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The Modified Variational Iteration Method with Hermite Polynomials for the Numerical Solution of Tenth Order Boundary Value Problem

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Abstract: In this study, the numerical solution of tenth-order boundary value problems was obtained by employing the modified variational iteration method with Hermite polynomials. The correction functional is corrected for the boundary value problem (BVP) in this proposed method, and the Lagrange multiplier is optimally constructed using variational theory to reduce iteration on the integral operator while minimizing computational time. There was no need for any form of discretization or linearization with this method. The proposed modification also includes the generation of Hermite polynomials for the given boundary value problem and their use as the approximation's basis function. Four numerical examples were also provided to demonstrate the proposed method's effectiveness and reliability. Furthermore, we compared the results to some previously published findings. Tables 1, 2, and 3 show that our proposed method produces a better approximation to the exact solution than the Kasi Viswanadham & Sreenivasulu method, and Table 4 shows that our proposed method produces a better approximation to the exact solution in a few iterations than the Ali, Esra, Dumitru & Mustafa, and Iqbal et al. approaches, Rehman, Pervaiz, and Hakeem techniques (as can be seen from the tables of results). The calculations were carried out using the Maple 18 software.

Keywords: Modified Variational Iteration Method, Boundary Value Problems, Hermite Polynomials, Approximate Solutions

1. Introduction

Consider a generalized boundary value problem of the form:

$$a_n \frac{d^n}{dx^n} v + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} v + a_{n-2} \frac{d^{n-2}}{dx^{n-2}} v \ldots a_1 \frac{d}{dx} v + a_0 v = f(x), a < x < b, \quad (1)$$

with boundary conditions

$$v(a) = A_1, v'(a) = A_2, v''(a) = A_3 \ldots v^n(a) = A_i, v(b) = B_1, v'(b) = B_2, v''(b) = B_3 \ldots v^n(b) = B_i. \quad (2)$$

where $a_n, a_{n-1}, \ldots, a_0$ are constants, $f(x)$ continuous on $[a, b]$ and $A_i, i = 1,2,3 \ldots n$ and $B_i, i = 1,2,3 \ldots n$. These kinds of problems are important in the mathematical modeling of real-world circumstances like viscoelastic flow, heat transfer, and other engineering sciences. Several numerical techniques for solving problems of this type have been developed over the years. For the solution of tenth order boundary value problems Kasi Viswanadham & Sreenivasulu [1] employed the Galerkin Method with Septic B-splines. Ali, Esra, Dumitru & Mustafa [2] also solved tenth order boundary value problem using the reproducing kernel Hilbert space

approach. Iqbal., Rehman, Pervaiz & Hakeem [3] solved linear tenth-order boundary value problems using polynomial and non-polynomial cubic spline approaches. Kasi Viswanadham & Reddy [4] solved ninth-order equations using the petrov-galerkin method. Reddy [5] solved ninth order boundary value problem using the collocation method. Akram & Sadaf [6] solved ninth-order boundary value problems using the homotopy analysis method. For eight-order boundary value problems. Noor & Mohyud-Din [7] developed the variational iteration decomposition method. For solving fifth-order boundary value problems, Noor & Mohyud-Din [8] designed and implemented the homotopy perturbation approach and the variational iteration method. Mohyud-Din & Yildirim [9] also used the homotopy perturbation method and variational iteration to solve ninth and tenth-order boundary value problems. For solving fifth-order and other higher-order boundary value problems, Noor & Mohyud-Din [10] developed and employed the Adomian decomposition approach and the variational iteration method. Siddiqi & Iftikhar [11] also solved seventh-order boundary value problems using the homotopy perturbation approach and the variational iteration method. Njoseh & Mamadu [12] recently introduced the power series approximation approach (PSAA) as a generalized method for this problem. Mamadu & Njoseh [13] also employed the tau technique and the tau-collocation approximation approach extensively for the solution of first and second ordinary differential equations. Caglar, Caglar, & Twizellll [14] are also interested in finding a numerical solution to a fifth-order boundary value problem using a sixth-degree B-spline. In addition, the Adomian decomposition method by Adomian [15] is used to solve the linear and nonlinear cases of this problems. For twelfth order boundary value problems, Yahya & Liu [16] used the differential transform method (DTM). Siddiqi & Twizell [17] presented spline solution of linear tenth order boundary value problem. Also, Siddiqi & Akram [18] used non-polynomial spline technique for solving tenth order boundary value problems. Abdulla & Mohammed [19] used variational iteration method for the solution of seventh order boundary value problem. Hamid & Khadigeh [20] proposed a computational method for the approximate solution of linear and nonlinear two-point boundary value problem using Bernstin polynomials. In this study, the variational iteration method with Hermite polynomials is used to solve a tenth-order boundary value problem. The correction functional is corrected for the BVP in this suggested method, and the Lagrange multiplier is ideally constructed using variational theory. The proposed strategy works well, and the findings thus far are positive and consistent. Finally, the solution is given in the form of an infinite series, which is usually convergent.

2. The Standard Variational Iteration Technique

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lv + Nv - g(x) = 0, \tag{3}$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the inhomogeneous term According to variational iteration method, we can construct a correction functional as follows

$$v_{i+1} = v_i(x) + \int_0^x \lambda(t)\left(Lv_i(t) + N\widetilde{v_i(t)} - g(t)\right)dt \tag{4}$$

where $\lambda(t)$ is a Lagrange multiplier, which can be identified optimally via variational iteration technique. The subscripts n denote the nth approximation, $\widetilde{v_i}$ is considered as a restricted variation. i.e., $\widetilde{v_i} = 0$. The relation (4) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. In this method, we need to determine the langrange multiplier $\lambda(t)$ optimally and hence the successive approximation of solution v will be readily obtained upon using the Langrange multiplier and our v_0, and the solution is given by

$$\lim_{i-\infty} v_i = v \tag{5}$$

The Lagrange Multiplier also play an important role in determining the solution of the problem, and can be defined as follows:

$$(-1)^n \frac{1}{(n-1)!}(t-x)^{n-1} \tag{6}$$

3. Hermite Polynomials

The Hermite polynomials of degree N is given as

$$H_N(x) = (-1)^N e^{x^2} \frac{d^N}{dx^N} e^{-x^2} \tag{7}$$

Replacing N with N-1, we have the following:

$$H_{N-1}(x) = (-1)^{N-1} e^{x^2} \frac{d^{N-1}}{dx^{N-1}} e^{-x^2} \tag{8}$$

Hence, the Hermite polynomials required for this work is given below:

$$H_{0,9}(x) = 1$$

$$H_{1,9}(x) = 2x$$

$$H_{2,9}(x) = 4x^2 - 2$$

$$H_{3,9}(x) = 8x^3 - 12x$$

$$H_{4,9}(x) = 16x^4 - 48x^2 + 12$$

$$H_{5,9}(x) = 32x^5 - 160x^3 + 120x \tag{9}$$

4. Modified Variational Iteration Method Using Hermite Polynomials (MVIMHP)

Using (2.1) and (2.2), we assume an approximate solution of the form

$$v_{i,N-1}(x) = \sum_{i=0}^{N-1} a_{i,N-1} H_{i,N-1}(x) \tag{10}$$

where $H_{i,N-1}(x)$ are Hermite polynomials, $a_{i,N-1}$ are constants to be determined, and N the degree of approximant.

Hence we obtain the following iterative method

$$v_{i+1,N-1}(x) = \sum_{i=0}^{N-1} a_{i,N-1}H_{i,N-1}(x) + \int_0^x \lambda(t)\Big(L\sum_{i=0}^{N-1} a_{i,N-1}H_{i,N-1}(t) + N\sum_{i=0}^{N-1} a_{i,N-1}H_{i,N-1}(t)\Big)dt \tag{11}$$

5. Numerical Applications

In this section, we solved four examples using the provided method. The numerical results further demonstrate the proposed scheme's accuracy and efficiency.

5.1. Example 1

Considers the following tenth order boundary value problem [1]

$$v^{(10)} + v = -10(2x\sin x - 9\cos x), -1 \le x \le 1 \tag{12}$$

$$v(-1) = v(1) = 0, v'(1) = -v'(-1) = 2\cos 1, v''(-1) = v''(1) = 2\cos 1 - 4\sin 1$$

$$v'''(-1) = -v'''(1) = 6\cos 1 + 6\sin 1,$$

$$v^{(4)}(-1) = v^{(4)}(1) = -12\cos 1 + 8\sin 1. \tag{13}$$

The exact solution for the problem is $v = (x^2 - 1)\cos x$.
The correction functional for the boundary value problem (12) and (13) is given as

$$v_{i+1} = v_i(x) + \int_0^x \lambda(t)\Big(v^{(10)} + v + 10(2t\sin t - 9\cos t)\Big)dt \tag{14}$$

where, $\lambda(t) = \frac{(-1)^{10}(t-x)^9}{9!}$ is the Lagrange multiplier.

Using the modified variational iteration approach with Hermite polynomials, we assume an approximation solution of the form

$$v_{n,9}(x) = \sum_{i=0}^9 a_{i,9}H_{i,9}(x) \tag{15}$$

Hence, we get the following iterative formula:

$$v_{n+1,N-1}(x) = \sum_{i=0}^9 a_{i,9}H_{i,9}(x) + \int_0^x \frac{(t-x)^9}{9!}\left(\frac{d^{10}}{dt^{10}}\Big(\sum_{i=0}^9 a_{i,9}H_{i,9}(x)\Big) + \sum_{i=0}^9 a_{i,9}H_{i,9}(t) + 10(2t\sin t - 9\cos t)\right)dt \tag{16}$$

$$v_{n+1,N-1}(x) = a_{0,9}H_{0,9}(x) + a_{1,9}H_{1,9}(x) + a_{2,9}H_{2,9}(x) + a_{3,9}H_{3,9}(x) + a_{4,9}H_{4,9}(x) + a_{5,9}H_{5,9}(x) + a_{6,9}H_{6,9}(x) + a_{7,9}H_{7,9}(x) +$$

$$a_{8,9}H_{8,9}(x) + a_{9,9}H_{9,9}(x) + \int_0^x \frac{(t-x)^9}{9!}\left(\frac{d^{10}}{dt^{10}}\Big(\sum_{i=0}^9 a_{i,9}H_{i,9}(x)\Big) + \sum_{i=0}^9 a_{i,9}H_{i,9}(t) + 10(2t\sin t - 9\cos t)\right)dt \tag{17}$$

As a result of (9), iteration, and application of the boundary conditions (13), the values of the unknown constants can be determined as follows

$$a_{0,9} = -0.5847981771, a_{1,9} = 0, a_{2,9} = 0.0712890626, a_{3,9} = 0, a_{4,9} = -0.01831054688,$$
$$a_{5,9} = 0, a_{6,9} = 0.00036498264, a_{7,9} = 0 \; a_{8,9} = -0.0000055222284, \; a_{9,9} = 0$$

Consequently, the series solution is given as

$$v(x) = \frac{1}{12164510040883200}x^{20} + 1.500000000x^2 - 1.000000000 + 4.79509653610^{-14}x^{18} - 1.15185403710^{-11}x^{16} +$$
$$2.09914644510^{-9}x^{14} - 2.77660867810^{-7}x^{12} + 0.00002507716050x^{10} - 0.001413690470x^8 + 0.04305555549x^6 -$$
$$0.5416666665x^4 \tag{18}$$

5.2. Example 2

Considers the following tenth order boundary value problem [1]

$$v^{(10)} - (x^2 - 2x)v = 10\cos x - (x-1)^3\sin x, 1 \; -1 \le x \le \tag{19}$$

with boundary conditions

$$v(-1) = 2sin1, v(1) = 0, v'(-1) = -2cos1 - sin1, v'(1) = sin1,$$

$$v''(-1) = 2cos1 - 2sin1, v''(1) = 2cos1, v'''(-1) = 2cos1 + 3sin1,$$

$$v'''(1) = -3sin1, v^{(4)}(-1) = -4cos1 + 2sin1, v^{(4)}(1) = -4cos1. \tag{20}$$

The exact solution for the problem is $v(x) = (x-1)sinx$.

The correction functional for the boundary value problem (19) and (20) is given as

$$v_{i+1} = v_i(x) + \int_0^x \lambda(t) \left(v^{(10)} - (t^2 - 2t)v - 10cost + (t-1)^3 sint\right) dt \tag{21}$$

where $\lambda(t) = \frac{(-1)^{10}(t-x)^9}{9!}$ is the Lagrange multiplier.

Using the modified variational iteration approach with Hermite polynomials, we assume an approximation solution of the form

$$v_{n,9}(x) = \sum_{i=0}^9 a_{i,9} H_{i,9}(x) \tag{22}$$

Hence, we get the following iterative formula

$$v_{n+1,N-1}(x) = \sum_{i=0}^9 a_{i,9} H_{i,9}(x) + \int_0^x \frac{(t-x)^9}{9!} \left(\frac{d^{10}}{dt^{10}} \left(\sum_{i=0}^9 a_{i,9} H_{i,9}(t)\right) - (t^2 - 2t) \sum_{i=0}^9 a_{i,9} H_{i,9}(t) - 10cost + (t-1)^3 sint\right) dt \tag{23}$$

$$v_{n+1,N-1}(x) = a_{0,9} H_{0,9}(x) + a_{1,9} H_{1,9}(x) + a_{2,9} H_{2,9}(x) + a_{3,9} H_{3,9}(x) + a_{4,9} H_{4,9}(x) + a_{5,9} H_{5,9}(x) + a_{6,9} H_{6,9}(x) + a_{7,9} H_{7,9}(x) +$$
$$a_{8,9} H_{8,9}(x) + a_{9,9} H_{9,9}(x) + \int_0^x \frac{(t-x)^9}{9!} \left(\frac{d^{10}}{dt^{10}} \left(\sum_{i=0}^9 a_{i,9} H_{i,9}(t)\right) - (t^2 - 2t) \sum_{i=0}^9 a_{i,9} H_{i,9}(t) - 10cost + (t-1)^3 sint\right) dt \tag{24}$$

As a result of (9), iteration, and application of the boundary conditions (20), the values of the unknown constants can be determined as follows

$$a_{0,9} = 0.3893229167, a_{1,9} = -0.389404297, a_{2,9} = 0.1458333334, a_{3,9} = 0.01622178$$

$$, a_{4,9} = -0.00716145834, \ a_{5,9} = -0.0002034505 \ a_{6,9} = 0.000086805557,$$
$$a_{7,9} = 0.00000116258, a_{8,9} = -7.750495810^{-7} \ a_{9,9} = -5.382310^{-9}$$

Consequently, the series solution is given as

$$v(x) = -1.000000x - 3.1410^{-10} - \frac{1}{85151570286182400} x^{22} + 2.50521084010^{-8} x^{11} - 2.50521083410^{-8} x^{12} - 1.6059446110^{-10} x^{13} +$$
$$1.60590430510^{-10} x^{14} + 7.647178310^{-13} x^{15} - 7.6471588110^{-13} x^{16} - 2.8115622110^{-15} x^{17} - 0.000002755737600 x^9 -$$
$$0.0001984126925 x^8 + 0.0001984135168 x^7 + 0.00833333343 x^6 - 0.00833340723 x^5 - 0.166666672 x^4 + 0.166666815 x^3 +$$
$$1.0000000 x^2 + 2.8114469910^{-15} x^{18} + 8.223040810^{-18} x^{19} - 4.110291810^{-18} x^{20} + 4.30604259810^{-18} x^{21} \tag{25}$$

5.3. Example 3

Considers the following tenth order boundary value problem [1]

$$v^{(10)} - v'' + vx = (-8 + x - x^2)e^x, 0 \le x \le 1 \tag{26}$$

with boundary conditions

$$v(0) = 1, v(1) = 0, v'(0) = 0, v'(1) = -e, v''(0) = -1, v''(1) = -2e, v'''(0) = -2,$$

$$v'''(1) = -3e, v^{(4)}(0) = -3, v^{(4)}(1) = -4e. \tag{27}$$

The exact solution of the example is $v(x) = (1-x)e^x$.

The correction functional for the boundary value problem (26) and (27) is given as

$$v_{i+1} = v_i(x) + \int_0^x \lambda(t) \left(v^{(10)} - v'' + vt - (-8 + t - t^2)e^t\right) dt \tag{28}$$

where $\lambda(t) = \frac{(-1)^{10}(t-x)^9}{9!}$ is the Lagrange multiplier.

Using the modified variational iteration approach with Hermite polynomials, we assume an approximation solution of the form

$$v_{n,9}(x) = \sum_{i=0}^9 a_{i,9} H_{i,9}(x) \tag{29}$$

Hence, we get the following iterative formula

$$v_{i+1,N-1}(x) = \sum_{i=0}^{9} a_{i,9}H_{i,9}(x) + \int_0^x \frac{(t-x)^9}{9!}\left(\frac{d^{10}}{dt^{10}}\left(\sum_{i=0}^{9} a_{i,9}H_{i,9}(t)\right) - \frac{d^2}{dt^2}\left(\sum_{i=0}^{9} a_{i,9}H_{i,9}(t)\right) + t\left(\sum_{i=0}^{9} a_{i,9}H_{i,9}(t)\right) - (-8+t-t^2)e^t\right)dt \quad (30)$$

$$v_{i+1,N-1}(x) = a_{0,9}H_{0,9}(x) + a_{1,9}H_{1,9}(x) + a_{2,9}H_{2,9}(x) + a_{3,9}H_{3,9}(x) + a_{4,9}H_{4,9}(x) + a_{5,9}H_{5,9}(x) + a_{6,9}H_{6,9}(x) + a_{7,9}H_{7,9}(x) +$$
$$a_{8,9}H_{8,9}(x) + a_{9,9}H_{9,9}(x) + \int_0^x \frac{(t-x)^9}{9!}\left(\frac{d^{10}}{dt^{10}}\left(\sum_{i=0}^{9} a_{i,9}H_{i,9}(t)\right) - \frac{d^2}{dt^2}\left(\sum_{i=0}^{9} a_{i,9}H_{i,9}(t)\right) + t\left(\sum_{i=0}^{9} a_{i,9}H_{i,9}(t)\right) - (-8+t-t^2)e^t\right)dt \quad (31)$$

As a result of (9), iteration, and application of the boundary conditions (27), the values of the unknown constants can be determined as follows

$$a_{0,9} = 0.642090, a_{1,9} = -0.320964, a_{2,9} = -0.240558\ a_{3,9} = -0.066842,\ a_{4,9} = -0.011637, a_{5,9} = -0.0014975\ a_{6,9} =$$
$$-0.00014647,\ a_{7,9} = -0.00001240,\ a_{8,9} = -6.79 \times 10^{-7}\ a_{9,9} = -4.30 \times 10^{-8}$$

Consequently, the series solution is given as

$$v(x) = 5.185350903 \times 10^{-16}x^{19} - 1.246791975 \times 10^{-15}x^{18} - 2.251646721 \times 10^{-14}x^{17} - 7.172775291 \times 10^{-13}x^{16} - 1.070659303 \times 10^{-11}x^{15} -$$
$$1.490877143 \times 10^{-10}x^{14} - 1.927126629 \times 10^{-9}x^{13} - 2.296507403 \times 10^{-8}x^{12} - 2.505208977 \times 10^{-7}x^{11} - 0.000002408152469x^{10} -$$
$$0.001190912000x^7 - 0.006940544000x^6 - 0.03333491200x^5 - 0.1250121600x^4 - 0.3333324800x^3 - 0.4999886400x^2 +$$
$$0.000007680000x + 3.283801310 \times 10^{-17}x^{20} - 0.0000220160000x^9 - 0.0001738240000x^8 + 0.9999976800 \quad (32)$$

5.4. Example 4

Considers the following tenth order boundary value problem [2, 3]

$$v^{(10)} = -(80 + 19x + x^2)e^x, 0 \le x \le 1 \quad (33)$$

with boundary conditions

$$v(0) = 0, v(1) = 0, v''(0) = 0, v''(1) = -4e, v^{(4)}(0) = -8, v^{(4)}(1) = -16e,$$

$$v^{(6)}(0) = -24, v^{(6)}(1) = -36e, v^{(8)}(0) = -48, v^{(8)}(1) = -64e. \quad (34)$$

The exact solution of the example is $v(x) = x(1-x)e^x$.
The correct functional for the boundary value problem (33) and (34) is given as

$$v_{i+1} = v_i(x) + \int_0^x \lambda(t)\left(v^{(10)} + (80 + 19t + t^2)e^t\right)dt \quad (35)$$

where $\lambda(t) = \frac{(-1)^{10}(t-x)^9}{9!}$ is the Lagrange multiplier.

Using the modified Variational iteration approach with Hermite polynomials, we assume an approximation solution of the form

$$v_{n,9}(x) = \sum_{i=0}^{9} a_{i,9}H_{i,9}(x) \quad (36)$$

Hence, we get the following iterative formula

$$v_{i+1,N-1}(x) = \sum_{i=0}^{9} a_{i,9}H_{i,9}(x) + \int_0^x \frac{(t-x)^9}{9!}\left(\frac{d^{10}}{dt^{10}}\left(\sum_{i=0}^{9} a_{i,9}H_{i,9}(t)\right) + (80 + 19t + t^2)e^t\right)dt \quad (37)$$

$$v_{i+1,N-1}(x) = a_{0,9}H_{0,9}(x) + a_{1,9}H_{1,9}(x) + a_{2,9}H_{2,9}(x) + a_{3,9}H_{3,9}(x) + a_{4,9}H_{4,9}(x) + a_{5,9}H_{5,9}(x) + a_{6,9}H_{6,9}(x) +$$
$$a_{7,9}H_{7,9}(x) + a_{8,9}H_{8,9}(x) + a_{9,9}H_{9,9}(x) + \int_0^x \frac{(t-x)^9}{9!}\left(\frac{d^{10}}{dt^{10}}\left(\sum_{i=0}^{9} a_{i,9}H_{i,9}(t)\right) + (80 + 19t + t^2)e^t\right)dt \quad (38)$$

As a result of (9), iteration, and application of the boundary conditions (34), the values of the unknown constants can be determined as follows

$$a_{0,9} = -0.320312500, a_{1,9} = -0.1600748750, a_{2,9} = -0.3593750000,$$
$$a_{3,9} = -0.1668294287,\ a_{4,9} = -0.04036458333, a_{5,9} = -0.0006697591893, a_{6,9} = -0.0007812500000,\ a_{7,9} =$$
$$-0.00007866757655, a_{8,9} = -0.000004650297619\ a_{9,9} = -3.39085779810^{-7}$$

Consequently, the series solution is given as

$$v(x) = \frac{1}{11432810188800}x^{18} - \frac{1}{2032499589120}x^{17} - \frac{1}{93405312000}x^{16} - \frac{1}{6706022400}x^{15} - \frac{1}{518918400}x^{14} - \frac{1}{43545600}x^{13} -$$
$$\frac{1}{3991680}x^{12} - \frac{1}{403200}x^{11} - \frac{1}{45360}x^{10} - 0.0001736119193x^9 - 0.001190476190x^8 - 0.006944435253x^7 -$$
$$0.0333333333x^6 - 0.1250000441x^5 - 0.33333333333x^4 - 0.4999999069x^3 + 0.999999410x \quad (39)$$

5.5. Tables

Table 1. *(Error estimates): The result of the proposed method compared Galerkin Method with Septic B-splines [1].*

x	Exact solution	Approximate solution	Absolute Error by the proposed method	GMSB-s Error
-0.8	-0.2508144153	-0.2508144153	0.000000e-00	5.483627e-06
-0.6	-0.5282147935	-0.5282147936	1.000000e-10	9.536743e-07
-0.4	-0.7736912350	-0.7736912350	0.000000e-00	8.702278e-06
-0.2	-0.9408639147	-0.9408639147	0.000000e-00	2.980232e-07
0.0	-1.0000000000	-1.0000000000	0.000000e-00	1.955032e-05
0.2	-0.9408639147	-0.9408639147	0.000000e-00	2.920628e-05
0.4	-0.7736912350	-0.7736912350	0.000000e-00	2.169609e-05
0.6	-0.5282147935	-0.5282147936	1.000000e-10	7.390976e-06
0.8	-0.2508144153	-0.2508144153	0.000000e-00	7.450581e-07

Table 2. *(Error estimates): The result of the proposed method compared Galerkin Method with Septic B-splines [1].*

X	Exact solution	Approximate solution	Absolute Error by the proposed method	GMSB-s Error
-0.8	1.291240964	1.291240962	2.000000e-09	4.649162e-06
-0.6	0.9034279574	0.9034279578	4.000000e-10	1.329184e-05
-0.4	0.5451856792	0.5451856802	1.000000e-09	2.050400e-05
-0.2	0.2384031970	0.2384031975	5.000000e-10	9.477139e-06
0.0	0.0000000000	-3.140000e-10	3.140000e-10	2.731677e-06
0.2	-0.1589354646	-0.1589354659	1.300000e-09	1.458824e-05
0.4	-0.2336510054	-0.2336510066	1.200000e-09	2.110004e-05
0.6	-0.2258569894	-0.2258569898	4.000000e-10	1.908839e-05
0.8	-0.1434712182	-0.1434712168	1.400000e-09	1.342595e-05

Table 3. *(Error estimates): The result of the proposed method compared Galerkin Method with Septic B-splines [1].*

x	Exact solution	Approximate solution	Absolute Error by the proposed method	GMSB-s Error
0.1	0.9946538262	0.9946523875	1.438700e-06	1.537800e-05
0.2	0.9771222064	0.9771218640	3.424000e-07	4.452467e-05
0.3	0.9449011656	0.9449020952	9.296000e-07	3.331900e-05
0.4	0.8950948188	0.8950971305	2.311700e-06	3.552437e-05
0.5	0.8243606355	0.8243643494	3.713900e-06	9.477139e-06
0.6	0.7288475200	0.7288525498	5.029800e-06	2.586842e-05
0.7	0.6041258121	0.6041319544	6.142300e-06	3.975630e-05
0.8	0.4451081856	0.4451151184	6.932800e-06	3.531575e-05
0.9	0.2459603111	0.2459676009	7.289800e-06	2.214313e-05

Table 4. *(Error estimates): The result of the proposed method compared with Reproducing Kernel Hilbert Space Method [2, 3].*

x	Exact solution	Absolute Error by the proposed method	AE (RKHSM) [2]	AE (NPCSM) [3]	AE (PCSM) [3]
0.2	0.195424441	1.110000e-08	3.330000e-08	2.433000e-07	3.982000e-04
0.4	0.358037927	1.810000e-08	7.031000e-08	3.986000e-07	6.663000e-04
0.6	0.358037927	1.840000e-08	6.076000e-08	4.428000e-07	7.598000e-04
0.8	0.356086549	1.200000e-08	2.682000e-08	3.328000e-07	5.885000e-04

6. Conclusion

The modified Variational iteration method using Hermite polynomials was effectively employed in this research to obtain numerical solutions to tenth-order boundary value problems. Hermite polynomials are combined with the Variational iteration approach in the modification. The approach produces rapidly converging series solutions, which are common in physical issues. Tables 1 to 4 show that the suggested strategy outperforms methods in the literature. Finally, the numerical results demonstrated that the current method is a powerful mathematical instrument for solving the class of problems under consideration.

Acknowledgements

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