$r(t) = \beta\lambda \mathrm{e}^{-\lambda x}(1 + \mathrm{e}^{-\lambda x})^{-\beta-1} t \in R, \beta > 0, \lambda > 0 \quad (5)$$

Let $G(x)$ and $\vec{G}(x) = 1 - G(x)$ be the baseline cdf and survival function (sf) by replacing $W[G(x)]$ by $\log(\frac{\mathrm{G(x)}}{1-\mathrm{G(x)}})$ and $r(t)$ with (5) in equation (2), we define the cdf of the log dagum-x family by

$$F(x) = \left[1 + \left(\frac{G(x)}{1-G(x)}\right)^{-\lambda}\right]^{-\beta} \quad (6)$$

The log dagum family pdf is expressed as

$$f(x) = \left[1 + \left(\frac{G(x)}{1-G(x)}\right)^{-\lambda}\right]^{-\beta-1} \left(\frac{G(x)}{1-G(x)}\right)^{-\lambda-1} \frac{\lambda\beta g(x)}{[1-G(x)]^2} \quad (7)$$

Henceforth, we denote by X a random variable having density function (7). The basic motivations for using the log dagum-x family in practice are to construct heavy tailed distributions that are not longer-tailed for modeling real data, to generate distributions with symmetric, left-skewed, right-skewed and reversed-J shaped, to define special models with all types of the hazard rate function (hrf), to provide consistently better fits than other generated models under the same baseline distribution.

The corresponding cumulative density function (cdf) probability density function (pdf), hazard function (hrf) and survival function are given as

$$F_{LDW}(x, \lambda, \Theta, \beta) = (1 + (e^{\theta x^{\beta}} - 1)^{-\lambda})^{-\beta} \quad (8)$$

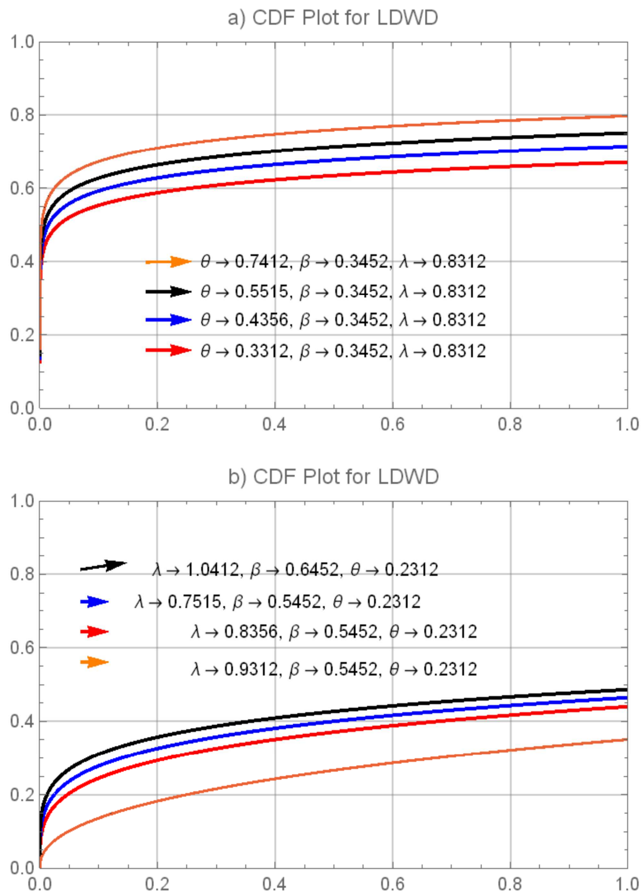


Figure 1. *Cumulative distribution plot of LDWD.*

Figure 1 gives the plots of the cumulative distribution function of the LDW distribution.

The plots of this figure shows that for fixed λ and $\beta$and changing θ the curve stretch out insignificantly towards right as θ increases. However, for fixed β and$\theta$and changing λ**.** Thecurve stretches out towards right significantly as λ increases.

And

$$f_{LDW}(x, \lambda, \Theta, \beta) = (1 + (e^{\theta x^{\beta}} - 1)^{-\lambda})^{-\beta-1}(-1e^{\theta x^{\beta}})^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^2 x^{\beta-1} \quad (9)$$

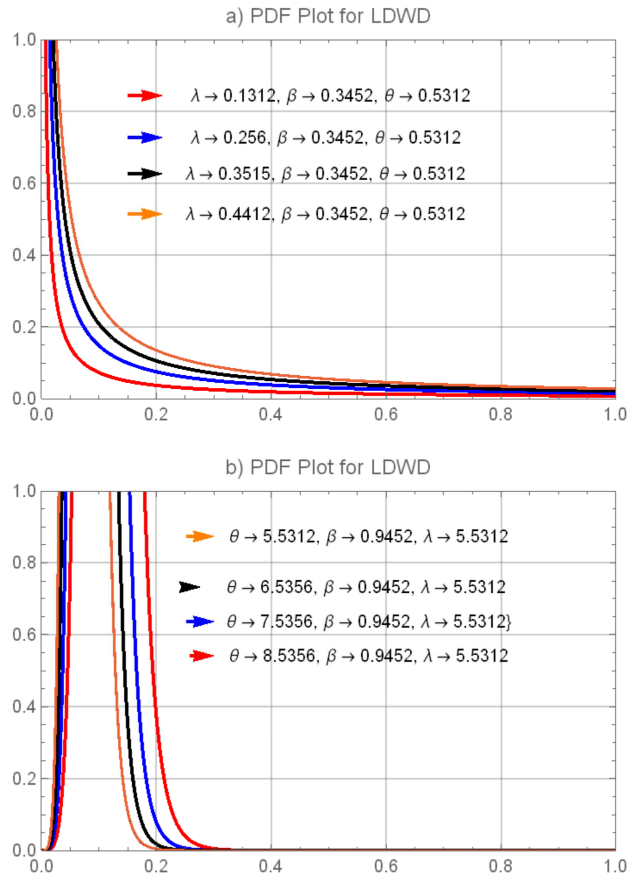
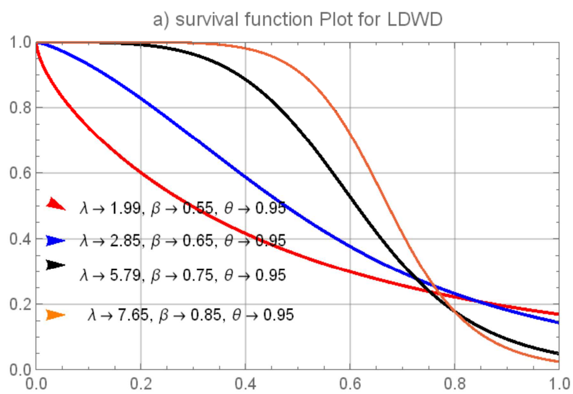


Figure 2. *Density plots of LDWD.*

Plots of Figure 2 display the density functions of the LDW distribution. Figure 2 portrays that changing λ against the fixed β and θ the density function decreases. But changing β against the fixed λ the nature of the curve towards right as θ increases, however in case of changing $\theta$with fixed β and λ shift the curve towards left.

$$S_{LDW}(x, \lambda, \theta, \beta) = 1 - (1 + (e^{\theta x^{\beta}} - 1)^{-\lambda})^{-\beta} \quad (10)$$



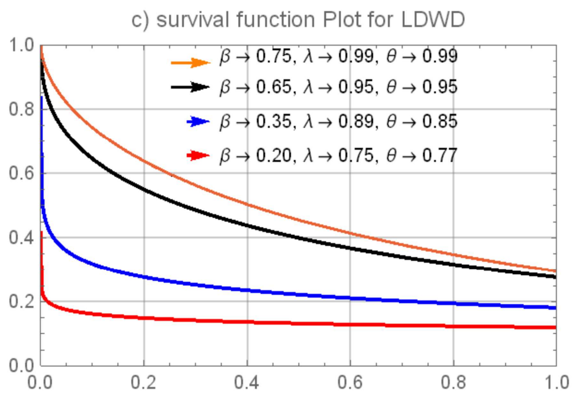


Figure 3. *Survival plot of LDWD.*

The graph of survival function increases for different values of parameters then suddenly starts gradually decreases and converges to zero.

$$h(x,\lambda,\beta)=\frac{\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^{\beta}}\right)^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^{2}x^{\beta-1}}{1-\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta}} \quad (11)$$

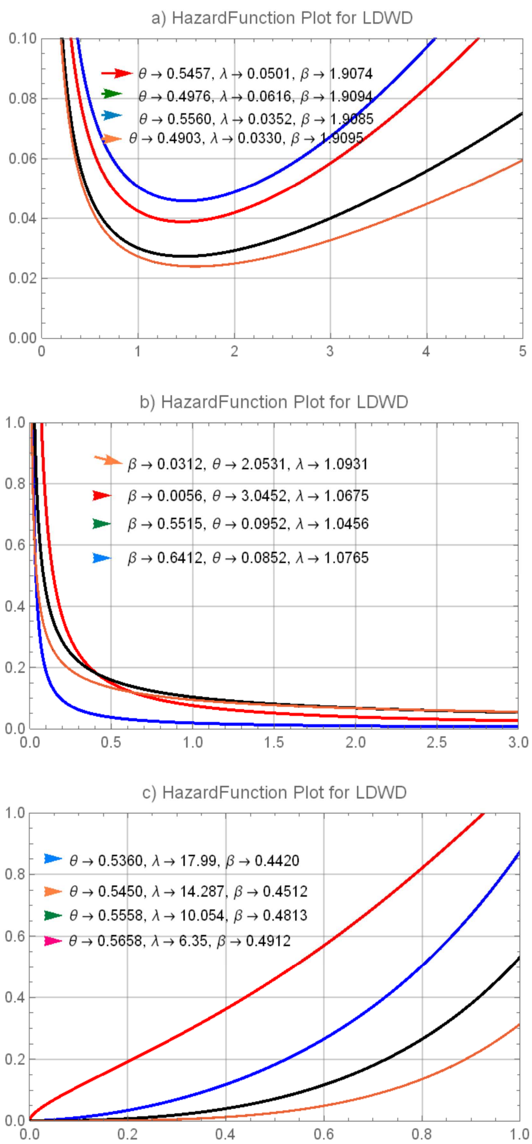


Figure 4. *Hazard plot of LDWD.*

The hazard function plots in Figure 4 also portray the declining circumstance of the product as time increases in terms of impulsive spikes at the end of either increasing or decreasing hazard rate. This implies that the hazard function is sensitive against different combinations of the parameters as time changes, which seems to be a refine image of non stationary process and hence the hazard curve does not remain stable as times passes. Moreover, Figure 4 displays increasing, decreasing bathtub hazard shapes.

3.1. Shape of Hazard Function

Shape of the density function can be described analytically, the critical point of the LDW density are the root of the equation

$$\frac{d}{dx}\left[\frac{\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^{\beta}}\right)^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^{2}x^{\beta-1}}{1-\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta}}\right]=0 \quad (12)$$

There may be more than one root.

3.2. Concavity

The concavity of hazard rate function $h''(x)$= 0

$$\frac{d^2}{dx^2}\left[\frac{\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^{\beta}}\right)^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^{2}x^{\beta-1}}{1-\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta}}\right]=0$$

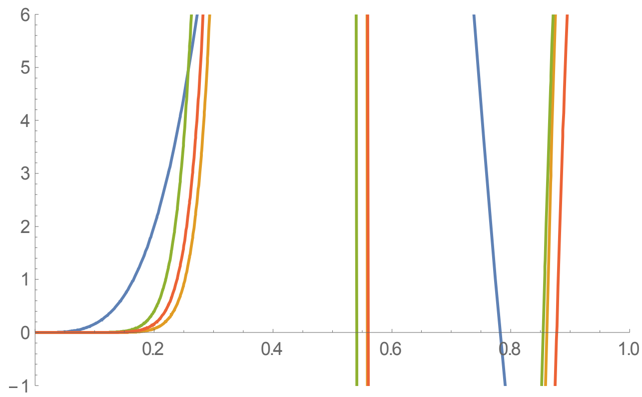


Figure 5. *Concavity of different value of parameters the hazard function is concave up and concave down where the point of concavity change is called point of inflection.*

4. Some Statistical Properties

In this section, we study some statistical properties of the LDW distribution, including Rth moments, L-moments, incomplete rth moments, quantile function and order statistics.

4.1. Moments of Ldwd

Let X is a particularly continuous non-negative random variable with PDF$f(X)$, and then the R$^{\text{th}}$ ordinary moment of the (LDW) distribution is given by:

$$E(x^r) = \int_0^\infty X^r f(X)dX$$

$$E(x^r) = \int_0^\infty x^r (1-(1-e^{\theta x^\beta})^{-\lambda})^{-\beta-1}(1-e^{\theta x^\beta})^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\beta^2 x^{\beta-1}dx \quad E(x^r) = \sum_{h=0}^{\infty}\frac{(-k)_h(-1)^h}{h!}\int_0^\infty X^r (1-e^{\theta x^\beta})^{\lambda\beta-1}e^{\theta x^\beta}\theta\lambda\beta^2 x^{\beta-1}dx$$

which, on substituting $y = e^{\theta x^\beta}$ and simplifying, reduces to

$$E(x^r)=\sum_{h=0}^{\infty}\frac{(-k)_h(-1)^h}{h!}\sum_{m=0}^{\lambda\beta-1}\frac{(-1)^m}{m!}\int_0^\infty (\ln y)^{\frac{r}{\beta}} y^m dy$$

Again substituting and simplifying

$$E(x^r) = \lambda\beta\frac{1}{\theta^{\frac{r}{\beta}}}\sum_{h=0}^{\infty}\frac{(-k)_h(-1)^h}{h!}\sum_{m=0}^{\lambda\beta-1}\frac{(-1)^m}{m!}\int_0^\infty (\frac{g}{(m+1)})^{\frac{r}{\beta}}e^{-g}\frac{dg}{(m+1)}$$

$$E(x^r) = \lambda\beta\frac{1}{\theta^{\frac{r}{\beta}}}\sum_{h=0}^{\infty}\frac{(-k)_h(-1)^h}{h!}\sum_{m=0}^{\lambda\beta-1}\frac{(-1)^m}{m!}\frac{1}{(m+1)^{\frac{r}{\beta}+1}}\Gamma(\frac{r}{\beta}+1) \tag{13}$$

4.2. L-Moments

The L-Moments of the LDW family is defined as

$$B_k = E(x(F(x)^k)$$

$$B_k = \int_0^\infty x\left(1-\left(1-e^{\theta x^\beta}\right)^{-\lambda}\right)^{-\beta k} f(x)dx$$

$$B_k = \int_0^\infty x\left(1-\left(1-e^{\theta x^\beta}\right)^{-\lambda}\right)^{-\beta k}((1-(1-e^{\theta x^\beta})^{-\lambda})^{-\beta-1}(1-e^{\theta x^\beta})^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\beta^2 x^{\beta-1}dx$$

$$B_k = \sum_{h=0}^{\infty}\frac{(-k)_h(-1)^h}{h!}\int_0^\infty x\left(1-e^{\theta x^\beta}\right)^{\lambda\beta(k+1)-1}e^{\theta x^\beta}\theta\lambda\beta^2 x^{\beta-1}dx$$

Let$e^{\theta x^\beta} = y, \beta\theta e^{\theta x^\beta}x^{\beta-1}dx = dy, x = \frac{1}{\theta^{\frac{1}{\beta}}}(lny)^{\frac{1}{\beta}}dy$

When $x\to 0$, $y\to 1$and$x\to\infty$, $y\to\infty$

$$B_k = \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}}\sum_{h=0}^{\infty}\frac{(-k)_h(-1)^h}{h!}\int_0^\infty (lny)^{\frac{1}{\beta}}(1-y)^{\lambda\beta(k+1)-1}dy$$

Expanding$(1-y)^{\lambda\beta(k+1)-1}$ using binomial expension

$$B_k = \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}}\sum_{h=0}^{\infty}\frac{(-k)_h(-1)^h}{h!}e^{m+1}\sum_{k=0}^{\infty}\sum_{m=0}^{(\lambda\beta(k+1)-1)}\binom{\lambda\beta(k+1)-1}{m}(-1)^{2m}\int_1^\infty (lny)^{\frac{1}{\beta}}(y)^m dy$$

After some substitution,

$$B_k = \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}}\sum_{h=0}^{\infty}\frac{(-k)_h(-1)^h}{h!}e^{m+1}\sum_{k=0}^{\infty}\sum_{m=0}^{(\lambda\beta(k+1)-1)}\binom{\lambda\beta(k+1)-1}{m}(-1)^{2m}\int_0^\infty e^{-z(m+1)}z^{\frac{1}{\beta}}dz$$

$$B_k == \frac{\lambda\beta}{\theta^{\frac{1}{\beta}}}\sum_{h=0}^{\infty}\frac{(-k)_h(-1)^h}{h!}e^{m+1}\sum_{k=0}^{\infty}\sum_{m=0}^{(\lambda\beta(k+1)-1)}\binom{\lambda\beta(k+1)-1}{m}(-1)^{2m}(\frac{\Gamma(\frac{1}{\beta}+1)}{(m+1)^{\frac{1}{\beta}+1}} \tag{14}$$

4.3. Incomplete Moments

The incomplete moments play an important role for measuring inequality, for example, income quintiles and Lorenz and Bonferroni curves. These curves depend on the first incomplete moment of the distribution.

The rth incomplete moment of the LDW family is defined as

$$M_r(y) = \int_0^y x^r f(x)dx$$

Using the PDF given in (8), we get

$$M_r(y) = \int_0^y x^r\,(1-(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta-1}(-1 + e^{\theta x^\beta})^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\beta^2x^{\beta-1}dx$$

After simplifying we get

$$M_r(y) = \left[\lambda\beta\theta^{k+1}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\binom{\lambda(n+1)-1}{n}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(n-1)^k\frac{q^{r+\frac{1}{\beta}+\frac{k}{\beta}}}{\frac{r}{\beta}+k+1}\right] \quad (15)$$

The first incomplete moment can be obtained by taking in (15) as r=1.

4.4. Quantile Points

The quantile points of LDW distribution are computed by numerically for different sets of values of the parameters as provided in table below by solving the equation

$$x = Q_u = \left[\frac{1}{\theta}\left(log\left(u^{-\frac{1}{\beta}}-1\right)^{-\frac{1}{\lambda}}+1\right)\right]^{1/\beta} \quad (16)$$

Table 2. *Quantile Points of the LDW Distribution.*

Parameters		0.75	0.80	0.85	0.90	0.95	0.99
θ=1, β=0.5, $\lambda = 1.5$	Q_u	0.6091	0.8158	1.1281	1.6615	2.8493	7.1275
θ=1, β=1.5, $\lambda = 1.5$	Q_u	1.2153	1.2980	1.5382	1.7623	1.7623	2.2477
θ=1.5, β=1.5, $\lambda = 1$	Q_u	1.1064	1.2029	1.3205	1.4763	1.7231	2.2341
θ=0.5, β=1.5, $\lambda = 2$	Q_u	2.0101	2.197224	2.4360	2.7725	3.357517	4.7867
θ=1, β=2, $\lambda = 2$	Q_u	1.1246	1.1677	1.2206	1.2914	1.4052	1.6474
θ=2, β=1, $\lambda = 2$	Q_u	0.5025	0.5493	0.6090	0.6931	0.8394	1.1966

4.5. Ordered Statistic

The pdf of the jth order statistic for a random sample of size n from a distribution function $F(x)$ and an associated pdf $f(x)$are given by:

$$f_{j,n}(x) = \frac{n!}{(j-1)(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j}f(x) \quad (17)$$

where $f(x)$ and $F(x)$ are the pdf and cdf of the LDWD, respectively. The pdf of the jth order statistics for a random sample of size n from the LDW distribution is, however, given as follows

$$f_{j,n}(x) = \frac{n!}{(j-1)(n-j)!}\left[\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta}\right]^{j-1}\left[1-\ [(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda})^{-\beta}\right]^{n-j}$$

$$(1+(e^{\theta x^\beta}-1)^{-\lambda})^{-\beta-1}(-1+e^{\theta x^\beta})^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\beta^2x^{\beta-1}$$

So, the pdf of minimum order statistics is obtained by substituting $j = 1$ we have:

$$f_{j,n}(x) = \frac{n!}{(j-1)(n-j)!}\left[1-\ [(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda})^{-\beta}\right]^{n-1}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-1}(-1+e^{\theta x^\beta})^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\beta^2x^{\beta-1}$$

While the corresponding pdf of maximum order statistics is obtained by making the substitution of $j = n$ in above equation

$$f_{j,n}(x) = \frac{n!}{(j-1)(n-j)!}\left[\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta}\right]^{n-1}\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right)^{-\beta-1}(-1+e^{\theta x^\beta})^{-\lambda-1}e^{\theta x^\beta}\theta\lambda\beta^2x^{\beta-1}$$

5. Entropies

Entropy is the measure of variation of random variable X. The theory of entropy has been used for the characterization of numerous standard probability distributions. Two popular entropy measures are the Shannon and Renyi entropies.

The Shannon entropy of a random variable with p.d.f. $f(x)$ is defined as

$$S_H = -\int_0^\infty f(x)logf(x)dx$$

$$logf(x) = -(\beta+1)\ln\left(1+\left(e^{\theta x^\beta}-1\right)^{-\lambda}\right) - (\lambda+1)\ln\left(-1+e^{\theta x^\beta}\right) + \theta x^\beta + \ln(\theta\lambda\beta^2) + (\beta-1)\ln x$$

$$S_H = (1-\lambda\beta)\int_0^{\infty} f(x)\ln\left(e^{\theta x^{\beta}}-1\right)dx - \theta\int_0^{\infty} f(x)\,x^{\beta}dx - \ln(\theta\lambda\beta^2)\int_0^{\infty} f(x)\,dx - (\beta-1)\int_0^{\infty} f(x)\,lnxdx$$

Where $\int_0^{\infty} f(x)\,dx = 1$, $\int_0^{\infty} f(x)\,lnxdx = E(lnx)$, $\int_0^{\infty} f(x)\,x^{\beta}dx = E\left(x^{\beta}\right)$

$$\ln\left(e^{\theta x^{\beta}}-1\right) = \sum_{k=0}^{\infty}\frac{(-\theta)^k x^{k\beta}}{k!}$$

$$S_H = (1-\lambda\beta)\sum_{k=0}^{\infty}\frac{(-\theta)^k \mu'_{k\beta}}{-k!} - \theta E\left(x^{\beta}\right) - ln\,(\theta\lambda\beta^2) - (\beta-1)E(lnx) \tag{18}$$

The Renyi entropy of a random variable X is defined by $I_k = \frac{1}{1-k}log\int_0^{\infty}\left(f(x)\right)^k dx$

$$I_k = \frac{1}{1-k}log\int_0^{\infty}\left((1+(e^{\theta x^{\beta}}-1)^{-\lambda})^{-\beta-1}(-1+e^{\theta x^{\beta}})^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^2x^{\beta-1}\right)^k dx$$

$$I_k = \frac{1}{1-k}\log(\theta^k\lambda^k\beta^{2k}\sum_{h=0}^{\infty}\frac{(-k)_h(-1)^k}{h!})\int_0^{\infty}(1-e^{\theta x^{\beta}})^{\lambda k(\beta+1))}(1-e^{\theta x^{\beta}})^{-k(\lambda+1)}e^{k\theta x^{\beta}}x^{k(\beta-)1}dx$$

$$I_k = \frac{1}{1-k}\log\left(\frac{\theta^{k-1}\lambda^k\beta^{2k-1}}{\theta^{(k-1)\left(1-\frac{1}{\beta}\right)}}\right)\sum_{h=0}^{\infty}\frac{(-k)_h(-1)^k}{h!}\sum_{k=0}^{\infty}\sum_{m=0}^{k(\lambda\beta-1)}(-1)^m\int_0^{\infty}e^{-z(m+k-1)}z^{\left(1-\frac{1}{\beta}\right)(k-1)}e^{-z}dz$$

$$I_k = (1-k)^{-1}\left(\frac{1}{\beta}-1\right)log\theta + klog\lambda + (2k-1)\log\beta\log\left(\sum_{h=0}^{\infty}\frac{(-k)_h(-1)^k}{h!}\sum_{k=0}^{\infty}\sum_{m=0}^{k(\lambda\beta-1)}(-1)^m\frac{\Gamma\left(1-\frac{1}{\beta}\right)(k-1)}{(m+k-2)^{\left(1-\frac{1}{\beta}\right)(k-1)}}\right)$$

6. Bonferroni and Lorenz Curves

In 1905, Max O. Lorenz represented a model for inequality of wealth distribution and C. E. Bonferroni in [1930] proposed a measure of income inequality. Both are used in financial mathematics to check equal distribution of wealth. Bonferroni and Lorenz curves are defined as follows:

Let X be a continuous random variable with probability density function $f(x)$ cumulative distribution function $F(x)$. Let $F^{-1}(.)$ denote the quantile function then the Bonferroni and Lorenz curves of a random variable X are defined by

$$E(p) = \frac{1}{p\mu}\int_0^q x\,f(x)dx$$

And

$$L(p) = \frac{1}{\mu}\int_0^q x\,f(x)dx$$

Respectively, here $\mu = E(x)$ and $q = F^{-1}(p)$. one can reduce (18) and (19) to

$$E(p) = \frac{1}{p\mu}\int_0^q x\,(1-(e^{\theta x^{\beta}}-1)^{-\lambda})^{-\beta-1}(-1+e^{\theta x^{\beta}})^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^2x^{\beta-1}dx$$

After some substitution,

$$E(p) = \frac{1}{p\mu}\frac{\lambda\beta}{\theta^{\frac{1}{\beta}}}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\int_0^{\theta q^{\beta}} z^{\frac{1}{\beta}}(1-e^z)^{-\lambda(n+1)-1}e^z dz$$

Expanding $(1-e^z)^{-\lambda(n+1)-1}$ by using binomial expansion

$$E(p) = \frac{1}{p\mu}\frac{\lambda\beta}{\theta^{\frac{1}{\beta}}}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\binom{\lambda(n+1)-1}{n}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(n-1)^k\int_0^{\theta q^{\beta}} z^{\frac{1}{\beta}+k}dzE(p)$$

$$E(p) = \frac{1}{p\mu}\left[\lambda\beta\theta^{k+1}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\binom{\lambda(n+1)-1}{n}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(n-1)^k\frac{q^{1+\frac{1}{\beta}+\frac{k}{\beta}}}{\frac{1}{\beta}+k+1}\right]$$

and

$$L(p)=\frac{1}{\mu}\left[\lambda\beta\theta^{k+1}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\binom{\lambda(n+1)-1}{n}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(n-1)^k\frac{q^{1+\frac{1}{\beta}+\frac{k}{\beta}}}{\frac{1}{\beta}+k+1}\right]$$

7. Characterization of Distribution by Truncated Moments

In this section, we present our proposed characterizations of three-parameter LDW distribution, with cdf (8) and pdf (9).

Many researchers such as Shakil et al. [28], [29] Ahsanullah et al. [9-11] and Rafique et al. [27] have studied the characterization by truncated moments. We characterize LDW distribution, the first characterization theorem is based on the relation between hazard rate and left truncated moment. The second characterization theorem is based on the relation between reverse hazard rate and right truncated moment. Applying these results, we have characterized LDW distribution by order statistics and upper record values also.

Proposition 1. Suppose the absolutely continuous random variable Y has cdf $F(y)$ with $F(0)=0, F(y)>0\ \forall\ y>0$, p.d.f $f(y)=F'(y)$, the

$$E[Y|Y\le y]=m(y)h(y), y>0$$

And where,

$$h(y)=\frac{f(y)}{F(y)}$$

$$m(y)=\frac{y(1-[1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda})]^{-\beta}+\int_0^y(1-[1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda})]^{-\beta}\mathrm{dt}}{[\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}(-1+e^{\theta x^{\beta}})^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^2x^{\beta-1}]}$$

Proof:

$$E[Y|Y\le y]=\frac{1}{F(x)}\int_0^y tf(t)dt$$

$$m(y)f(y)=\int_0^y tf(t)dt$$

$$m(y)=\frac{\int_0^y tf(t)dt}{f(y)}=\frac{-y(1-F(y)+\int_0^y(1-F(t))dt}{f(y)}$$

Substituting (8) and (9). Then it is easily seen that

$$m(y)=\frac{-y(1-\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta})+\int_0^y(1-[1+\left(e^{\theta t^{\beta}}-1\right)^{-\lambda})]^{-\beta}\mathrm{dt}}{\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}(-1+e^{\theta x^{\beta}})^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^2x^{\beta-1}}$$

Simple differentiation and simplification gives

$$m'(y)=y-m(y)A(y),$$

where

$$A(y)=\frac{f`(y)}{f(y)}=\frac{\theta^4\lambda^2\beta^5x^{4\beta-5}(1-\beta^2)\left(1+\left(\mathrm{e}^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-2}\left(\mathrm{e}^{\theta x^{\beta}}-1\right)^{-\lambda-1}\beta x^{\beta-1}(\lambda+1)\left(\mathrm{e}^{\theta x^{\beta}}-1\right)^{-\lambda-2}\mathrm{e}^{3\theta x^{\beta}}}{\left(1+\left(\mathrm{e}^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}(-1+\mathrm{e}^{\theta x^{\beta}})^{-\lambda-1}\mathrm{e}^{\theta x^{\beta}}\theta\lambda\beta^2x^{\beta-1}}$$

Then we have $m`(y)=y-m(y)\left\{\frac{\theta^4\lambda^2\beta^5x^{4\beta-5}(1-\beta^2)\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-2}\left(e^{\theta x^{\beta}}-1\right)^{-\lambda-1}\beta x^{\beta-1}(\lambda+1)\left(e^{\theta x^{\beta}}-1\right)^{-\lambda-2}e^{3\theta x^{\beta}}}{\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}(-1+e^{\theta x^{\beta}})^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^2x^{\beta-1}}\right\}$

From which we obtain $\frac{y-m`(y)}{m(y)}=\left\{\frac{\theta^4\lambda^2\beta^5x^{4\beta-5}(1-\beta^2)\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-2}\left(e^{\theta x^{\beta}}-1\right)^{-\lambda-1}\beta x^{\beta-1}(\lambda+1)\left(e^{\theta x^{\beta}}-1\right)^{-\lambda-2}e^{3\theta x^{\beta}}}{\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}(-1+e^{\theta x^{\beta}})^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^2x^{\beta-1}}\right\}$

we have $\frac{f`(y)}{f(y)}=\frac{y-m`(y)}{m(y)}$

It follows that

$$\frac{f`(y)}{f(y)}=\left\{\frac{\theta^4\lambda^2\beta^5x^{4\beta-5}(1-\beta^2)\left(1+\left(\mathrm{e}^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-2}\left(\mathrm{e}^{\theta x^{\beta}}-1\right)^{-\lambda-1}\beta x^{\beta-1}(\lambda+1)\left(\mathrm{e}^{\theta x^{\beta}}-1\right)^{-\lambda-2}\mathrm{e}^{3\theta x^{\beta}}}{\left(1+\left(\mathrm{e}^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}(-1+\mathrm{e}^{\theta x^{\beta}})^{-\lambda-1}\mathrm{e}^{\theta x^{\beta}}\theta\lambda\beta^2x^{\beta-1}}\right\}$$

On integrating the above expression with respect to 'y' and simplifying, we obtain

$$lnf(y) = \ln[C\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^{\beta}}\right)^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^{2}x^{\beta-1}]$$

Since C is determined by $\int_0^{\infty} f(y)\,) = 1$, we have the pdf.

Proposition 2. Suppose the absolutely continuous random variable Y has cdf $F(y)$ with

$$F(0) = 0, F(y) > 0\ \forall\, y > 0, \text{p.d.f}\ f(y) = F'(y), \text{then}$$

$$E[Y|Y \geq y] = s(y)h(y), y > 0$$

And where, $h(y) = \frac{f(y)}{1-F(y)}$

$$s(y) = \frac{y(1-[1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda})]^{-\beta} + \int_{y}^{\infty}(1-[1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda})]^{-\beta}dt}{[\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^{\beta}}\right)^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^{2}x^{\beta-1}]}$$

Proof:

$$E[Y|Y \geq y] = \frac{1}{1-F(x)}\int_{y}^{\infty} tf(t)dt$$

$$s(y)f(y) = \int_{y}^{\infty} tf(t)dt$$

$$s(y) = \frac{\int_y^{\infty} tf(t)dt}{f(y)} = \frac{y(1-F(y)+\int_y^{\infty}(1-F(t))dt}{f(y)}$$

Substituting (8) and (9). Then it is easily seen that

$$s(y) = \frac{y(1-\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta}) + \int_{y}^{\infty}(1-[1+\left(e^{\theta t^{\beta}}-1\right)^{-\lambda})]^{-\beta}dt}{\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^{\beta}}\right)^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^{2}x^{\beta-1}}$$

Simple differentiation and simplification gives $s'(y) = -y - s(y)A(y)$, where

$$A(y) = \frac{f`(y)}{f(y)} = \frac{\theta^4\lambda^2\beta^5x^{4\beta-5}(1-\beta^2)\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-2}\left(e^{\theta x^{\beta}}-1\right)^{-\lambda-1}\beta x^{\beta-1}(\lambda+1)\left(e^{\theta x^{\beta}}-1\right)^{-\lambda-2}e^{3\theta x^{\beta}}}{\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^{\beta}}\right)^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^{2}x^{\beta-1}}$$

Then we have $s`(y) = -y - s(y)\left\{\frac{\theta^4\lambda^2\beta^5x^{4\beta-5}(1-\beta^2)\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-2}\left(e^{\theta x^{\beta}}-1\right)^{-\lambda-1}\beta x^{\beta-1}(\lambda+1)\left(e^{\theta x^{\beta}}-1\right)^{-\lambda-2}e^{3\theta x^{\beta}}}{\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^{\beta}}\right)^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^{2}x^{\beta-1}}\right\}$

From which we obtain $\frac{-y+s`(y)}{s(y)} = \left\{\frac{\theta^4\lambda^2\beta^5x^{4\beta-5}(1-\beta^2)\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-2}\left(e^{\theta x^{\beta}}-1\right)^{-\lambda-1}\beta x^{\beta-1}(\lambda+1)\left(e^{\theta x^{\beta}}-1\right)^{-\lambda-2}e^{3\theta x^{\beta}}}{\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^{\beta}}\right)^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^{2}x^{\beta-1}}\right\}$

we have $\frac{f`(y)}{f(y)} = \frac{-y+s`(y)}{s(y)}$

It follows that

$$\frac{f`(y)}{f(y)} = \left\{\frac{\theta^4\lambda^2\beta^5x^{4\beta-5}(1-\beta^2)\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-2}\left(e^{\theta x^{\beta}}-1\right)^{-\lambda-1}\beta x^{\beta-1}(\lambda+1)\left(e^{\theta x^{\beta}}-1\right)^{-\lambda-2}e^{3\theta x^{\beta}}}{\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^{\beta}}\right)^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^{2}x^{\beta-1}}\right\}$$

On integrating the above expression with respect to 'y' and simplifying, we obtain

$$lnf(y) = \ln[C\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}\left(-1+e^{\theta x^{\beta}}\right)^{-\lambda-1}e^{\theta x^{\beta}}\theta\lambda\beta^{2}x^{\beta-1}]$$

Since C is determined by $\int_0^{\infty} f(y)\ ) = 1$, we have the pdf.

Proposition 3: If $X_1, X_2, \ldots, X_n$ be the n independent copies of the random variable X with absolutely continuous distribution function $F(x)$ and pdf $f(x)$, and if $X_{1,n} \le X_{2,n} \le \ldots \le X_{n,n}$ be the corresponding order statistics, then it is known from Ahsanullah et al. [9], chapter 5, or Arnold et al. [12], chapter 2, that $X_{j,n} \mid X_{k,n} = x$, for $1 \le k < j \le n$, is distributed as the $(j-k)th$ order statistics from independent observations from the random variable V having the pdf $f_V(v \mid x)$ where $f_V(v \mid x) = \frac{f(v)}{1-F(x)}$, $0 \le v < x$, and $X_{i.,n} \mid X_{k,n} = x, 1 \le i < k \le n$, is distributed as ith order statistics from k independent observations from the random variable W having the pdf $f_W(w \mid x)$ where $f_W(w \mid x) = \frac{f(w)}{F(x)}, w < x$. Let $S_{k-1} = \frac{1}{k-1}\left(X_{1,n} + X_{2,n} + \ldots + X_{k-1,n}\right)$, and $T_{k,n} = \frac{1}{n-k}\left(X_{k+1,n} + X_{k+2,n} + \ldots + X_{n.n}\right)$. Now, suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $\omega = \inf\{x \mid F(x) > 0\}$, and $\delta = \sup\{x \mid F(x) < 1\}$. We also assume that $f(x)$ is a differentiable for all x, and $E(X)$ exists. Taking $\omega = 0$ and $\delta = \infty$, we have $E(S_{k-1} \mid X_{k,n} = x) = m(x)h(x)$, where $m(x)$ and $h(x)$ are respectively given by the expressions in Proposition 1, if and only if X has the pdf (9).

Proof: It is known that $E(S_{k-1} \mid X_{k,n} = x) = E(X \mid X \le x)$; see Ahsanullah et al. [9], and David and Nagaraja [18]. Hence, by Proposition 1, the result follows.

Proposition 4. Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $\omega = 0$ and $\delta = \infty$. We also assume that $f(x)$ is a differentiable for all x, and $E(X)$ exists. Then $E(T_{k,n} \mid X_{k,n} = x) = s(x)h(x)$, where $s(x)$ and $h(x)$ are respectively given by the expressions in Proposition 2, if and only if X has the pdf (9).

Proof: Since $E(T_{k,n} \mid X_{k,n} = x) = E(X \mid X \ge x)$, see Ahsanullah et al. [9], and David and Nagaraja [18], the result follows from Proposition 2.

Proposition 5. Let be a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and pdf $f(x)$. If $Y_n = \max(X_1, X_2, \ldots, X_n)$ for $n \ge 1$ and $Y_j > Y_{j-1}, j > 1$, then X_j is called an upper record value of $\{X_n, n \ge 1\}$. The indices at which the upper records occur are given by the record times $\{U(n) > \min(j \mid j > U(n+1), X_j > X_{U(n-1)}, n > 1)\}$ and $U(1) = 1$. Let the nth upper record value be denoted by $X(n) = X_{U(n)}$. For details on record values, see Ahsanullah [8].

Now, Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $\omega = 0$ and $\delta = \infty$. We also assume that $f(x)$ is a differentiable for all x, and $E(X)$ exists. Then $E(X(n+1) \mid X(n) = x) == s(x)h(x)$, where $s(x)$ and $h(x)$ are respectively given by the expressions in Proposition 2, if and only if X has the pdf (9).

Proof: It is known from Ahsanullah et al. [8] and Nevzorov [24] that $E(X(n+1) \mid X(n) = x) = E(X \mid X \ge x)$. Then, the result follows from Proposition 2.

8. Maximum Likelihood Estimation

Several approaches for parameter estimation have been proposed in the literature but the maximum likelihood method is the most commonly employed.

Let $X_1, X_2, \ldots, X_m$ be a random sample of size n of the LDW distribution then the total log-likelihood (LL) function is given

$$L(\lambda,\theta,\beta) = \beta^{2n}\prod_{i=1}^{n}\left(1+\left(e^{\theta x^{\beta}}-1\right)^{-\lambda}\right)^{-\beta-1}\lambda^{n}\prod_{i=1}^{n}\left(-1+e^{\theta x^{\beta}}\right)^{-\lambda-1}\theta^{n}\prod_{i=1}^{n}e^{\theta x^{\beta}}\prod_{i=1}^{n}x^{\beta-1}\quad L(\lambda,\theta,\beta)$$

$$= 2n\mathrm{Log}[\beta] + n\mathrm{Log}[\theta] + n\mathrm{Log}[\lambda] + \theta\sum_{I=1}^{n}x_I^{\beta} + (\beta-1)\sum_{I=1}^{n}\mathrm{Log}[x_I] - (\lambda+1)\sum_{I=1}^{n}\mathrm{Log}\left[e^{x_I^{\beta}\theta}-1\right] + \lambda(\beta + 1)\sum_{I=1}^{n}\mathrm{Log}(1+\left(e^{x_I^{\beta}\theta}-1\right))$$

The First derivatives of the log-likelihood function are given as follow

$$\frac{\delta L(\lambda,\theta,\beta)}{\delta\beta}=\frac{2n}{\beta}+\theta\beta\sum_{I=1}^{n}x_I{}^{\beta-1}+\sum_{I=1}^{n}\mathrm{Log}[x_I]-(\lambda+1)\sum_{I=1}^{n}\theta\beta x_I{}^{\beta-1}+2\lambda\sum_{I=1}^{n}\beta x_I{}^{\beta-1}\theta$$

$$\frac{\delta L(\lambda,\theta,\beta)}{\delta\lambda}=\frac{n}{\lambda}-\sum_{I=1}^{n}\mathrm{Log}[e^{x_I{}^{\beta}\theta}-1]+\lambda(\beta+1)\sum_{I=1}^{n}\mathrm{Log}(1+\left(e^{x_I{}^{\beta}\theta}-1\right))$$

$$\frac{\delta L(\lambda,\theta,\beta)}{\delta\theta}=\frac{n}{\theta}+\sum_{I=1}^{n}x_I{}^{\beta}-(\lambda+1)\sum_{I=1}^{n}x_I{}^{\beta}+\lambda(\beta+1)\sum_{I=1}^{n}x_I{}^{\beta}$$

Equating equations to zero and solving them numerically, one can obtain the estimates of the unknown parameters.

9. Simulation Study

This section deals the simulation study. In proposed model we generated random variables by using CDF of LDWD with four different value of parameters for n=25, 50, 100. Parameters are estimated with method of MLE by using each generated random variable.

In statistical study, bias states to the tendency of a measurement process to over or under estimate the value of population parameters. Squared error is a function which obtained from square values of bias. MSE is always constructive. Bias shows the contrasts between estimated values of parameter variation from true value of parameter.

By using the estimated parameters, we calculated Bias and MSE of LDWD. All simulations were done on computational software `Mathematica.

The analysis computes the coming values:

Average bias of the simulated estimates:

$$\frac{1}{1000}\sum_{i=1}^{1000}(\Theta^{*}-\Theta)$$

Average mean square error (MSE) of the simulated estimates:

$$\frac{1}{1000}\sum_{i=1}^{1000}(\Theta^{*}-\Theta)^2$$

The results are reported in Tables 3 and 4.

Table 3. *Average mean of Bias and MSE values for estimators$\hat{\theta}$,$\hat{\beta}$ and $\hat{\lambda}$ of data 1.*

n	**Bias ($\hat{\theta}$)**	**Bias($\hat{\beta}$)**	**Bias($\hat{\lambda}$)**	**MSE($\hat{\theta}$)**	**MSE($\hat{\beta}$)**	**MSE($\hat{\lambda}$)**
25	0.65234	−0.08837	0.56572	0.56572	0.40559	0.86286
50	0.35310	0.23613	0.17950	0.29754	0.52918	0.15562
100	0.27740	0.46278	−0.01169	0.11882	0.72056	0.04713

Table 4. *Average mean of Bias and MSE values for estimators$\hat{\theta}$,$\hat{\beta}$and $\hat{\lambda}$of data 2.*

n	**Bias ($\hat{\theta}$)**	**Bias ($\hat{\beta}$)**	**Bias ($\hat{\lambda}$)**	**MSE (θ)**	**MSE ($\hat{\beta}$)**	**MSE ($\hat{\lambda}$)**
25	1.00745	−0.44500	−0.34131	1.26654	0.25397	0.21862
50	0.57483	−0.23206	−0.39724	0.39676	0.09400	0.19138
100	0.43445	0.41399	−0.68022	0.28879	0.63077	0.50529

10. Evaluation Measures and Practical Data Examples

In this section, we illustrate the usefulness of the log dagum weibull distribution and compare the results with the weibull distribution, gamma distribution, Lomax distribution, exponentiated exponential distribution, Nadarajah exponentiated exponential distributions by means of four real data sets. We will check goodness of fit of our model with some test statistics like AD test, CVM test, K-S test and p-value. All calculations are executed on computational software MATHEMATICA.

In order to demonstrate the proposed methodology, we consider four different practical data sets described below with their analysis. Moreover, perfection of competing models is also tested via the Kolmogrov-Simnorov (K S), the Anderson Darling (A^{*}) and the Cramer-von Misses (W^{*}) statistics. The mathematical expressions for the statistics are given by

$$KS=\max\{\frac{i}{m}-z_i,z_i-\frac{i-1}{m}\}$$

$$A^{*}=\left(\frac{2.25}{m^2}-\frac{0.75}{m}+1\right)\{-1-\frac{1}{m}\sum_{1}^{m}(2i-1)\ln(z_i(z_{m-i+1}))\}$$

$$w^{*}=\sum_{1}^{m}\left(z_i-\frac{2i-1}{2m}\right)^2+\frac{1}{12m}$$

Data set 1. Thefirstdata set of leukemia-free survival times of 50 patients with Autologous transplant. Data sets are presented in the following tables 0.03, 0.493, 0.855, 1.184, 1.283, 1.480, 1.776, 2.138, 2.500, 2.763, 2.993, 3.224, 3.421, 4.178, 4.441, 5.691, 5.855, 6.941, 6.941, 7.993, 8.882, 8.882, 9.145, 11.480, 11.513, 12.105, 12.796, 12.993, 13.849, 16.612, 17.138, 20.066, 20.329, 22.368, 26.776, 28.717, 28.717, 32.928, 33.783, 34.211, 34.770, 39.539, 41.118, 45.033, 46.053, 46.941, 48.289, 57.401, 58.322, 60.625.

Table 5. *AD, CVM, The K-S statistics and p-values for the data set 1.*

Distributions	A^*	W^*	K-S	p-value
LDWD	0.403996	0.0651719	0.076948	0.943568
EED	0.362828	0.0483839	0.084435	0.868171
WD	0.411538	0.0562415	0.0868536	0.845013
GD	0.369975	0.0496265	0.0847622	0.86513
LD	2.504843	0.3799524	0.19666206	0.04182
NEED	0.666096	0.0962511	0.0906376	0.805953

Table 6. *Information Criteria of Different Distributions for Data 1.*

Distributions	AIC	AICC	BIC	HQIC	CAIC
LDWD	394.682	395.140	398.235	396.37	395.140
EED	394.954	395.209	398.778	396.41	395.209
WD	395.433	395.689	399.257	396.89	395.689
GD	395.057	395.312	398.881	396.51	395.312
LD	394.783	395.304	400.5187	396.97	395.304
NEED	396.045	396.301	399.869	397.50	396.301

Data set 2: Second data set Lifetime of 50 devices is 0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

We fit the LDW model and other competitive models such as the Exponentiated exponential distribution (EED), Weibull distribution (WD), Gamma distribution (GD), NEED Nadarajah Exponentiated exponential distribution and Lomax distribution (LD) to data sets.

Table 7. *The K-S statistics and p-values for the data set 2.*

Distributions	A^*	W^*	K-S	p-value
LDWD	0.41395	0.06328	0.07134	0.9135
EED	0.36282	0.04838	0.08444	0.8682
WD	0.41153	0.05624	0.08685	0.8450
GD	0.36997	0.04962	0.08476	0.8651
LD	8.09533	1.66869	0.3377	0.00002
NEED	8.11488	1.67229	0.322722	0.00006

Table 8. *Information Criteria of Different Distributions for Data 2.*

Model	AIC	AICC	BIC	CAIC
LDWD	455.064	455.586	460.800	455.586
EED	483.99	484.246	487.814	484.246
WD	486.004	486.259	489.828	486.259
GD	484.38	484.636	488.204	484.636
NEED	516.033	519.857	516.289	516.857
LD	474.0873	474.3427	477.9114	474.3427

Data set 3: This data set consists of 100 uncensored data on breaking stress of carbon fibres (in Gba), 0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.17, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.57, 1.59, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.73, 1.8, 1.84, 1.84, 1.87, 1.89, 1.92, 2, 2.03, 2.03, 2.05, 2.12, 2.17, 2.17, 2.17, 2.35, 2.38, 2.41, 2.43, 2.48, 2.48, 2.5, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.76, 2.77, 2.79, 2.81, 2.81, 2.82, 2.83, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.51, 3.56, 3.6, 3.65, 3.68, 3.68, 3.68, 3.7, 3.75, 4.2, 4.38, 4.42, 4.7, 4.9, 4.91, 5.08, 5.56.

Table 9. *Information criteria of different distributions for data 3.*

Model	AIC	BIC	AICC	HQIC	CAIC
LDWD	288.62	296.43	288.8685	296.4883	296.4883
GD	290.4673	295.6775	290.5909	292.576	290.5909
WD	289.06	296.87	289.3086	292.2217	289.3086
EED	296.3646(	301.574	296.488	298.4733	296.4883
NEED	393.8472	399.0575	393.9709	395.9559	393.9709
LD	474.0873	477.9114	474.3427	475.54356	474.3427

Table 10. *The K-S statistics and p-values for the data sets 3.*

Distributions	A^*	W^*	K-S	p-value
LDWD	0.39666	0.06508	0.0618	0.8395
EED	1.2341	0.2303	0.1077	0.19618
WD	18.9521	3.7772	0.3341	4.02837×10^{-10}
GD	200.5016	32.9885	0.9996	2.22044×10^{-16}
LD	79.3018	17.3623	0.8210	-2.22044×10^{-16}
NEED	16.9307	3.35163	0.3170	3.73137×10^{-9}

Data set 4: This data consist times to failure of eighteen electronic devices used to show how the proposed distribution can be appliedin practice. 5, 11, 21, 31, 46, 75, 98, 122, 145, 165, 196, 224, 245, 293, 321, 330, 350, 420.

Table 11. *The K-S statistics and p-values for the data sets 4.*

Distributions	A^*	W^*	K-S	p-value
LDWD	0.1725	0.02361	0.0840	0.9996
EED	0.4456	0.07077	0.12138	0.9535
WD	0.4609	0.0644	0.1132	0.9752
GD	0.4487	0.06986	0.1206	0.956104
LD	28.2328	5.0981	0.9157	1.5487×10^{-13}
NEED	2.46950	0.4826	0.28141	0.115548

Table 12. *Information Criteria of Different Distributions for Data 4.*

Model	AIC	BIC	HQIC	AICC	CAIC
LDWD	208.2915	210.9626	208.6598	210.0058	210.0057
GD	226.1	227.9	226.9	229.9	229.9
WD	395.433	397.214	395.6789	396.233	396.233
EED	225.2528	227.0335	225.4983	226.05277	226.0527
NEED	237.8595	239.6403	238.10512	238.6596	238.65956
LD	341.4154	343.1962	341.6609	342.2154	342.2154

Analysis

The measures of goodness of fit including the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (A^*), Cram´er–von Mises (W^*) and Kolmogrov-Smirnov (K-S) statistics are computed to compare the fitted models. In general, the smaller values of these statistics better fit to the data. The required computations are carried out in the Mathematica 11.0.

Tables 4 to 12 represents that the LDWD model produces the heigh p-value and the smallest test statistics value and therefore fitted better than the others (WD, GD, LD, EED, NEED) for the estimated parameter the pdf of the distributions have been superimposed on the histogram of four data sets provided as figures 6, 7 and 8.

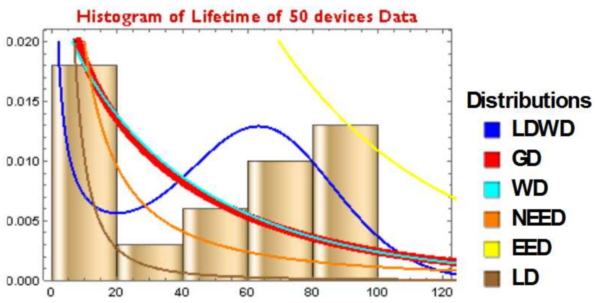


Figure 6. *Fitted densities for data 1.*

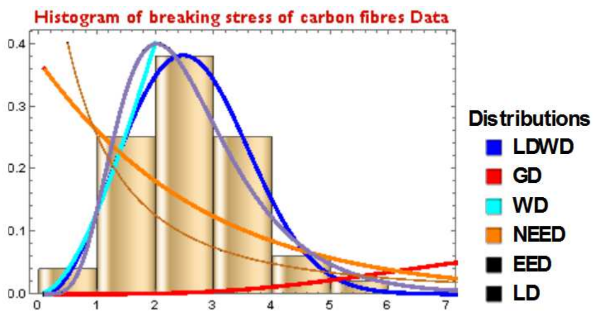


Figure 7. *Fitted densities for data 2.*

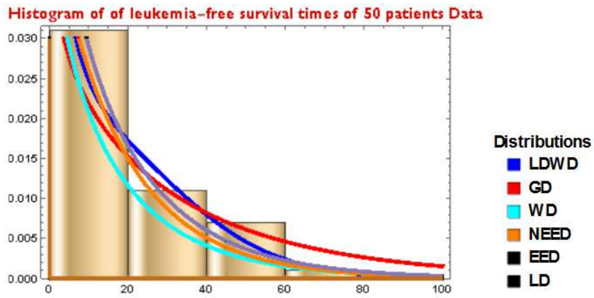


Figure 8. *Fitted densities for data 3.*

11. Conclusion

There has been an increased interest in defining new generated classes of univariate continuous distributions. The extended distributions have attracted several statisticians to develop new models. Current article proposes the new Log-Dagum T-X family of distributions. Some of its mathematical properties have been derived; estimation of the parameters has been discussed. Also, computations of Quantile points and Characterizations have been done. The application of LDW distributions has been studied using four real life-time data sets. It has been observed that the proposed special model is consistently better fit than other competing models. It is expected that this new family and its generated models will attract wider application in several areas such as engineering, survival and lifetime data, hydrology, economy.

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