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The Unified Format of Trapezoid and Parabola Quadrature Formula and Its Complex Formula

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Abstract: In numerical integration, classical trapezoidal formula and parabolic formula play an important role in the theory and application of numerical integration, but trapezoidal formula and parabolic formula are relatively independent quadrature formulas, and the reasoning of error formula requires that the integrand function be second-order differentiable and fourth-order differentiable respectively, these conditions limit the wide application of the formula. For this reason, recent relevant documents have studied the error estimation of trapezoidal formula and parabolic formula under the condition that the integrand has a continuous first derivative in the integral interval except for the most limited points, but sometimes the integral integrand of practical problems can be derived almost everywhere, and the breakpoints between its derivatives are countable. In this paper, the unified integral formula format and its complex quadrature formula of two classical quadrature formulas are constructed firstly, and then appropriately relaxed the limiting conditions of the integrand function, under the condition that the integral interval is almost everywhere differentiable and the non-differentiable points are the first kind of discontinuities. Finally, the error estimation of the quadrature formula is studied. The research results weaken the restrictions of the integrand, thus expand the conditions for the use of the complex trapezoidal quadrature formula and the complex parabolic quadrature formula, and modify and improve the existing literature results.

Keywords: Numerical Integration, Trapezoid Formula, Parabola Formula, Quadrature Formula, Error Formula

1. Introduction

The research on the approximate calculation of definite integral is of great significance. The classical quadrature formulas include trapezoidal quadrature formula and its complex quadrature formula, parabolic quadrature formula and its complex quadrature formula, Newton Cotes quadrature formula and Gauss quadrature formula [1, 2]. The error estimation of these quadrature formulas requires the higher-order derivative of the integrand function. The trapezoidal quadrature formula requires the integrand function to be the second-order derivative, the parabolic quadrature formula requires the integrand function to be the fourth derivative, which limits the application of the classical quadrature formula method to a great extent.

When the integrand function $f(x)$ is a first-order differentiable function, Dragomir [3], Pearce et al. [4] and Kirmaci [5] respectively studied the middle quadrature formula, complex quadrature formula and their error estimates, parabolic quadrature formula, complex quadrature formula and their error estimates of the first-order differentiable function; When the integrand function $f(x)$ is an integrable but non-differentiable function, Zheng Liu et al. [6] studied the trapezoidal quadrature formula under the condition that the integrand function is non differentiable in 2004. Zhizhu Lai et al. [7, 8] and Zheng Liu et al. [6] studied the quadrature formula corresponding to the classical quadrature formula under the condition that the integrand function has continuous derivatives everywhere outside at most finite points in the integral interval.

If the integrand function of the integral does not have good properties, the approximate calculation of the integral can only get an approximate value of the integral, but the truncation error of the approximate value may not be obtained, or there is no theoretical support, which brings inconvenience to the practical work.

According to paper [6, 9-16], the conditions of the quadrature formula are:

Condition I: Let the function $f:[a,b] \subseteq R \to R$ be a continuous function, and at most finite points $\{y_k\}_{k=1}^{k=n}$ in the open interval (a,b), there is a continuous derivative everywhere, and there is a positive number M such that $|f'(x)| \le M$[3-5].

In order to obtain the unified format of trapezoidal quadrature formula and parabolic quadrature formula, as well as their complex quadrature formula and corresponding error estimation problems, the general assumption is:

Condition II: Let the function $f:[a,b] \subseteq R \to R$ be differentiable in interval $[a,b]$, except at most multiple points $\{y_k\}_{k=1}^{k=\infty}$ that can be listed, and the points in the point set $\{y_k\}_{k=1}^{k=\infty}$ are the first kind of discontinuities of the derivative function $f'(x)$.

Obviously, the left and right derivations $f'_{\pm}(y_i)$ of all points in the set exist, so there is a positive number M, which makes $|f'_{\pm}(x)| \le M,\ x \in [a,b]$.

2. A Unified Scheme of Trapezoidal and Parabolic Quadrature Formulas and Its Error Estimation

2.1. Unified Format of Trapezoidal and Parabolic Quadrature Formulas

In order to establish the unified format of trapezoidal quadrature formula and parabolic quadrature formula, two lemmas are introduced:

In 1998, Dragomir established the following integral equation for differentiable functions in paper [3]:

Lemma 1 [3]: Let the function $f: I \subseteq R \to R$, and the function $f(x)$ be differentiable in the open interval I°, $a,b \in I,\ a < b$. If $f' \in L_1([a,b])$, then

$$\int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] = \frac{(b-a)^2}{2}\int_0^1 (1-2t)\,f'(ta+(1-t)b)dt, \tag{1}$$

where $L_p([a,b])$ represents the set of Lebesgue integrable functions $|f|^p$ on interval $[a,b]$.

In 2004, Kirmaci established the integral equation of differentiable functions:

Lemma 2 [5]: Let the function $f: I \subseteq R \to R$ be differentiable in the open interval I°, $a,b \in I,\ a < b$. If $f' \in L_1([a,b])$, then

$$(b-a)f\left(\frac{a+b}{2}\right) - \int_a^b f(x)dx = (b-a)^2\left(\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (1-t)f'(ta+(1-t)b)dt\right). \tag{2}$$

The following theorem generalizes the results of lemma 1 and lemma 2:

Theorem 1: Let the function $f:[a,b] \subseteq R \to R$ satisfy the condition II in the interval $[a,b]$, $\lambda \ge 0$, if $f' \in L_1([a,b])$, then

$$\frac{b-a}{\lambda+2}\left[f(a)+\lambda f\left(\frac{a+b}{2}\right)+f(b)\right] - \int_a^b f(x)dx$$
$$= \frac{(b-a)^2}{4} \times \left[\int_0^1 \left(t-\frac{2}{\lambda+2}\right) f'\left(\left(1-\frac{t}{2}\right)a+\frac{t}{2}b\right)dt + \int_0^1 \left(t-\frac{\lambda}{\lambda+2}\right) f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right]. \tag{3}$$

By using the results of theorem 1, the unified format of trapezoidal quadrature formula and parabolic quadrature formula can be obtained.

Theorem 2 (Unified format of trapezoidal and parabolic quadrature formulas): Let the function $f:[a,b] \subseteq R \to R$ satisfy the condition II in the interval $[a,b], \lambda \ge 0$, if $f' \in L_1([a,b])$, then there is quadrature formula:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{\lambda+2}\left[f(a)+\lambda f\left(\frac{a+b}{2}\right)+f(b)\right]. \tag{4}$$

According to the unified format of theorem 2, trapezoidal quadrature formula and parabolic quadrature formula can be obtained.

Corollary 1: Let the function $f:[a,b] \subseteq R \to R$ satisfy the condition II in the interval $[a,b], \lambda \ge 0$, if $f' \in L_1([a,b])$, then

(1) if $\lambda = 0$, then there is trapezoidal quadrature formula:

$$\int_a^b f(x)\,dx \approx T = \frac{b-a}{2}[f(a)+f(b)], \tag{5}$$

(2) if $\lambda = 1$, then there is quadrature formula:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3}\left[f(a)+f\left(\frac{a+b}{2}\right)+f(b)\right], \tag{6}$$

(3) if $\lambda = 4$, then there is trapezoidal quadrature formula:

$$\int_a^b f(x)\,dx \approx S = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]. \tag{7}$$

2.2. Unified Format Error Estimation

The error estimate of the unified format (4) is:

Theorem 3: Let the function $f: [a, b] \subseteq R \to R$ satisfy the condition II in the interval $[a, b]$, $\lambda \geq 0$, there is a positive number M, such that $\left|f'_{\pm}(x)\right| \leq M$, if $f' \in L_1([a, b])$, then

$$\left|\int_a^b f(x)dx - \frac{b-a}{\lambda+2}\left[f(a) + \lambda f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leq \frac{(\lambda^2+4)(b-a)^2 M}{4(\lambda+2)^2} \tag{8}$$

Prove According to theorem 1, there is

$$\left|\int_a^b f(x)dx - \frac{b-a}{\lambda+2}\left[f(a) + \lambda f\left(\frac{a+b}{2}\right) + f(b)\right]\right|$$

$$\leq \frac{(b-a)^2}{4}\left[\int_0^1\left|t - \frac{2}{\lambda+2}\right|\left|f'\left((1-t)a + t\frac{a+b}{2}\right)\right|dt + \int_0^1\left|t - \frac{\lambda}{\lambda+2}\right|\left|f'\left((1-t)\frac{a+b}{2} + tb\right)\right|dt\right]$$

$$\leq \frac{(b-a)^2 M}{4}\left[\int_0^1\left|t - \frac{2}{\lambda+2}\right|dt + \int_0^1\left|t - \frac{\lambda}{\lambda+2}\right|dt\right] \leq \frac{(b-a)^2(\lambda^2+4)M}{4(\lambda+2)^2}.$$

Certificate completion.

Corollary 2: Let the function $f: [a, b] \subseteq R \to R$ satisfy the condition II in the interval $[a, b]$, $\lambda \geq 0$, there is a positive number M, such that $\left|f'_{\pm}(x)\right| \leq M$, if $f' \in L_1([a, b])$, then

(1) if $\lambda = 0$, then the error estimate of trapezoidal quadrature formula is:

$$\left|\int_a^b f(x)dx - \frac{b-a}{2}[f(a) + f(b)]\right| \leq \frac{(b-a)^2 M}{4} \tag{9}$$

(2) if $\lambda = 1$, then the error estimate of quadrature formula (6) is:

$$\left|\int_a^b f(x)dx - \frac{b-a}{3}\left[f(a) + f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leq \frac{5(b-a)^2 M}{36} \tag{10}$$

(3) if $\lambda = 4$, then the error estimate of parabolic quadrature formula is:

$$\left|\int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leq \frac{5(b-a)^2 M}{36} \tag{11}$$

3. The Complex Formula of Unified Scheme and Its Error Estimation

3.1. The Unified Complex Quadrature Formula

Theorem 4: Let the function $f: [a, b] \subseteq R \to R$ be differentiable in the interval $[a, b]$, except at most multiple points $\{y_k\}_{k=1}^{k=\infty}$ that can be listed, and the function $f(x)$ satisfy condition II, $n \in N_+$, if $f' \in L_1([a, b])$, $\lambda \geq 0$, then complex quadrature formula

$$\int_a^b f(x)\ \mathrm{d}x \approx \frac{1}{\lambda+2}\left[\Delta x_1 f(a) + \sum_{k=1}^{n-1}(\Delta x_k + \Delta x_{k+1})f(x_k) + \lambda\sum_{k=1}^{n}\Delta x_k f\left(\frac{x_{k-1}+x_k}{2}\right) + \Delta x_n f(b)\right], \tag{12}$$

where $\Delta x_i = x_i - x_{i-1} (i = 1,2,\cdots,n)$ represents the length of the ith cell.

Prove Applying theorem 2 on the interval $[x_{k-1}, x_k](k = 1,2,\cdots,n)$, can be got

$$\int_{x_{k-1}}^{x_k} f(x)\ \mathrm{d}x \approx \frac{\Delta x_k}{\lambda+2}\left[f(x_{k-1}) + \lambda f\left(\frac{x_{k-1}+x_k}{2}\right) + f(x_k)\right],\ k = 1,2,\cdots,n \tag{13}$$

sum formula (13) with respect to $k = 1,2,\cdots,n$, get

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{1}{\lambda+2}\sum_{k=1}^{n}\Delta x_k\left[f(x_{k-1}) + \lambda f\left(\frac{x_{k-1}+x_k}{2}\right) + f(x_k)\right]$$

$$= \frac{1}{\lambda+2}\left[\Delta x_1 f(a) + \sum_{k=1}^{n-1}(\Delta x_k + \Delta x_{k+1})f(x_k) + \lambda\sum_{k=1}^{n}\Delta x_k f\left(\frac{x_{k-1}+x_k}{2}\right) + \Delta x_n f(b)\right],$$

Certificate completion.

3.2. Error Estimation of the Unified Complex Quadrature Formula

Theorem 5: Let the function $f: [a, b] \subseteq R \to R$ satisfy the condition II within $[a, b]$, $n \in N_+$, there is a positive number M, such that $\left|f'_{\pm}(x)\right| \leq M$, $x \in [a, b]$, $\lambda \geq 0$, if $f' \in L_1([a, b])$, then

$$\left|\int_a^b f(x)\ \mathrm{d}x - \frac{1}{\lambda+2}\left[\Delta x_1 f(a) + \sum_{k=1}^{n-1}(\Delta x_k + \Delta x_{k+1}) f(x_k) + \lambda \sum_{k=1}^{n} \Delta x_k f\left(\frac{x_{k-1}+x_k}{2}\right) + x_n f(b)\right]\right|$$

$$+4\sum_{k=1}^{n} \Delta x_k\, f\left(\frac{x_{k-1}+x_k}{2}\right) + \Delta x_n f(b)\Bigg]\Bigg| \le \frac{(\lambda^2+4)M}{4(\lambda+2)^2}\sum_{k=1}^{n}(\Delta x_k)^2, \tag{14}$$

where $\Delta x_i = x_i - x_{i-1} (i = 1,2,\cdots,n)$ represents the length of the ith cell.

Prove Applying theorem 3 and theorem 4 on the interval $[x_{k-1}, x_k](k = 1,2,\cdots,n)$, can be got

$$\left|\int_a^b f(x)\ \mathrm{d}x - \frac{1}{\lambda+2}\left[\Delta x_1 f(a) + \sum_{k=1}^{n-1}(\Delta x_k + \Delta x_{k+1}) f(x_k) + \lambda \sum_{k=1}^{n} \Delta x_k f\left(\frac{x_{k-1}+x_k}{2}\right) + x_n f(b)\right]\right|$$

$$\le \sum_{k=1}^{n}\left|\frac{\Delta x_k}{\lambda+2}\left[f(x_{k-1}) + \lambda f\left(\frac{x_{k-1}+x_k}{2}\right) + f(x_k)\right]\right| \le \frac{(\lambda^2+4)M}{4(\lambda+2)^2}\sum_{k=1}^{n}(\Delta x_k)^2.$$

Certificate completion.

4. The Complex Formula of Trapezoid and Parabola Quadrature Formula and Its Error Estimation

From above, the complex trapezoidal quadrature formula can be obtained.

4.1. Non-equidistant Complex Trapezoidal Quadrature Formula

Theorem 6: Let the function $f\colon [a,b] \subseteq R \to R$ satisfy the condition II within $[a,b]$, $n \in N_+$, if $f' \in L_1([a,b])$, then complex quadrature formula

$$\int_a^b f(x)dx \approx \frac{1}{2}\left[\Delta x_1 f(a) + \sum_{k=1}^{n-1}(\Delta x_k + \Delta x_{k+1}) f(x_k) + \Delta x_n f(b)\right] \tag{15}$$

where $\Delta x_i = x_i - x_{i-1} (i = 1,2,\cdots,n)$ represents the length of the ith cell.

Prove Take $\lambda = 0$ in theorem 4, can be proved.

Certificate completion.

The procedure and algorithm steps of quadrature formula (15) are as follows:

Step 1: input a, b, n;

Step 2: output parameter I;

Step 3: set $h(k) = x(k) - x(k-1)$; $s := 0$;

Step 4: loop $= 1{:}\,n-1$;

$$s \coloneqq s + \big(h(k) + h(k+1)\big) * f\big(x(k)\big);$$

Step 5: output $I = (h(1) * f(a) + s + h(n) * f(b))/2$.

The complex trapezoidal quadrature formula can be obtained.

4.2. Equidistant Complex Trapezoidal Quadrature Formula

Corollary 3: Let the function $f\colon [a,b] \subseteq R \to R$ satisfy the condition II within $[a,b]$, $n \in N_+$, there is a positive num M, such that $\left|f'_{\pm}(x)\right| \le M$, $x \in [a,b]$, if $f' \in L_1([a,b])$, then

$$\int_a^b f(x)\ \mathrm{d}x \approx \frac{b-a}{2n}\left[f(a) + 2\sum_{k=1}^{n-1} f(x_k) + f(b)\right], \tag{16}$$

where $x_k = a + k\frac{b-a}{n} (k = 0,1,2,\cdots,n)$.

The program algorithm steps of quadrature formula (16) are as follows:

Step 1: input a, b, n;

Step 2: output parameter I;

Step 3: set $h = (b-a)/n$;

Step 4: loop $k = 0,1,\cdots,n$;

$$x(k) = a + k * h;$$

Step 5: set $s = 0$;

Step 6: set $k = 1,2,\cdots,n-1$.

$$s = s + f\big(x(k)\big);$$

Step 7: output $I = (b-a) * \big(f(a) + 2 * s + f(b)\big) * (\lambda+2)$.
The complex parabola quadrature formula is derived as follows.

4.3. Quadrature Formula of Non-equidistant Complex Parabola

Theorem 7: Let the function $f: [a,b] \subseteq R \to R$ satisfy the condition II within $[a,b]$, $n \in N_+$, if $f' \in L_1([a,b])$, then complex parabola quadrature formula

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{1}{6}\Big[\Delta x_1 f(a) + \textstyle\sum_{k=1}^{n-1}(\Delta x_k + \Delta x_{k+1})\, f(x_k) + 4\sum_{k=1}^{n} \Delta x_k\, f\left(\frac{x_{k-1}+x_k}{2}\right) + \Delta x_n f(b)\Big], \tag{17}$$

where $\Delta x_i = x_i - x_{i-1} (i = 1,2,\cdots,n)$ represents the length of the *i*th cell.
Prove Take $\lambda = 0$ in theorem 4, can be proved.
Certificate completion.
The procedure and algorithm steps of quadrature formula (17) are as follows:
Step 1: input a, b, n;
Step 2: output parameter I;
Step 3: set $T = 0$;
Step 4: loop $k = 1,2,\cdots,n$;

$$h(k) = x(k) - x(k-1);$$

Step 5: set $s1 = 0, s2 = 0$;
Step 6: loop $k = 1,2,\cdots,n-1$.

$$s1 = s1 + \big(h(k) + h(k+1)\big) * f\big(x(k)\big);$$

Step 7: loop $k = 1,2,\cdots,n$;

$$s2 = s2 + h(k) * f((x(k-1) + x(k))/2);$$

Step 8: output $I = (h(1) * f(a) + s1 + 4 * s2 + h(n) * f(b))/6$.

4.4. Quadrature Formula of Equidistant Complex Parabola

Corollary 4: Let the function $f: [a,b] \subseteq R \to R$ satisfy the condition II within $[a,b]$, $n \in N_+$, there is a positive num M, such that $\left|f'_{\pm}(x)\right| \le M$, $x \in [a,b]$, if $f' \in L_1([a,b])$, then

$$\int_a^b f(x)\ \mathrm{d}x \approx \frac{b-a}{6n}\Big[f(a) + 2\textstyle\sum_{k=1}^{n-1} f(x_k) + 4\sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right) + f(b)\Big], \tag{18}$$

where $x_k = a + k\frac{b-a}{n} (k = 0,1,2,\cdots,n)$.
The procedure and algorithm steps of quadrature formula (18) are as follows:
Step 1: input a, b, n;
Step 2: output parameter T;
Step 3: set $h = (b-a)/n$;
Step 4: loop $k = 0,1,\cdots,n$;

$$x(k) = a + k * h;$$

Step 5: set $s1 = 0, s2 = 0$;
Step 6: loop $k = 1,2,\cdots,n-1$;

$$s1 = s1 + f\big(x(k)\big);$$

Step 7: loop $k = 1,2,\cdots,n$;

$$s2 = s2 + f((x(k-1) + x(k))/2);$$

Step 8: output $T = (b-1) * (f(a) + 2 * s1 + 4 * s2 + f(b))/(\lambda + 2)$.
Now the error estimate of the non-equidistant complex trapezoidal quadrature formula is given.

4.5. Error Estimation of Non-equidistant Complex Trapezoidal Quadrature Formula

Theorem 8: Let the function $f: [a,b] \subseteq R \to R$ satisfy the condition II within $[a,b]$, $n \in N_+$, there is a positive num M, such that $|f'_{\pm}(x)| \le M,\ x \in [a,b]$, if $f' \in L_1([a,b])$, then

$$\left|\int_a^b f(x)\ \mathrm{d}x - \frac{1}{2}[\Delta x_1 f(a) + \sum_{k=1}^{n-1}(\Delta x_k + \Delta x_{k+1}) f(x_k) + \Delta x_n f(b)]\right| \le \frac{M}{4}\sum_{k=1}^{n}(\Delta x_k)^2, \tag{19}$$

where $\Delta x_i = x_i - x_{i-1}(i = 1,2,\cdots,n)$ represents the length of the *i*th cell.
Prove Take $\lambda = 0$ in theorem 5, can be proved.
Certificate completion.

4.6. Error Estimation of Equidistant Complex Trapezoidal Quadrature Formula

Corollary 5: Let the function $f: [a,b] \subseteq R \to R$ satisfy the condition II within $[a,b]$, $n \in N_+$, there is a positive num M, such that $|f'_{\pm}(x)| \le M,\ x \in [a,b]$, if $f' \in L_1([a,b])$, then

$$\left|\int_a^b f(x)dx - \frac{b-a}{2n}[f(a) + 2\sum_{k=1}^{n-1} f(x_k) + f(b)]\right| \le \frac{(b-a)^2 M}{4n} \tag{20}$$

where $x_k = a + k\frac{b-a}{n}(k = 0,1,2,\cdots,n)$.
The error estimation of the quadrature formula of non-equidistant parabola is given below:

4.7. Error Estimation of Quadrature Formula of Non-equidistant Complex Parabola

Theorem 9: Let the function $f: [a,b] \subseteq R \to R$ satisfy the condition II within $[a,b]$, $n \in N_+$, there is a positive num M, such that $|f'_{\pm}(x)| \le M,\ x \in [a,b]$, if $f' \in L_1([a,b])$, then

$$\left|\int_a^b f(x)\ \mathrm{d}x - \frac{1}{6}[\Delta x_1 f(a) + \sum_{k=1}^{n-1}(\Delta x_k + \Delta x_{k+1}) f(x_k) + 4\sum_{k=1}^{n}\Delta x_k f\left(\frac{x_{k-1}+x_k}{2}\right) + \Delta x_n f(b)]\right| \le \frac{5M}{36}\sum_{k=1}^{n}(\Delta x_k)^2, \tag{21}$$

where $\Delta x_i = x_i - x_{i-1}(i = 1,2,\cdots,n)$ represents the length of the *i*th cell.
Prove Take $\lambda = 4$ in theorem 6, can be proved.
Certificate completion.

4.8. Error Estimation of Equidistant Complex Parabolic Quadrature Formula

Corollary 6: Let the function $f: [a,b] \subseteq R \to R$ satisfy the condition II within $[a,b]$, $n \in N_+$, there is a positive num M, such that $|f'_{\pm}(x)| \le M,\ x \in [a,b]$, if $f' \in L_1([a,b])$, then

$$\left|\int_a^b f(x)\ \mathrm{d}x - \frac{b-a}{6n}[f(a) + 2\sum_{k=1}^{n-1} f(x_k) + 4\sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right) + f(b)]\right| \le \frac{5(b-a)^2 M}{36n}, \tag{22}$$

where $x_k = a + k\frac{b-a}{n}(k = 0,1,2,\cdots,n)$.
Theorem 10: Let the function $f: [a,b] \subseteq R \to R$ satisfy the condition II within $[a,b]$, $n \in N_+$, there is a positive num M, such that $|f'_{\pm}(x)| \le M,\ x \in [a,b]$, if $f' \in L_1([a,b])$, then complex quadrature formula

$$\int_a^b f(x)\ \mathrm{d}x \approx \frac{1}{3}[\Delta x_1 f(a) + \sum_{k=1}^{n-1}(\Delta x_k + \Delta x_{k+1}) f(x_k) + \sum_{k=1}^{n}\Delta x_k f\left(\frac{x_{k-1}+x_k}{2}\right) + \Delta x_n f(b)], \tag{23}$$

and

$$\left|\int_a^b f(x)\,\mathrm{d}x - \frac{1}{3}\left[\Delta x_1 f(a) + \sum_{k=1}^{n-1}(\Delta x_k + \Delta x_{k+1}) f(x_k) + \sum_{k=1}^{n}\Delta x_k f\left(\frac{x_{k-1}+x_k}{2}\right) + \Delta x_n f(b)\right]\right| \le \frac{5M}{36}\sum_{k=1}^{n}(\Delta x_k)^2, \tag{24}$$

where $\Delta x_i = x_i - x_{i-1}(i = 1,2,\cdots,n)$ represents the length of the *i*th cell.
Prove Take $\lambda = 1$ in theorem 6, can be proved.
Certificate completion
Corollary 7: Under the condition of theorem 10, then

$$\int_a^b f(x)\ \mathrm{d}x \approx \frac{b-a}{3n}\left[f(a) + 2\sum_{k=1}^{n-1} f(x_k) + \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right) + f(b)\right], \tag{25}$$

and

$$\left|\int_a^b f(x)\ \mathrm{d}x - \frac{b-a}{3n}\left[f(a) + 2\sum_{k=1}^{n-1} f(x_k) + \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right) + f(b)\right]\right| \le \frac{5(b-a)^2 M}{36n}, \tag{26}$$

where $x_k = a + k\frac{b-a}{n}(k = 0,1,2,\cdots,n)$.
Under the condition of equidistant partition, it can be seen from corollary 6 and corollary 7 that the upper bounds of the

errors of the two quadrature formulas both are $\frac{5(b-a)^2 M}{36n}$, but the truncation errors of the complex parabola quadrature formula (25) and the equidistant complex parabola quadrature formula are not necessarily equal.

5. Conclusion

This paper first constructed the unified format of trapezoidal quadrature formula and parabolic quadrature formula and their complex quadrature formula. On this basis, the complex quadrature formula and parabolic quadrature formula are derived by unified method. When the integrand function meets condition (II), their error estimation problems are obtained, and the existing literature results are modified and improved. Thus, the application range of trapezoidal quadrature formula and parabolic quadrature formula is expanded, which lays a theoretical foundation for the construction of integral quadrature formula in the future research.

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