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Methodology Article

Homotopy Perturbation Method for Solving Nonlinear Fractional Reaction Diffusion Systems

Khaled Abdalla Ishag[1, *], Mohamed Ahmed Abdallah[2], Abulfida Mohamed Ahmed[3]

[1]Department of Basic Science, Faculty of Engineering Science, Omdurman Islamic University, Omdurman, Sudan
[2]Department of Basic Science, Faculty of Engineering, University of Sinnar, Sinnar, Sudan
[3]Department of Mathematics, Faculty of Education, University of Nyala, Nyala, Sudan

Email address:
Khalid.a.ishag@oiu.edu.sd (K. A. Ishag), Mohamed.ah.abd@hotmail.com (M. A. Abdallah), fida11@hotmail.com (A. M. Ahmed)
[*]Corresponding author

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Abstract: This study presents the Homotopy Perturbation Method (HPM) for nonlinear fractional reaction diffusion systems, the fractional derivatives are described in the caputo's' fractional operator. The study focus on three systems of fractional reaction diffusion equations in one, two and three dimensions, in this method, the solution considered as the sum of an infinite series. Which converges rapidly to exact solution. The Homotopy Perturbation Method is no need to use Adomian's polynomials to calculate the nonlinear terms; we test the proposed method to solve nonlinear fractional systems of redaction diffusion equations in one dimension, two dimensions and three dimensions. To show the efficiency and accuracy of this method, we compared the results of the fractional derivatives orders with ordinary derivative order index α_1=α_2=1 for nonlinear fractional reaction diffusion systems. Approximate solutions for different values of fractional derivatives index α_1=0.5 and α_2=0.5 together with non-fractional derivative index α_1=1 and α_2=1 and absolute errors are represented graphically in two and three dimensions. In addition, the graphical represented the solutions, which had been given by MATLAB program. From all numerical results, we can conclude the efficiency of the proposed method for solving different types of nonlinear fractional systems of partial differential equations over existing methods.

Keywords: Fractional Calculus, Diffusion Equations, Homotopy Perturbation Method, Approximate Solutions

1. Introduction

Recently, it has turned out that many phenomena in engineering and other sciences can be described by models using mathematical tools from fractional calculus (FC), fractional calculus owes its origin to a question of whether the meaning of a derivative to an integer order could be extended to still be valid when n is not an integer. In 1819, Lacroix [1] became the first mathematician to publish a paper that mentioned a fractional derivative. Diffusion phenomena is one the most important topic in heat transfer, especially in Mechanics Engineering. In this work, we consider the fractional nonlinear reaction diffusion system [2, 3], is given by

$$\begin{cases} D_t^{\alpha_1} u = \nabla^2 u + u(a(t,x) - b(t,x)u + c(t,x)v) \\ D_t^{\alpha_2} v = \nabla^2 v + v(d(t,x) + e(t,x)u - f(t,x)v) \end{cases} \quad (1)$$

where $n-1 < \alpha_1 \le n,\ n-1 < \alpha_2 \le n,\ a, b, c, d, e, and\ f$ are constants, ∇^2 denotes Laplacian with respect to the variables $x = (x_1, x_2, x_3)$ and $u(x,t), v(x,t)$ is solution of Eq. (1).

2. Preliminaries and Fractional Calculus

In this section, gives some important definitions, such as the gamma function and basic definitions of the fractional derivatives.

2.1. Gamma Function

Gamma function $\Gamma(n)$ is simply the generalization of factorial to complex and real arguments. The gamma function can be defined as [4]

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt = (n-1)!, n \in IN \quad (2)$$

which is convergent for $n > 0$. A recurrence formula for gamma function are [5, 6]

$$\Gamma(n+1) = n\Gamma(n)\ for\ n \in IR^+ \quad (3)$$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}\ for\ n \in IR^- \quad (4)$$

2.2. Fractional Derivatives

Definition (1): Riemann-Liouville Fractional Integral Operator

Suppose that $\alpha > 0$, $n-1 < \alpha \le n$, the Riemann-Lioville fractional integral define as [5, 6, 7]

$$D^{-\alpha}(f(t)) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-u)^{\alpha-1}f(u)du \quad (5)$$

Fractional integral for polynomial and fractional derivative [6, 7, 8, 9]

$$D^{-\alpha}(t^n) = \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}t^{\alpha+n} \quad (6)$$

$$D^{-\alpha}(D^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0), n-1 < \alpha \le n \quad (7)$$

Definition (2): Caputo Fractional Differential Operator

Suppose that $\alpha > 0$, $n-1 < \alpha \le n$, the Caputo fractional differential define as [6, 10, 11, 12, 13, 14]

$$D_c^{-\alpha}(f(t)) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^n(u)}{(t-u)^{\alpha-n+1}}du \quad (8)$$

Definition (3): The Mittag- Leffler Function

Suppose $\alpha > 0, \beta > 0$, then the Mittag-Leffler function define by [7, 14]

$$E_{\alpha,\beta}(t) = \sum_{k=0}^\infty \frac{t^k}{\Gamma(\alpha k+\beta)} \quad (9)$$

3. Homotopy Perturbation Method

The Homotopy Perturbation Method (HPM), which provides an analytical approximate solution [15]. In this section, we extened HPM to Eq. (1) according to this method we construct the following simple homotopy [5, 7]

$$\begin{bmatrix} D_t^{\alpha_1}u = p(\nabla^2 u + u(a(t,x) - b(t,x)u + c(t,x)v)) \\ D_t^{\alpha_2}v = p(\nabla^2 v + v(d(t,x) + e(t,x)u - f(t,x)v)) \end{bmatrix} \quad (10)$$

where $p \in [0,1]$ is an embedding parameter. In case $p = 0$, Eq. (10) is fractional differential equations, which is easy to solve; when $p = 1$, Eq. (10) turns out to be the original system in Eq. (1). The basic assumption is that the solution can be written as a power series in p [8-10]

$$u(x,t) = \sum_{n=0}^\infty p^n u_n(x,t) \quad (11)$$

$$v(x,t) = \sum_{n=0}^\infty p^n v_n(x,t) \quad (12)$$

If $p \to 1$, we obtain the analytical approximate solution of Eq. (11) and Eq. (12)

$$u = \lim_{p\to1}\sum_{n=0}^\infty p^n u_n(x,t) = \sum_{n=0}^\infty u_n(x,t) = u_0 + u_1 + \cdots \quad (13)$$

$$v = \lim_{p\to1}\sum_{n=0}^\infty p^n v_n(x,t) = \sum_{n=0}^\infty v_n(x,t) = v_0 + v_1 + \cdots \quad (14)$$

4. The Steady State Solution

Consider the following steady state problem:

$$\begin{bmatrix} D_t^{\alpha_1}u = \nabla^2 u + au - bu^2 + cuv \\ D_t^{\alpha_2}v = \nabla^2 v + dv + evu - fv^2 \end{bmatrix} \quad (15)$$

where $a, b, c, d, e, and\ f$ are positive constants and $u(t,x), v(t,x)$ is a solution of Eq. (11). The steady state solution satisfies the following equations [1, 16].

$$f(u,v) = au - bu^2 + cuv \quad (16)$$

$$g(u,v) = dv + evu - fv^2 \quad (17)$$

We compute the Jacobian

$$A = \begin{bmatrix} f_u\ g_u \\ f_v\ g_v \end{bmatrix} = \begin{bmatrix} a - 2bu + cv\ ev \\ cu\ d + eu - 2fv \end{bmatrix}$$

$$A = \begin{bmatrix} (a - bu + cv) - bu\ ev \\ cu\ (d + eu - fv) - fv \end{bmatrix}$$

Now we find intersection points from

$$a - bu + cv = 0 \quad (18)$$

$$d + eu - fv = 0 \quad (19)$$

From Eq. (14), in u axis the point is $(\frac{a}{b}, 0)$ and from Eq. (15) in v axis the point is $(0, \frac{d}{f})$, by using simultaneous solution we obtain $(u^*, v^*) = (\frac{af+dc}{bf-ec}, \frac{bd+ea}{bf-ec})$.

If $\frac{b}{e} > \frac{c}{f}$ there are four equilibriums points (0,0), $(\frac{a}{b}, 0)$, $(0, \frac{d}{f})$ and (u^*, v^*).

4.1. Stability Steady

The stability steady of this problem, describing by using eigenvalue problems [1]

$$A - \lambda I = \begin{pmatrix} (a - bu + cv) - bu - \lambda\ ev \\ cu\ (d + eu - fv) - fv - \lambda \end{pmatrix}$$

4.2. The Stability

In this section, we discuss these points (0,0), $\left(\frac{a}{b}, 0\right)$,

$\left(0,\frac{d}{f}\right)$ and (u^*,v^*)

i. $\det(A-\lambda I)|_{(0,0)} = \begin{vmatrix} a-\lambda & 0 \\ 0 & d-\lambda \end{vmatrix} = 0$, $\lambda_1,\lambda_2 > 0$ this point unstable

ii. $\det(A-\lambda I)|_{\left(\frac{a}{b},0\right)} = \begin{vmatrix} -a-\lambda & 0 \\ \frac{ca}{b} & d+e\frac{a}{b}-\lambda \end{vmatrix} = 0$, $\lambda_1 < 0, \lambda_2 > 0$ this point unstable

iii. $\det(A-\lambda I)|_{\left(0,\frac{d}{f}\right)} = \begin{vmatrix} a+c\frac{d}{f}-\lambda & e\frac{d}{f} \\ 0 & -d-\lambda \end{vmatrix} = 0$, $\lambda_1 > 0, \lambda_2 < 0$ this point unstable

iv. $\det(A-\lambda I)|_{(u^*,v^*)} = \begin{vmatrix} -bu^*-\lambda & ev^* \\ cu^* & -fv^*-\lambda \end{vmatrix} = 0$, the characteristic equation is $\lambda^2+(bu^*+fv^*)\lambda+(bf-ce)u^*v^* = 0$, then

$$\lambda_{1,2} = \frac{-(bu^*+fv^*)\pm\sqrt{(bu^*+fv^*)^2-4(bf-ce)u^*v^*}}{2}$$

If $\frac{b}{c} > \frac{e}{f}$, the $Re(\lambda_{1,2}) < 0$ under this condition the problem is stable [1].

5. Numerical Results

Example 4.1: Fractional nonlinear reaction diffusion system in one dimension

$$D_t^{\alpha_1}u = u_{xx}+au-bu^2+cuv$$

$$D_t^{\alpha_2}v = v_{xx}+dv-ev^2+fvu$$

BCs: $u(x,0) = x$ & $v(x,0) = x^2, 0 \le x \le L$

Define the HPM

$$D_t^{\alpha_1}u = p(u_{xx}+au-bu^2+cuv)$$

$$D_t^{\alpha_2}v = p(v_{xx}+dv-ev^2+fvu)$$

Integration both sides, obtain

$$D_t^{-\alpha_1}D_t^{\alpha_1}u = pD_t^{-\alpha_1}(u_{xx}+au-bu^2+cuv)$$

$$D_t^{-\alpha_2}D_t^{\alpha_2}v = pD_t^{-\alpha_2}(v_{xx}+dv-ev^2+fvu)$$

$$u(x,t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}\frac{\partial^k u(x,0)}{\partial t^k} = pD_t^{-\alpha_1}(u_{xx}+au-bu^2+cuv)$$

$$v(x,t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}\frac{\partial^k v(x,0)}{\partial t^k} = pD_t^{-\alpha_2}(v_{xx}+dv-ev^2+fvu)$$

Now define $u(x,t) = \sum_{m=0}^{\infty}p^m u_m$, $v(x,t) = \sum_{m=0}^{\infty}p^m v_m$ & $0 < \alpha_1,\alpha_2 \le 1$

$$\sum_{m=0}^{\infty}p^m u_m = u(x,0)+p\,D_t^{-\alpha_1}[(\sum_{m=0}^{\infty}p^m u_m)_{xx}+a\sum_{m=0}^{\infty}p^m u_m-b(\sum_{m=0}^{m}p^m\,u_m)^2+c(\sum_{m=0}^{m}p^m\,u_m)(\sum_{m=0}^{m}p^m\,v_m)]$$

$$\sum_{m=0}^{\infty}p^m v_m = v(x,0)+p\,D_t^{-\alpha_1}[(\sum_{m=0}^{\infty}p^m v_m)_{xx}+d\sum_{m=0}^{\infty}p^m v_m-e(\sum_{m=0}^{m}p^m\,v_m)^2+f(\sum_{m=0}^{m}p^m\,u_m)(\sum_{m=0}^{m}p^m\,v_m)]$$

Coefficients of p

$$p^0: u_0 = u(x,0) = x$$

$$v_0 = v(x,0) = x^2$$

$$p^1: u_1 = D_t^{-\alpha_1}[(u_0)_{xx}+au_0-bu_0^2+cu_0v_0]$$

$$u_1 = (ax-bx^2+cx^3)\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}$$

$$v_1 = D_t^{-\alpha_2}[(v_0)_{xx}+dv_0-ev_0^2+fu_0v_0]$$

$$v_1 = (2+dx^2-ex^4+fx^3)\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}$$

$$p^2: u_2 = D_t^{-\alpha_1}[(u_1)_{xx}+au_1-2bu_0u_1+cu_0v_1+cv_0u_1]$$

$$u_2 = D_t^{-\alpha_1}\left[(-2b+(6c+a^2)x-3abx^2+(2ac-2b^2)x^3-3bcx^4+c^2x^5)\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}+(2cx+cdx^3-c^2x^5+cfx^4)\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}\right]$$

$$u_2 = (-2b+(6c+a^2)x-3abx^2+(2ac-2b^2)x^3-3bcx^4+c^2x^5)\frac{t^{2\alpha_1}}{\Gamma(2\alpha_1+1)}+(2cx+cdx^3-c^2x^5+cfx^4)\frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}$$

$$v_2 = D_t^{-\alpha_2}[(v_1)_{xx}+dv_1-2ev_0v_1+fu_0v_1+fv_0u_1]$$

$$v_2 = D_t^{-\alpha_2}\left[(4d+8fx-(16e-d^2)x^2-3abx^2+3dfx^3-(3de-f^2)x^4-3efx^5+2e^2x^6)\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}+(afx^3-bfx^4+cfx^5)\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}\right]$$

$$v_2 = (4d+8fx-(16e-d^2)x^2-3abx^2+3dfx^3-(3de-f^2)x^4-3efx^5+2e^2x^6)\frac{t^{2\alpha_2}}{\Gamma(2\alpha_2+1)}+(afx^3-bfx^4+cfx^5)\frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}$$

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$$u(x,t) = \lim_{p \to 1} \sum_{m=0}^{\infty} p^m u_m = u_0 + u_1 + u_2 + \cdots$$

$$v(x,t) = \lim_{p \to 1} \sum_{m=0}^{\infty} p^m v_m = v_0 + v_1 + v_2 + \cdots$$

$$u(x,t) \cong x + (ax - bx^2 + cx^3)\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)} + (-2b + (6c + a^2)x - 3abx^2 + (2ac - 2b^2)x^3 - 3bcx^4 + c^2x^5)\frac{t^{2\alpha_1}}{\Gamma(2\alpha_1+1)} + (2cx + cdx^3 - c^2x^5 + cfx^4)\frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}$$

$$v(x,t) \cong x^2 + (2 + dx^2 - ex^4 + fx^3)\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} + (4d + 8fx - (16e - d^2)x^2 - 3abx^2 + 3dfx^3 - (3de - f^2)x^4 - 3efx^5 + 2e^2x^6)\frac{t^{2\alpha_2}}{\Gamma(2\alpha_2+1)} + (afx^3 - bfx^4 + cfx^5)\frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}$$

Table 1. *The result with time 0.001, the length 10,* $a = 1, b = 2, c = 0.01, d = 1, e = 2$ & $f = 0.02$.

x	$\alpha_1 = \alpha_2 = 1$		$\alpha_1 = \alpha_2 = 0.5$	
	$u(x,y,t)$	$v(x,y,t)$	$u(x,y,t)$	$v(x,y,t)$
0.0000	-0.0000	0.0754	-0.0000	0.0754
1.0000	1.0359	1.1110	1.0359	1.1110
2.0000	2.0705	4.2089	2.0705	4.2089
3.0000	3.1041	9.3420	3.1041	9.3420
4.0000	4.1368	16.4661	4.1368	16.4661
5.0000	5.1688	25.5208	5.1688	25.5208
6.0000	6.2004	36.4319	6.2004	36.4319
7.0000	7.2317	49.1140	7.2317	49.1140
8.0000	8.2631	63.4732	8.2631	63.4732
9.0000	9.2947	79.4115	9.2947	79.4115
10.0000	10.3268	96.8305	10.3268	96.8305
0.0000	-0.0000	0.0754	-0.0000	0.0754

Table 1 shows the approximate solution of fractional nonlinear reaction diffusion system in one dimension, it is noted that only the third order of the Hopotopy perturbation solution. Figure 1 and Figure 2: The surface of system diffusion equation in one dimension is convergence between fractional order and ordinary order, in Figure 3: we get small difference between ordinary order with multiple fractional orders.

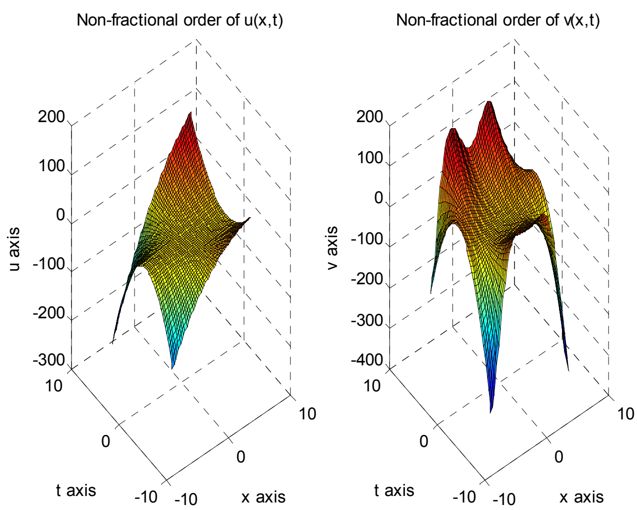

Figure 1. *Graphical presentation of system with non-fractional order.*

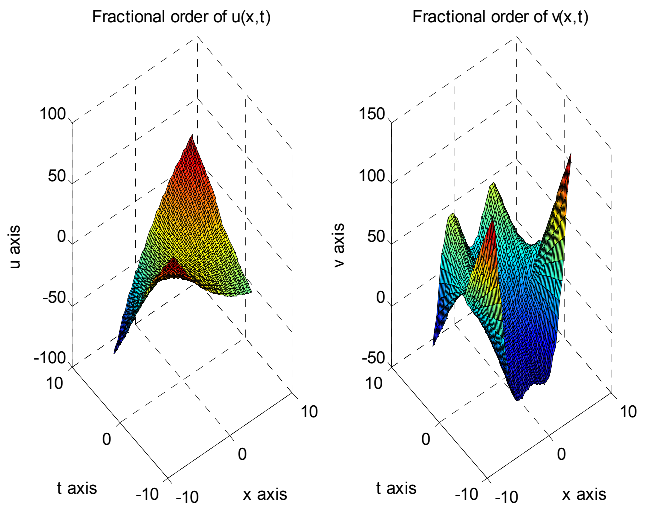

Figure 2. *Graphical presentation of system with fractional order.*

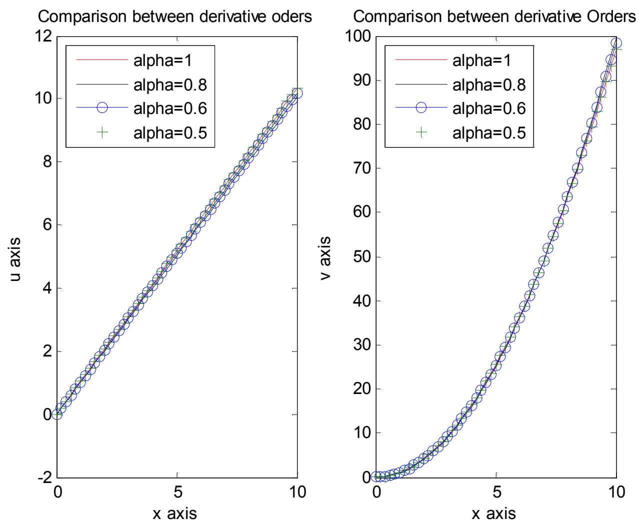

Figure 3. *Comparison between derivative orders system.*

Example 4.2: Fractional nonlinear reaction diffusion system in two dimension

$$D_t^{\alpha_1} u = u_{xx} + u_{yy} + au - bu^2 + cuv$$

$D_t^{\alpha_2} v = v_{xx} + v_{yy} + dv - ev^2 + fvu$

BCs: $u(x, y, 0) = \sinh(x + y)$, $v(x, y, 0) = \cosh(x + y), 0 \le x \le L$

Define the HPM

$$D_t^{\alpha_1} u = p\big(u_{xx} + u_{yy} + au - bu^2 + cuv\big)$$

$$D_t^{\alpha_2} v = p\big(v_{xx} + v_{yy} + dv - ev^2 + fvu\big)$$

Integration both sides, we obtain

$$D_t^{-\alpha_1} D_t^{\alpha_1} u = pD_t^{-\alpha_1}\big(u_{xx} + u_{yy} + au - bu^2 + cuv\big)$$

$$D_t^{-\alpha_2} D_t^{\alpha_2} v = pD_t^{-\alpha_2}\big(v_{xx} + v_{yy} + dv - ev^2 + fvu\big)$$

$$u(x, y, t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \frac{\partial^k u(x,y,0)}{\partial t^k} = pD_t^{-\alpha_1}\big(u_{xx} + u_{yy} + au - bu^2 + cuv\big)$$

$$v(x, y, t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \frac{\partial^k v(x,y,0)}{\partial t^k} = pD_t^{-\alpha_2}\big(v_{xx} + v_{yy} + dv - ev^2 + fvu\big)$$

Now define $u(x, y, t) = \sum_{m=0}^{\infty} p^m u_m$, $v(x, y, t) = \sum_{m=0}^{\infty} p^m v_m$ & $0 < \alpha_1, \alpha_2 \le 1$

$$\sum_{m=0}^{\infty} p^m u_m = u(x, y, 0) + p\, D_t^{-\alpha_1}[(\sum_{m=0}^{\infty} p^m u_m)_{xx} + (\sum_{m=0}^{\infty} p^m u_m)_{yy} + a \sum_{m=0}^{\infty} p^m u_m - b(\sum_{m=0}^{m} p^m\, u_m)^2 + c(\sum_{m=0}^{m} p^m\, u_m)(\sum_{m=0}^{m} p^m\, v_m)]$$

$$\sum_{m=0}^{\infty} p^m v_m = v(x, y, 0) + p\, D_t^{-\alpha_1}[(\sum_{m=0}^{\infty} p^m v_m)_{xx} + (\sum_{m=0}^{\infty} p^m v_m)_{yy} + d \sum_{m=0}^{\infty} p^m v_m - e(\sum_{m=0}^{m} p^m\, v_m)^2 + f(\sum_{m=0}^{m} p^m\, u_m)(\sum_{m=0}^{m} p^m\, v_m)]$$

coefficients of p

$$p^0: u_0 = u(x, y, 0) = \sinh(x + y)$$

$$v_0 = v(x, y, 0) = \cosh(x + y)$$

$$p^1: u_1 = D_t^{-\alpha_1}[(u_0)_{xx} + (u_0)_{yy} + au_0 - bu_0^2 + cu_0 v_0]$$

$$u_1 = [(2 + a)\sinh(x + y) - b\sinh^2(x + y) + c\sinh(x + y)\cosh(x + y)]\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}$$

$$v_1 = D_t^{-\alpha_2}[(v_0)_{xx} + (v_0)_{yy} + dv_0 - ev_0^2 + fu_0 v_0]$$

$$v_1 = [(2 + d)\cosh(x + y) - e\cosh^2(x + y) + f\sinh(x + y)\cosh(x + y)]\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}$$

$$p^2: u_2 = D_t^{-\alpha_1}[(u_1)_{xx} + (u_1)_{yy} + au_1 - 2bu_0 u_1 + cu_0 v_1 + cv_0 u_1]$$

$$v_2 = D_t^{-\alpha_2}[(v_1)_{xx} + (v_1)_{yy} + dv_1 - 2ev_0 v_1 + fu_0 v_1 + fv_0 u_1]$$

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$$u(x, y, t) = \lim_{p\to 1} \sum_{m=0}^{\infty} p^m u_m = u_0 + u_1 + u_2 + \cdots$$

$$v(x, y, t) = \lim_{p\to 1} \sum_{m=0}^{\infty} p^m v_m = v_0 + v_1 + v_2 + \cdots$$

$$u(x, y, t) \cong \sinh(x + y) + [(2 + a)\sinh(x + y) - b\sinh^2(x + y) + c\sinh(x + y)\cosh(x + y)]\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}$$

$$v(x, y, t) \cong \cosh(x + y) + [(2 + d)\cosh(x + y) - e\cosh^2(x + y) + f\sinh(x + y)\cosh(x + y)]\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}$$

Table 2 shows the approximate solution of fractional nonlinear reaction diffusion system in two dimension, it is noted that only the second order of the Hopotopy perturbation solution. Figure 4 and Figure 5: The surface of system diffusion equation in two dimension be affected when changed order equation between fractional and ordinary, in

Figure 6: we get small difference between ordinary order with multiple fractional orders.

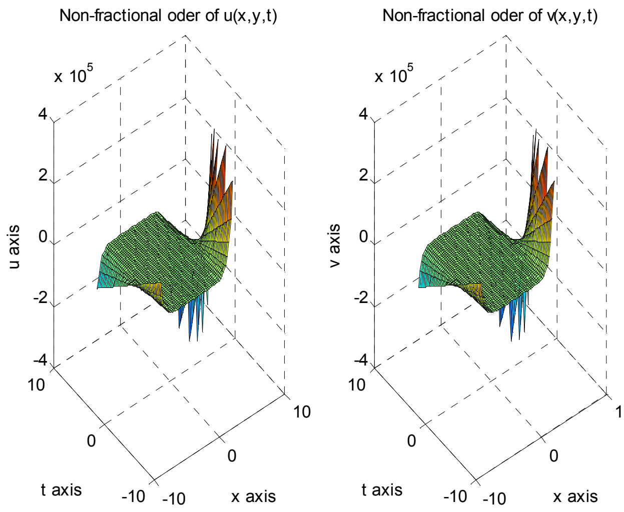

Figure 4. *Graphical presentation of system with non-fractional order.*

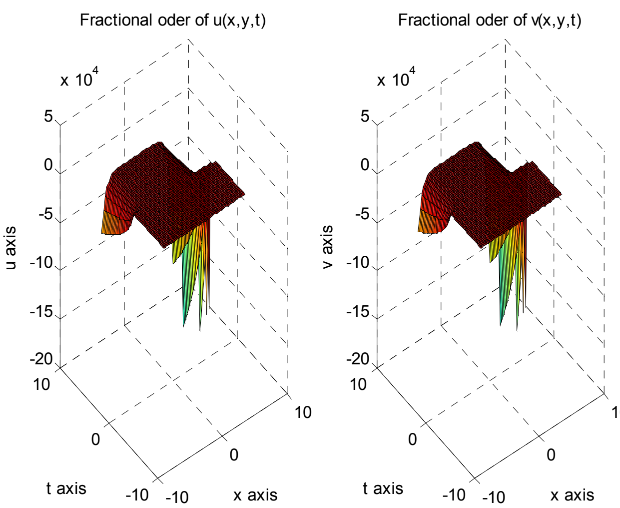

Figure 5. *Graphical presentation of system with fractional order.*

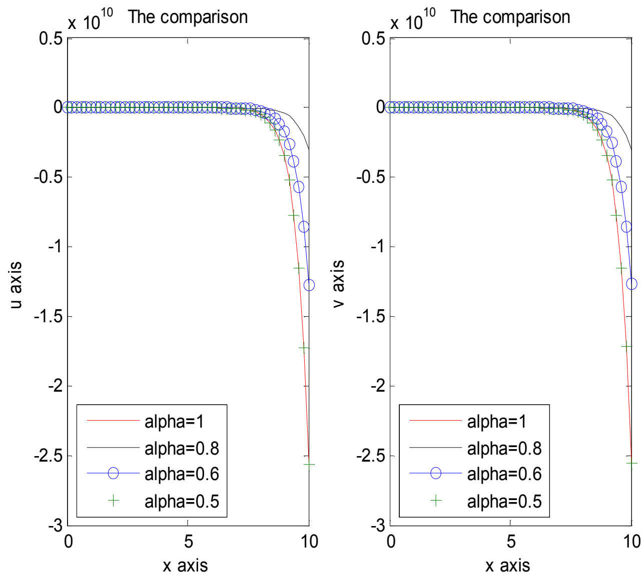

Figure 6. *Comparison between derivative orders system.*

Table 2. *The result with time 0.001, the length 10* $y = 1, a = 1, b = 2, c = 0.01, d = 1, e = 2$ & $f = 0.02$.

x	$\alpha_1 = \alpha_2 = 1$		$\alpha_1 = \alpha_2 = 0.5$	
	$u(x,y,t)$	$v(x,y,t)$	$u(x,y,t)$	$v(x,y,t)$
1.0e+010 *				
0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.0000	-0.0000	-0.0000	-0.0000	-0.0000
0.0000	-0.0001	-0.0001	-0.0001	-0.0001
0.0000	-0.0009	-0.0009	-0.0009	-0.0009
0.0000	-0.0064	-0.0063	-0.0064	-0.0063
0.0000	-0.0470	-0.0468	-0.0470	-0.0468
0.0000	-0.3474	-0.3457	-0.3474	-0.3457
0.0000	-2.5673	-2.5544	-2.5673	-2.5544
1.0e+010 *				

Example 4.3: Fractional nonlinear reaction diffusion system in three dimension

$$D_t^{\alpha_1} u = u_{xx} + u_{yy} + u_{zz} + au - bu^2 + cuv$$

$$D_t^{\alpha_2} v = v_{xx} + v_{yy} + v_{zz} + dv - ev^2 + fvu$$

$$\text{BCs: } u(x,y,z,0) = x + y + z,$$

$$v(x,y,z,0) = xyz, 0 \le x \le L$$

Define the HPM

$$D_t^{\alpha_1} u = p\left(u_{xx} + u_{yy} + u_{zz} + au - bu^2 + cuv\right)$$

$$D_t^{\alpha_2} v = p\left(v_{xx} + v_{yy} + v_{zz} + dv - ev^2 + fvu\right)$$

Integration both sides, we obtain

$$D_t^{-\alpha_1} D_t^{\alpha_1} u = pD_t^{-\alpha_1}\left(u_{xx} + u_{yy} + u_{zz} + au - bu^2 + cuv\right)$$

$$D_t^{-\alpha_2} D_t^{\alpha_2} v = pD_t^{-\alpha_2}\left(v_{xx} + v_{yy} + v_{zz} + dv - ev^2 + fvu\right)$$

$$u(x,y,z,t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}\frac{\partial^k u(x,y,z,0)}{\partial t^k}=pD_t^{-\alpha_1}\left(u_{xx}+u_{yy}+u_{zz}+au-bu^2+cuv\right)$$

$$v(x,y,z,t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}\frac{\partial^k v(x,y,z,0)}{\partial t^k}=pD_t^{-\alpha_2}\left(v_{xx}+v_{yy}+v_{zz}+dv-ev^2+fvu\right)$$

Now define $u(x,y,z,t)=\sum_{m=0}^{\infty}p^m u_m,\ v(x,y,z,t)=\sum_{m=0}^{\infty}p^m v_m\ \&\ 0<\alpha_1,\alpha_2\le 1$

$$\sum_{m=0}^{\infty}p^m u_m=u(x,y,z,0)+p\,D_t^{-\alpha_1}\left[\sum_{m=0}^{\infty}p^m\nabla^2 u_m+a\sum_{m=0}^{\infty}p^m u_m-b\left(\sum_{m=0}^{m}p^m\,u_m\right)^2+c\left(\sum_{m=0}^{m}p^m\,u_m\right)\left(\sum_{m=0}^{m}p^m\,v_m\right)\right]$$

$$\sum_{m=0}^{\infty}p^m v_m=v(x,y,z,0)+p\,D_t^{-\alpha_1}\left[\sum_{m=0}^{\infty}p^m\nabla^2 v_m+d\sum_{m=0}^{\infty}p^m v_m-e\left(\sum_{m=0}^{m}p^m\,v_m\right)^2+f\left(\sum_{m=0}^{m}p^m\,u_m\right)\left(\sum_{m=0}^{m}p^m\,v_m\right)\right]$$

coefficients of p

$$p^0: u_0=u(x,y,0)=x+y+z$$

$$v_0=v(x,y,0)=\text{xyz}$$

$$p^1: u_1=D_t^{-\alpha_1}\left[(u_0)_{xx}+(u_0)_{yy}+(u_0)_{zz}+au_0-bu_0^2+cu_0v_0\right]$$

$$=\left[a(x+y+z)-b(x+y+z)^2+cxyz(x+y+z)\right]\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}$$

$$v_1=D_t^{-\alpha_2}\left[(v_0)_{xx}+(v_0)_{yy}+v_0)_{zz}+dv_0-ev_0^2+fu_0v_0\right]$$

$$=\left[dxyz-ex^2y^2z^2+fxyz(x+y+z)\right]\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}$$

$$p^2: u_2=D_t^{-\alpha_1}\left[(u_1)_{xx}+(u_1)_{yy}+(u_1)_{zz}+au_1-2bu_0u_1+cu_0v_1+cv_0u_1\right]$$

$$v_2=D_t^{-\alpha_2}\left[(v_1)_{xx}+(v_1)_{yy}+(v_1)_{zz}+dv_1-2ev_0v_1+fu_0v_1+fv_0u_1\right]$$

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$$u(x,y,z,t)=\lim_{p\to 1}\sum_{m=0}^{\infty}p^m u_m=u_0+u_1+u_2+\cdots$$

$$v(x,y,z,t)=\lim_{p\to 1}\sum_{m=0}^{\infty}p^m v_m=v_0+v_1+v_2+\cdots$$

$$u(x,y,z,t)\cong x+y+z+\left[a(x+y+z)-b(x+y+z)^2+cxyz(x+y+z)\right]\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}$$

$$v(x,y,t)\cong xyz+\left[dxyz-ex^2y^2z^2+fxyz(x+y+z)\right]\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}$$

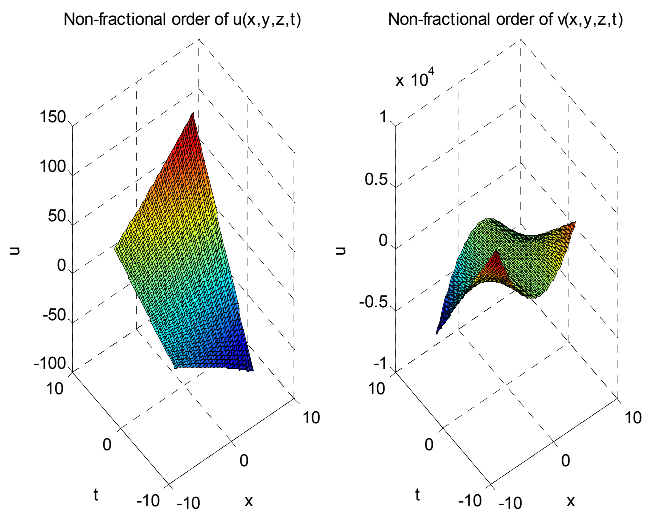

Figure 7. *Graphical presentation of system with non-fractional order.*

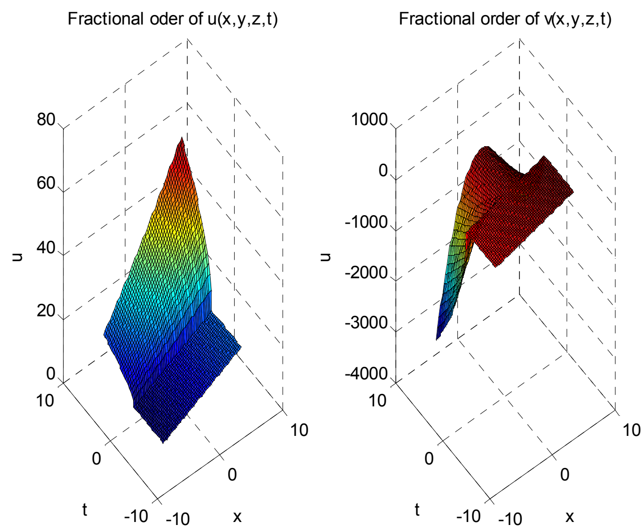

Figure 8. *Graphical presentation of system with fractional order.*

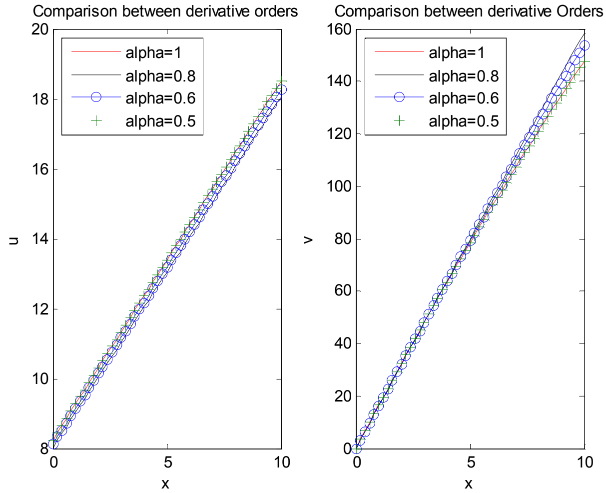

Figure 9. *Comparison between derivative orders system.*

Table 3. *The result with time 0.001, the length 10* $y = 8, z = 4,, a = 1, b = 2, c = 0.01, d = 1, e = 2$ & $f = 0.02$.

x	$\alpha_1 = \alpha_2 = 1$		$\alpha_1 = \alpha_2 = 0.5$	
	$u(x,y,t)$	$v(x,y,t)$	$u(x,y,t)$	$v(x,y,t)$
0.0000	8.2398	0.0000	8.2398	0.0000
1.0000	9.2685	16.3985	9.2685	16.3985
2.0000	10.2969	32.4339	10.2969	32.4339
3.0000	11.3250	48.1062	11.3250	48.1062
4.0000	12.3528	63.4154	12.3528	63.4154
5.0000	13.3804	78.3615	13.3804	78.3615
6.0000	14.4076	92.9444	14.4076	92.9444
7.0000	15.4346	107.1643	15.4346	107.1643
8.0000	16.4613	121.0211	16.4613	121.0211
9.0000	17.4877	134.5147	17.4877	134.5147
10.0000	18.5138	147.6453	18.5138	147.6453

Table 3 shows the approximate solution of fractional nonlinear reaction diffusion system in three dimension, it is noted that only the third order of the Hopotopy perturbation solution. Figure 7 and Figure 8: The surface of system diffusion equation in three-dimension is convergence between fractional order and ordinary order, in Figure 9: we get small difference between ordinary order with multiple fractional orders.

6. Conclusions

Homotopy perturbation Method (HPM) has been successfully applied to obtains analytical approximate solution for fractional nonlinear reaction diffusion systems. It is easy to recognize that HPM is powerful mathematical tool for solving different kinds of linear and/or nonlinear fractional partial differential equations the HPM is no need to use Adomian's polynomials to calculate the nonlinear terms. The mathematical models is very important step to solve physical problem, we have concluded that the fractional derivative of reaction diffusion systems are more accurate than ordinary derivative order. From all numerical results, we can concluded the efficiency of the proposed method for solving different types of nonlinear fractional partial differential equations so we recommended researchers would use Homotopy perturbation Method when derivation the mathematical models for fractional derivatives phenomena.

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