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A Class of Generalized Operator Quasi-Equilibrium Problems

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Abstract: In this work, introduce and study a generalized operator quasi-equilibrium problems (in short, OQEP) in the setting of topological vector spaces. We prove some new existence results for the solution of this problem by applying $C(f)$-quasiconvex, escaping sequence in Hausdörff topological vector spaces. The results of this paper can generalize and unify previously known corresponding results of this area.

Keywords: Operator Quasi-equilibrium Problem, $C(f)$-quasiconvex, Escaping Sequence

1. Introduction and Preliminaries

The study of equilibrium problems was introduced by Blum and Oettle [2] in 1994. In 2005, Kazmi and Raouf [12] have intensively studied a class of operator equilibrium problems and established some existence results for solutions of this problem. In this work we establish some existence theorems for solutions of a new class of generalized operator quasi-equilibrium problems (in short, OQEP). The results of this paper can be viewed as a generalization and improvement of many well-known results in the literature, see for example ([5, 11-14]).

Throughout the paper, unless otherwise specified, we use the following notations.

Let W and Z be Hausdörff topological vector spaces; $L(W,Z)$ be a space of all continuous linear operators from W to Z, endowed with the topology of point-wise convergence (w.r.t.p.c.) and let $B \subset L(W,Z)$ be a nonempty convex set. Let $C : B \to \Pi(Z)$ be a multi-valued map such that for each $f \in B$, $C(f)$ is a solid, convex and open cone and $0 \notin C(f)$ and let $C_0(f) = C(f) \cup \{0\}$, where $\Pi(Z)$ denotes the family of all subsets of Z.

Let Z be an ordered topological vector space with an ordering cone $C(f)$. Note that $C(f) \neq Z$. It is clear that the cone $C(f)$ for each $f \in B$ defines on Z a partial ordering $\leq_{C(f)}$ as follows; $g \leq_{C(f)} h$ if and only if $h - g \in C(f), \forall h, g \in C(f)$.

Let $S : B \to \Pi(B)$ be a multi-valued map with non-empty values. We denote by $\mathcal{F}(B)$ the family of multi-valued maps from $B \times B$ to $\Pi(Z)$. Let $F \in \mathcal{F}(B)$.

In this paper, we consider the following *generalized operator quasi-equilibrium problem* (OQEP). Find $f \in B$ such that

$$f \in cl_B S(f) \quad \text{and} \quad F(f,g) \nsubseteq -C(f), \quad \forall g \in S(f), \quad (1)$$

where $cl_B S(f)$ denotes the closure of $S(f)$ in B.

We remark that, for suitable choices of F, B, S, W, Z and C, OQEP (1) reduces to the problems presented in ([5, 12-14]) and the references therein. If $B \subset W$, then (1) reduces to *vector equilibrium problem* studied by Khaliq and Raouf [8], Khaliq [6], Khaliq and Krishan [7], Kazmi [10-11], Ansari and Yao [1] and the references therein. We omit the details.

We need the following definitions and results.

Definition 1.1. Let $P : D \subset L(W,Z) \to \Pi(Z)$ be a multi-valued mapping. Then

(i) The *graph* of a multi-valued map $P : D \subset L(W,Z) \to \Pi(Z)$ define as

$$G(f) = \{(f,z) \in D \times Z : f \in D, \ z \in P(f)\}.$$

(ii) Let P be a multi-valued map from range of P to D. The *inverse* P^{-1} of P defined by

$$f \in P^{-1}(z) \text{ if and only if } z \in P(f).$$

(iii) If, for each $f \in D$ and any open set V in Z containing $P(f)$,there exists an open neighborhood U of f in D such that $P(f) \subseteq V$ for all $f \in U$, then P is called *upper semicontinuous* on D.

Let $B = \bigcup_{n=1}^{\infty} B_n$, be a subset of $L(W,Z)$ where $\{B_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty and compact sets such that $B_n \subseteq B_{n+1}$ for all $n \in N$. A sequence $\{f_n\}_{n=1}^{\infty}$ in B is said to be escaping from B (relative to $\{B_n\}_{n=1}^{\infty}$) if for each n there is an $m \geq M, \ \ f_m \notin B_n$.

Let $S, T : B \to \Pi(L(W,Z))$ be multi-valued maps then the multi-valued maps $cl_B S, coS, S \bigcap T : B \to \Pi(L(W,Z))$ are defined as $(cl_B S)(f) = cl_B S(f)$, $(coS)(f) = coS(f)$ and $(S \bigcap T)(f) = S(f) \bigcap T(f)$, for all $f \in B$, where $coS(f)$ denotes the convex hull of $S(f)$

To prove the existence of solution of OQEP(1), we shall use the following theorems which are the special cases of Theorem 2 in Ding, Kim and Tan [3] and Theorem 2 in Ding, Kim and Tan [4].

Theorem 1.1. Let B be a non-empty, compact and convex subset of a Hausdörff topological vector space E. Suppose that $A,\ cl_E A,\ P : B \to \Pi(B)$ are multi-valued maps such that for each $x \in B,\ A(x)$ is non-empty and convex set, for each $y \in B,\ A^{-1}(y)$ is open set in B, $cl_E A$ is upper semicontinuous, for each $x \in B,\ x \notin coP(x)$ and for each $y \in B,\ P^{-1}(y)$ is open in B. Then there exists $x^* \in B$ such that $x^* \in cl_B A(x^*)$ and $A(x^*) \cap P(x^*) = \emptyset$.

Theorem 1.2. Let B be a non-empty and convex subset of a locally convex Hausdörff topological vector space E and D be a non-empty and compact subset of B. Suppose that $A,\ P : B \to \Pi(D)$ and $cl_E A : B \to \Pi(B)$ be multi-valued maps such that for each $x \in B,\ A(x)$ is non-empty and convex set, for each $y \in B,\ A^{-1}(y)$ is open set in B, $cl_E A$ is upper semicontinuous, for each $x \in B,\ x \notin coP(x)$ and for each $y \in D,\ P^{-1}(y)$ is open in B. Then there exists $x^* \in B$ such that $x^* \in cl_B A(x^*)$ and $A(x^*) \cap P(x^*) = \emptyset$.

2. Existence Results

Now we give existence results for OQEP(1).

Definition 2.1. Let $C : B \to \Pi(Z)$ be a multi-valued map such that for each $f \in B,\ C(f)$ is a multi-valued map with convex cone values in Z. A multi-valued bifunction $F \in \mathcal{F}(B)$, where $\mathcal{F}(B)$ the family of multi-valued maps from $B \times B$ to $\Pi(Z)$, is called $C(f)$-quasiconvex, if for all $f,\ g_1,\ g_2 \in K$ and $\lambda \in [0,1]$, $g_\lambda = \lambda g_1 + (1-\lambda) g_2$, we have

$$F(f, g_\lambda) \subseteq F(f, g_1) - C(f)$$

or

$$F(f, g_\lambda) \subseteq F(f, g_2) - C(f).$$

Theorem 2.1. Let $B \subset L(W,Z)$ be a non-empty, compact and convex set. Let $C : B \to \Pi(Z)$ be a multi-valued map such that for each $f \in B,\ C(f)$ is a solid, convex and open cone, $0 \notin C(f)$. Let $(Z, C(f))$ be an ordered topological vector space. Let $F \in \mathcal{F}(B)$ and $S,\ cl_B S : B \to \Pi(B)$ be multi-valued maps such that for each $f \in B,\ S(f)$ is non-empty convex, for each $g \in B,\ S^{-1}(g)$ is open in B and $cl_B S$ is upper semicontinuous. Assume that

(i) F is $C(f)$-quasiconvex;
(ii) graph of $G(f) = Z \backslash (-C(f))$ is closed for all $f \in B$;
(iii) for each $g \in B,\ F(\cdot, g)$ is upper semi-continuous with compact values on B;
(iv) $F(f,f) \subseteq C_0(f),\ \forall f \in B$.

Then there exists $f_0 \in B$ such that

$$f_0 \in cl_B S(f_0) \text{ and } F(f_0, g) \nsubseteq -C(f_0) \ \forall g \in S(f_0).$$

Proof: For each $f \in B$, we define a multi-valued map $P : B \to \Pi(B)$ by

$$P(f) = g \in B : F(f,g) \subseteq -C(f).$$

We show that $f \notin coP(f)$, for each $f \in B$. Suppose that $f \in coP(f)$, for some $f \in B$. Then there exists $f_0 \in B$ such that $f_0 \in coP(f_0)$. This implies that f_0 can be expressed as

$$f_0 = \Sigma_{j \in J} \alpha_j g_j \text{ with } \alpha_j \geq 0, \Sigma_{j \in J} \alpha_j = 1,\ j = 1,2,3,....,n,$$

where $\{g_j : j \in N\}$ be a finite subset of B, $J \subset N$ be an arbitrary non-empty subset where N denotes the set of natural numbers. This follows

$$F(f_0, g_j) \subseteq -C(f_0) \text{ for all } j = 1,2,3,...,n.$$

Since F is $C(f)$-quasiconvex, we have

$$\begin{aligned} F(f_0, g_j) &\subseteq F(f_0, f_0) + C(f_0) \\ &\subseteq C_0(f_0) + C(f_0) \\ &\subseteq C(f_0) \end{aligned}$$

for all $j = 1,2,...,n$.

$$F(f_0, g_j) \subseteq C(f_0)$$

which is a contradiction. Hence $f \notin coP(f)$ for each $f \in B$. Now we show that $P^{-1}(g)$ is open in B, which is equivalent to show that $[P^{-1}(g)]^c = B \backslash P^{-1}(g)$ is closed.

Indeed, we have

$$\begin{aligned} P^{-1}(g) &= \{f \in B : g \in P(f)\} \\ &= \{f \in B : F(f,g) \subseteq -C(f)\} \end{aligned}$$

$$[P^{-1}(g)]^c = \{f \in B : F(f,g) \nsubseteq -C(f)\}.$$

By assumptions (ii) and (iii), we claim that $[P^{-1}(g)]^c$ is closed in B, for all $g \in B$.

Indeed, let $\{f_\lambda\}_{\lambda \in \Gamma}$ be a net in $[P^{-1}(g)]^c$ such that $\{f_\lambda\}$ converges to f (w.r.t.p.c.). Then we have $F(f_\lambda, g) \nsubseteq -C(f_\lambda)$ for each $g \in B$, that is, there exists $h_\lambda \in F(f_\lambda, g)$ such that $h_\lambda \notin -C(f_\lambda)$ or $h_\lambda \in G(f_\lambda)$ for all $\lambda \in \Gamma$. Let $A = \{f_\lambda\} \cup \{f\}$. Then A is compact and $h_\lambda \in F(A, g)$ which is compact. Therefore h_λ converges to h (w.r.t.p.c.). Then, by upper semi continuity of $F(\cdot, g)$, we have $h \in F(f, g)$. Also since $G(\cdot)$ has a closed graph in $B \times Z$, we have $h \in G(f)$. Consequently, $h \in F(f, g)$ and $h \notin -C(f)$, i.e., $F(f, g) \nsubseteq -C(f)$. Hence $f \in [P^{-1}(g)]^c$ and so $[P^{-1}(g)]^c$ is closed in B for all $g \in B$.

Thus it follows that all the hypothesis of Theorem 1.1 are satisfied. Hence there exists $f_0 \in B$ such that

$$f_0 \in cl_B S(f_0) \text{ and } S(f_0) \cap P(f_0) = \emptyset$$

which implies that there exists $f_0 \in B$ such that

$$f_0 \in cl_B S(f_0) \text{ and } F(f_0, g) \nsubseteq -C(f_0), \forall g \in S(f_0). \quad \square$$

Theorem 2.2. Let $B \subset L(W, Z)$ be a non-empty subset such that $B = \bigcup_{n=1}^{\infty} B_n$ where $\{B_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty, compact and convex subsets of B. Let $C : B \to \Pi(Z)$ be a multi-valued map such that for each $f \in B$, $C(f)$ is a solid, convex and open cone, $0 \notin C(f)$. Let $(Z, C(f))$ be an ordered topological vector space. Let $F \in \mathcal{F}(B)$ and S, $cl_B S : B \to \Pi(B)$ be multi-valued maps such that for each $f \in B$, $S(f)$ is non-empty convex for each $g \in B$, $S^{-1}(g)$ is open in B and $cl_B S$ is upper semicontinuous. Assume that

(i) F is $C(f)$-quasiconvex;
(ii) graph of $G(f) := Z \backslash (-C(f))$ is closed for all $f \in B$;
(iii) for each $g \in B$, $F(., g)$ is upper semi-continuous with compact values on B;
(iv) $F(f, f) \subseteq C_0(f)$, $\forall f \in B$;
(v) for each sequence $\{f_n\}_{n=1}^{\infty}$ in B with $f_n \in B_n$, $n \in N$ which is escaping from B relative to $\{B_n\}_{n=1}^{\infty}$, there exists $m \in N$ and $g_m \in B_m \bigcap S(f_m)$ such that for each $f_m \in cl_B S(f_m)$

$$F(f_m, g_m) \subseteq -C(f_m).$$

Then there exists $f_0 \in B$ such that

$$f_0 \in cl_B S(f_0) \text{ and } F(f_0, g) \nsubseteq -C(f_0), \ \forall g \in S(f_0).$$

Proof: Since for each $n \in N$, B_n is compact and convex subset in $L(W, Z)$, applying Theorem 2.1, we have for all $n \in N$, there exists $f_n \in B_n$ such that

$$f_n \in cl_B S(f_n) \text{ and } F(f_n, h) \nsubseteq -C(f_n) \tag{2}$$

for all $h \in S(f_n)$

Suppose that the sequence $\{f_n\}_{n=1}^{\infty}$ in B be escaping from relative to $B = \bigcup_{n=1}^{\infty} B_n$. By assumption (iv) there exists $m \in N$ and $h_m \in B_m \bigcap S(f_m)$ such that for each $f_m \in cl_B S(f_m)$,

$$F(f_m, h_m) \subseteq -C(f_m),$$

which contradicts (2). Hence $\{f_n\}_{n=1}^{\infty}$ is not an escaping sequence from B relative to $\{B_n\}_{n=1}^{\infty}$. Thus using the similar arguments, which have been used by Qun [15] in proving Theorem 2, there exist $r \in N$ and $f_0 \in B_r$ such that $f_n \to f_0$ (w.r.t.p.c.) and $F(f_0, g) \subseteq G(f_0)$. Since $cl_B S : B \to \Pi(B)$ is upper semicontinuous with compact values, hence there exists $f_0 \in B$ such that

$$f_0 \in cl_B S(f_0) \text{ and } F(f_0, g) \nsubseteq -C(f_0), \forall g \in S(f_0). \quad \square$$

Theorem 2.3. Let B be a non-empty for each convex subset of a locally convex Hausdorff topological vector space $L(W, Z)$ and D be a nonempty compact subset of B. Let $C : B \to \Pi(Z)$ be a multi-valued map such that for each $f \in B$, $C(f)$ is a solid, convex and open cone, $0 \notin C(f)$. Let $(Z, C(f))$ be an ordered Hausdorff topological vector space. Let $F \in \mathcal{F}(B)$ and S, $cl_B S : B \to \Pi(B)$ be multi-valued maps such that for each $f \in B$, $S(f)$ is non-empty convex, for each $g \in B$, $S^{-1}(g)$ is open in B and $cl_B S$ is upper semicontinuous. Assume that

(i) F is $C(f)$-quasiconvex;
(ii) The graph of $G(f) := Z \backslash (-C(f))$ is closed for all $f \in B$;
(iii) for each $g \in B$, $F(\cdot, g)$ is upper semi-continuous with compact values on B;
(iv) $F(f, f) \subseteq C_0(f)$, $\forall f \in B$.

Then there exists $f_0 \in B$ such that

$$f_0 \in cl_B S(f_0) \text{ and } F(f_0, g) \nsubseteq -C(f_0), \ \forall g \in S(f_0).$$

Proof: Let $P : B \to \Pi(B)$ be a multi-valued map define by

$$P(f) := \{g \in D : F(f, g) \subseteq -C(f)\} \ \forall f \in B.$$

Then by using the same argument, which we have used in proving Theorem 2.1, we have $f \notin B$ for each $f \in B$ and $P^{-1}(g)$ is open for each $g \in D$. Thus all the conditions of Theorem 1.2 are satisfied. Hence there exists $f_0 \in B$ such that

$$f_0 \in cl_B S(f_0) \text{ and } S(f_0) \cap P(f_0) = \emptyset$$

which implies that there exists $f_0 \in B$ such that

$$f_0 \in cl_B S(f_0) \text{ and } F(f_0, g) \nsubseteq -C(f_0), \forall g \in S(f_0). \quad \square$$

3. Application

As an immediate application of theorems proved in Section 2, we obtain as special cases, the following existence results for the solutions of *generalised operator quasi-variational-like inequality problem* (in short, GOQVLIP) and *generalized operator variational-like inequality problem* (in short, GOVLIP).

Theorem 3.1. Let W,Y and Z be three Hausdörff topological vector spaces and B be a non-empty, compact and convex subset of $L(W, Z)$. Let $(Y, C(f))$ be an ordered Hausdörff topological vector space. Let $M : B \times Z \to \Pi(L(W, Y))$ be a multi-valued map; $\eta : B \times B \to L(W, Y)$ be bifunction; $T : B \to \Pi(Z)$ and $S : B \to \Pi(B)$ be multi-valued maps. Define

$$F(f, g) = \mathrm{Max}\langle M(f, \nu), \eta(g, f)\rangle$$

for all $f, g \in B$, where $\nu \in T(f)$ be assume that all the other conditions of Theorem 2.1 holds. Then the GOQVLIP of finding $f_0 \in B$ such that for each $g \in S(f_0)$ there exists $\nu_0 \in T(f_0)$ satisfying

$$f_0 \in cl_B S(f_0) \text{ and } \operatorname{Max} \langle M(f_0, \nu_0), \eta(g, f_0)\rangle \nsubseteq -C(f_0),$$

has a solution.

Corollary 3.1. If $S(f) = B$ for all $f \in B$, in Theorem 3.1. Then the GOVLIP of finding $f_0 \in B$ such that for each $g \in B$, there exists $\nu_0 \in T(f_0)$ satisfying

$$f_0 \in B \text{ and } \operatorname{Max} \langle M(f_0, \nu_0), \eta(g, f_0)\rangle \nsubseteq -C(f_0),$$

has a solution.

Remark 3.1. If $B \subset W$ in Corollary 3.1, then we obtain the existence results of compact and non-compact settings of [9].

4. Conclusion

In this work, we studied a new class which is known as a generalized operator quasi-equilibrium problem and establish existance results for using escaping sequence and $C(f)$-qusasiconvex, in the setting of topological vector spaces. The operator quasi-equilibrium is a generalization and improvement, include not only scalar and vector equilibrium and variational inequalities problems as special cases, but have sufficient evidence for their importance to study, see ([5, 12])

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