

The Cauchy Operator and the Homogeneous Hahn Polynomials

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To cite this article:

Qiuxia Hu, Xinhao Huang, Zhizheng Zhang. The Cauchy Operator and the Homogeneous Hahn Polynomials. *American Journal of Applied Mathematics*. Vol. 9, No. 3, 2021, pp. 64-69. doi: 10.11648/j.ajam.20210903.11

Received: March 31, 2021; **Accepted:** April 23, 2021; **Published:** May 27, 2021

Abstract: The Cauchy operator plays important roles in the theory of basic hypergeometric series. As some applications, our purpose is mainly to show new proofs of the Mehler's formula, the Rogers formula and the generating function for the homogeneous Hahn polynomials $\Phi_n^{(\alpha)}(x, y|q)$ by making use of the Cauchy operator and its properties. In addition, some interesting results are also obtained, which include a formal extension of the generating function for $\Phi_n^{(\alpha)}(x, y|q)$.

Keywords: The Cauchy Operator, The Hahn Polynomials, Mehler's Formula, Rogers Formula

1. Introduction

Following [9], the q-shifted factorial is defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k),$$

and $(a_1, a_2, \dots, a_m)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$, where m is a positive integer and n is nonnegative integer or ∞ .

The basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s \left[\begin{matrix} a_1, & a_2, & \dots, & a_r \\ b_1, & b_2, & \dots, & b_s \end{matrix} \middle| q; z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r)_n}{(q, b_1, b_2, \dots, b_s)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n.$$

The q-exponential function is given by

$$\sum_{n=0}^{\infty} \frac{z^n}{(q)_n} = \frac{1}{(z)_\infty}, \quad |z| < 1. \quad (1)$$

The q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, & 0 \leq k \leq n, \\ 0, & k \geq n. \end{cases}$$

The q-Hahn polynomials $\Phi_n^{(\alpha)}(x|q)$ were first studied by Hahn, and then by Al-Salam and Carlitz [1, 10, 11]. Now, we restate the definition of the q-Hahn polynomials as follows.

Definition 1.1. The q-Hahn polynomials are defined by

$$\Phi_n^{(\alpha)}(x|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (\alpha)_k x^k.$$

According to the q-Hahn polynomials, we can easily obtain the homogeneous Hahn polynomials $\Phi_n^{(\alpha)}(x, y|q)$. In this paper, we need to give the following definition of the homogeneous Hahn polynomials, again.

Definition 1.2. The homogeneous Hahn polynomials are defined by

$$\Phi_n^{(\alpha)}(x, y|q) = y^n \Phi_n^{(\alpha)}\left(\frac{x}{y} \middle| q\right) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (\alpha)_k x^k y^{n-k}.$$

Evidently, $\Phi_n^{(\alpha)}(x, 1|q) = \Phi_n^{(\alpha)}(x|q)$.

The usual q-differential operator is defined by

$$D_{q,x} \{f(x)\} = \frac{f(x) - f(xq)}{x},$$

and we further define $D_{q,x}^0 \{f(x)\} = f(x)$, and for $n \geq$

$1, D_{q,x}^n \{f(x)\} = D_{q,x} \{D_{q,x}^{n-1} \{f(x)\}\}$. By q -differential operator $D_{q,x}$, VY. B. Chen and NS. S. Gu gave the argumentation operator $T(bD_{q,x}) = \sum_{n=0}^{\infty} \frac{b^n D_{q,x}^n}{(q)_n}$ [4, 5].

Then, VY. B. Chen and NS. S. Gu defined the Cauchy operator [3],

$$T(a, b; D_{q,x}) = \sum_{n=0}^{\infty} \frac{(a)_n b^n D_{q,x}^n}{(q)_n}.$$

Lemma 1.1.

$$D_{q,x}^k \{x^n\} = \begin{cases} x^{n-k} (q^{n-k+1})_k, & 0 \leq k \leq n, \\ 0, & k > n. \end{cases} \quad (2)$$

$$D_{q,x}^k \left\{ \frac{1}{(xt)_{\infty}} \right\} = \frac{t^k}{(xt)_{\infty}}, \quad (3)$$

$$D_{q,x}^k \left\{ \frac{x^n}{(xt)_{\infty}} \right\} = \frac{x^n}{(xt)_{\infty}} \sum_{m=0}^{\min\{k,m\}} \begin{bmatrix} k \\ m \end{bmatrix} (q^{-n})_m (xt)_m (-1)^m t^{k-m} q^{nm-m^2+\binom{m}{2}} \quad (4)$$

$$T(\alpha, x; D_{q,y}) \{y^n\} = \Phi_n^{(\alpha)}(x, y|q) \quad (5)$$

$$T(\alpha, x; D_{q,y}) \left\{ \frac{y^n}{(yt)_{\infty}} \right\} = \frac{y^n (\alpha xt)_{\infty}}{(yt, xt)_{\infty}} \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{(\alpha, yt)_m}{(\alpha xt)_m} \left(\frac{x}{q} \right)^m. \quad (6)$$

Three Cauchy operator identities established by VY. B. Chen and NS. S. Gu are restated as the following lemmas [3].

Lemma 1.2. We have

$$T(a, b; D_{q,x}) \left\{ \frac{1}{(xt)_{\infty}} \right\} = \frac{(abt)_{\infty}}{(bt, xt)_{\infty}}, \quad (7)$$

provided $|bt| < 1$.

Lemma 1.3. We have

$$T(a, b; D_{q,x}) \left\{ \frac{1}{(xs, xt)_{\infty}} \right\} = \frac{(abt)_{\infty}}{(bt, xs, xt)_{\infty}} {}_2\phi_1 \left[\begin{matrix} a, & xt \\ & abt \end{matrix} \middle| q, bs \right], \quad (8)$$

provided $\max\{|bs|, |bt|\} < 1$.

Lemma 1.4. We have

$$T(a, b; D_{q,x}) \left\{ \frac{(xv)_{\infty}}{(xs, xt)_{\infty}} \right\} = \frac{(abs, xv)_{\infty}}{(bs, xs, xt)_{\infty}} {}_3\phi_2 \left[\begin{matrix} a, & xs, & v/t \\ & abs, & xv \end{matrix} \middle| q, bt \right], \quad (9)$$

provided $\max\{|bs|, |bt|\} < 1$.

In [3], the authors derived Heine's ${}_2\phi_1$ transformation formula and Sears' ${}_3\phi_2$ transformation formula by the symmetric property of some parameters in the above operator identities (8)(9). And further they obtain extensions of the Askey-Wilson integral, the Askey-Roy integral, Sears' two-term summation formula.

In this paper, our main purpose is to make use of the above Cauchy operator identities (4)-(9) to give new proofs of the Mehler's formula, the Rogers formula and the generating function for the homogeneous Hahn polynomials. In addition,

Later, many researchers further study the Cauchy operator. For details of q -differential operator, the readers refer to [6, 7, 8, 14, 15, 16].

Based on the definitions for $D_{q,x}$ and $T(a, b; D_{q,x})$ and the Leibniz rule for $D_{q,x}$, we can easily obtain the following formulas.

some interesting results are also derived, which include a formal extension of the generating function for $\Phi_n^{(\alpha)}(x, y|q)$.

2. Some Applications

In Liu's recent paper [13], the following first two known results were proved by using the theory of analytic functions of several complex variables. In this section, we first make use of the Cauchy operator identities (4) and (9) to prove them.

Theorem 2.1. (The generating function for $\Phi_n^{(\alpha)}(x, y|q)$). If $\max\{|xt|, |yt|\} < 1$, then, we have that

$$\sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x, y|q) \frac{t^n}{(q)_n} = \frac{(\alpha xt)_{\infty}}{(xt, yt)_{\infty}}. \quad (10)$$

Proof. The identity (1) can be rewritten as follows:

$$\sum_{n=0}^{\infty} \frac{(ty)^n}{(q)_n} = \frac{1}{(ty)_{\infty}}.$$

Applying $T(\alpha, x; D_{q,y})$ to both sides of the above identity with respect to y , we get

$$\sum_{n=0}^{\infty} T(\alpha, x; D_{q,y}) \{y^n\} \frac{t^n}{(q)_n} = T(\alpha, x; D_{q,y}) \left\{ \frac{1}{(ty)_{\infty}} \right\}.$$

By means of (4) and (7), we obtain our desired result (10). This proof is completed.

We recall the q -Mehler formula for the homogeneous Hahn polynomials, which was found by Al-Salam and Carlitz [1] (see also [2, 12]).

Theorem 2.2. (The Mehler formula for $\Phi_n^{(\alpha)}(x, y|q)$). For $\max\{|xtu|, |xtv|, |yту|, |ytv|\} < 1$, then, we have that

$$\sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x, y|q) \Phi_n^{(\beta)}(u, v|q) \frac{t^n}{(q)_n} = \frac{(\alpha xtv, \beta ytu)_{\infty}}{(xtv, ytv, ytu)_{\infty}} {}_3\phi_2 \left[\begin{matrix} \alpha, & \beta, & ytv \\ \alpha xtv, & \beta ytu \end{matrix} \middle| q, xtu \right]. \quad (11)$$

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x, y|q) \Phi_n^{(\beta)}(u, v|q) \frac{t^n}{(q)_n} &= \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x, y|q) T(\beta, u; D_{q,v}) \{v^n\} \frac{t^n}{(q)_n} \\ &= T(\beta, u; D_{q,v}) \left\{ \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x, y|q) \frac{(tv)^n}{(q)_n} \right\}. \end{aligned}$$

By the identity (9) and (10), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x, y|q) \Phi_n^{(\beta)}(u, v|q) \frac{t^n}{(q)_n} &= T(\beta, u; D_{q,v}) \left\{ \frac{(\alpha xtv)_{\infty}}{(xtv, ytv)_{\infty}} \right\} \\ &= \frac{(\alpha xtv, \beta ytu)_{\infty}}{(xtv, ytv, ytu)_{\infty}} {}_3\phi_2 \left[\begin{matrix} \alpha, & \beta, & ytv \\ \alpha xtv, & \beta ytu \end{matrix} \middle| q, xtu \right]. \end{aligned}$$

This completes the proof.

Let $y = 1, v = 1$ in (11). Thus, we get the following result [12].

Corollary 2.1. For $\max\{|xtu|, |xt|, |yту|, |yt|\} < 1$, then, we have that

$$\sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x, y|q) \Phi_n^{(\beta)}(u|q) \frac{t^n}{(q)_n} = \frac{(\alpha xt, \beta ytu)_{\infty}}{(xt, yt, ytu)_{\infty}} {}_3\phi_2 \left[\begin{matrix} \alpha, & \beta, & yt \\ \alpha xt, & \beta ytu \end{matrix} \middle| q, xtu \right]. \quad (12)$$

Proof. Here, the Cauchy operator is used to prove the above identity. First, letting $v = 1$ in (11), we yield that

$$\begin{aligned} \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x, y|q) \Phi_n^{(\beta)}(u|q) \frac{t^n}{(q)_n} &= \sum_{n=0}^{\infty} \Phi_n^{(\beta)}(u|q) T(\alpha, x; D_{q,y}) \{y^n\} \frac{t^n}{(q)_n} \\ &= T(\alpha, x; D_{q,y}) \left\{ \sum_{n=0}^{\infty} \Phi_n^{(\beta)}(u|q) \frac{(ty)^n}{(q)_n} \right\} \\ &\stackrel{by(10)}{=} T(\alpha, x; D_{q,y}) \left\{ \frac{(\beta uty)_{\infty}}{(uty, ty)_{\infty}} \right\} \\ &\stackrel{by(9)}{=} \frac{(\alpha xt, \beta ytu)_{\infty}}{(xt, yt, ytu)_{\infty}} {}_3\phi_2 \left[\begin{matrix} \alpha, & \beta, & yt \\ \alpha xt, & \beta ytu \end{matrix} \middle| q, xtu \right]. \end{aligned}$$

Then, taking $y = 1$ in the above identity, we derive our desired result. This completes the proof.

Theorem 2.3. (The Rogers formula for $\Phi_n^{(\alpha)}(x, y|q)$). For $\max\{|xs|, |xt|\} < 1$, then, we have that

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Phi_{n+m}^{(\alpha)}(x, y|q) \frac{t^n}{(q)_n} \frac{s^m}{(q)_m} = \frac{(\alpha xt)_{\infty}}{(xt, ys, yt)_{\infty}} {}_2\phi_1 \left[\begin{matrix} \alpha, & yt \\ & \alpha xt \end{matrix} \middle| q, xs \right]. \quad (13)$$

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Phi_{n+m}^{(\alpha)}(x, y|q) \frac{t^n}{(q)_n} \frac{s^m}{(q)_m} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} T(\alpha, x; D_{q,y}) \{y^{n+m}\} \frac{t^n}{(q)_n} \frac{s^m}{(q)_m} \\ &= T(\alpha, x; D_{q,y}) \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(ty)^n}{(q)_n} \frac{(sy)^m}{(q)_m} \right\} \stackrel{by(1)}{=} T(\alpha, x; D_{q,y}) \left\{ \frac{1}{(yt, ys)_{\infty}} \right\} \\ &\stackrel{by(8)}{=} \frac{(\alpha xt)_{\infty}}{(xt, ys, yt)_{\infty}} {}_2\phi_1 \left[\begin{matrix} \alpha, & yt \\ & \alpha xt \end{matrix} \middle| q, xs \right]. \end{aligned}$$

This completes the proof.

Next, we make use of Cauchy operator properties to derive some interesting results relevant to $\Phi_n^{(\alpha)}(x, y|q)$.

Theorem 2.4. (A formal extension of the identity (10)). For $\max\{|ty|, |tx|\} < 1$, then, we have that

$$\sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x, y|q) \frac{(a)_n t^n}{(q)_n} = \frac{(aty)_{\infty}}{(ty)_{\infty}} {}_2\phi_1 \left[\begin{matrix} \alpha, & a \\ & aty \end{matrix} \middle| q, xt \right]. \quad (14)$$

Proof. We begin with the q-binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a)_n (ty)^n}{(q)_n} = \frac{(aty)_{\infty}}{(ty)_{\infty}}.$$

Applying $T(\alpha, x; D_{q,y})$ to both sides of the above identity with respect to the variable y , we get

$$\sum_{n=0}^{\infty} \frac{(a)_n t^n}{(q)_n} T(\alpha, x; D_{q,y}) \{y^n\} = T(\alpha, x; D_{q,y}) \left\{ \frac{(aty)_{\infty}}{(ty)_{\infty}} \right\}.$$

By means of (4) and (9), we obtain our desired result. This completes the proof.

Theorem 2.5. For $\max\{|xt|, |yt|\} < 1$, then, we have that

$$\sum_{k=0}^{\infty} \Phi_{n+k}^{(\alpha)}(x, y|q) \frac{t^k}{(q)_k} = \frac{y^n (\alpha xt)_{\infty}}{(yt, xt)_{\infty}} \sum_{m=0}^n \left[\begin{matrix} n \\ m \end{matrix} \right] \frac{(\alpha, yt)_m}{(\alpha xt)_m} \left(\frac{x}{q} \right)^m. \quad (15)$$

Proof.

$$\begin{aligned} \sum_{k=0}^{\infty} \Phi_{n+k}^{(\alpha)}(x, y|q) \frac{t^k}{(q)_k} &= \sum_{k=0}^{\infty} T(\alpha, x; D_{q,y}) \{y^{n+k}\} \frac{t^k}{(q)_k} \\ &= T(\alpha, x; D_{q,y}) \left\{ y^n \sum_{k=0}^{\infty} \frac{(yt)^k}{(q)_k} \right\} \stackrel{by(1)}{=} T(\alpha, x; D_{q,y}) \left\{ \frac{y^n}{(yt)_{\infty}} \right\} \\ &\stackrel{by(3)}{=} \frac{y^n}{(yt)_{\infty}} \sum_{k=0}^{\infty} \sum_{m=0}^k \frac{(\alpha)_k (q^{-n})_m (ty)_m}{(q)_m (q)_{k-m}} (-1)^m x^k t^{k-m} q^{nm-m^2+\binom{m}{2}} \\ &= \frac{y^n}{(yt)_{\infty}} \sum_{m=0}^n \left[\begin{matrix} n \\ m \end{matrix} \right] (\alpha, yt)_m (x/q)^m \sum_{k=0}^{\infty} \frac{(\alpha q^m)_k}{(q)_k} (xt)^k \\ &\stackrel{by(1)}{=} \frac{y^n}{(yt)_{\infty}} \sum_{m=0}^n \left[\begin{matrix} n \\ m \end{matrix} \right] (\alpha, yt)_m (x/q)^m \frac{(\alpha xt q^m)_{\infty}}{(xt)_{\infty}} = \frac{y^n (\alpha xt)_{\infty}}{(yt, xt)_{\infty}} \sum_{m=0}^n \left[\begin{matrix} n \\ m \end{matrix} \right] \frac{(\alpha, yt)_m}{(\alpha xt)_m} \left(\frac{x}{q} \right)^m. \end{aligned}$$

This completes the proof.

Theorem 2.6. For $\max\{|uxt|, |vyt|, |vxt|\} < 1$, then, we have that

$$\begin{aligned} & \sum_{n=0}^{\infty} \Phi_{n+k}^{(\alpha)}(x, y|q) \Phi_n^{(\beta)}(x, y|q) \frac{t^n}{(q)_n} \\ &= \frac{y^k (\beta uyt, \alpha xtv)_{\infty}}{(uyt, vyt, vxt)_{\infty}} \sum_{m=0}^k \sum_{n=0}^{\infty} \begin{bmatrix} k \\ m \end{bmatrix} \frac{(uyt)_m (\beta)_n (\alpha, vyt)_{m+n}}{(q)_n (\alpha xtv, \beta uyt)_{m+n}} (x/q)^m (uxt)^n. \end{aligned} \quad (16)$$

Proof.

$$\begin{aligned} & \sum_{n=0}^{\infty} \Phi_{n+k}^{(\alpha)}(x, y|q) \Phi_n^{(\beta)}(x, y|q) \frac{t^n}{(q)_n} = \sum_{n=0}^{\infty} \Phi_{n+k}^{(\alpha)}(x, y|q) T(\beta, u; D_{q,v}) \{v^n\} \frac{t^n}{(q)_n} \\ &= T(\beta, u; D_{q,v}) \left\{ \sum_{n=0}^{\infty} \Phi_{n+k}^{(\alpha)}(x, y|q) \frac{(vt)^n}{(q)_n} \right\} \\ &\stackrel{by(15)}{=} T(\beta, u; D_{q,v}) \left\{ \frac{y^k (\alpha xtv)_{\infty}}{(y tv, xtv)_{\infty}} \sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix} \frac{(\alpha, ytv)_m}{(\alpha xtv)_m} (x/q)^m \right\} \\ &= y^k \sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix} (\alpha)_m (x/q)^m T(\beta, u; D_{q,v}) \left\{ \frac{(\alpha xtv^m)_\infty}{(x tv, y tq^m v)_\infty} \right\} \\ &\stackrel{by(9)}{=} y^k \sum_{m=0}^k \begin{bmatrix} k \\ m \end{bmatrix} (\alpha)_m (x/q)^m \frac{(\beta uyt q^m, \alpha xtv^m)_\infty}{(uyt q^m, y tq^m v, xtv)_\infty} \sum_{n=0}^{\infty} \frac{(\beta, y tq^m v, \alpha q^m)_n}{(q, \beta uyt q^m, \alpha xtv^m v)_n} (uxt)^n \\ &= \frac{y^k (\beta uyt, \alpha xtv)_{\infty}}{(uyt, vyt, vxt)_{\infty}} \sum_{m=0}^k \sum_{n=0}^{\infty} \begin{bmatrix} k \\ m \end{bmatrix} \frac{(uyt)_m (\beta)_n (\alpha, vyt)_{m+n}}{(q)_n (\alpha xtv, \beta uyt)_{m+n}} (x/q)^m (uxt)^n. \end{aligned}$$

This completes the proof.

3. Conclusions

The Cauchy operator plays very important roles in the theory of basic hypergeometric series and q-orthogonal polynomials. In terms of the Cauchy operator, some known identities can be proved, again. Meanwhile, some new results can be obtained too. These results may be helpful in the theory of basic hypergeometric series and q-orthogonal polynomials.

Acknowledgements

The authors are grateful to the editor and the referee for some valuable suggestions to improve the quality of this paper. This work is supported by Innovation and Entrepreneurship Training Program for College Students in Henan Province (No. SA2100002211) and the Natural Science Foundation of Henan Province (No. 212300410211).

This work is supported by Research project of 2020 teacher education curriculum reform of Henan Provincial Department of Education 2020-JSJYYB-053£©.

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