A Maximization Problem Involving a Fractional Laplace Type Operator

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Abstract: Fractional Laplacian is an important nonlocal operator which has many applications in different kinds of differential equations. Recently, optimization problems involving the fractional Laplacian have been studied a lot by many authors. However, most of these papers are focusing on the optimization problems related to the first eigenvalue of the equation. Optimization problems related to the energy functional of the equation have not been investigated well enough. In this paper, we are going to study a maximization problem related to the energy functional of an equation involving a fractional Laplace type operator. Firstly, by using suitable variational framework in a fractional Sobolev space, we can show that a fractional equation has a solution which is in fact the global minimum of the corresponding energy functional. Moreover, by using reduction to absurdity we can obtain the uniqueness of the solution of the fractional equation. Then, we focus on a maximization problem related to the equation which takes the energy functional as the objective functional. Finally, by carefully analysing the properties of an arbitrarily choosen minimizing sequence and the tools of the rearrangement theory, we can prove that the maximization problem is solvable.

Keywords: Maximization, Fractional Laplace, Rearrangement

1. Introduction

Since Burton's theory of rearrangement was established in the papers [1-2], there have been many applications in optimization problems related to different kinds of equations. Cuccu et al studied the optimization problems involving the p-Laplacian in [3-5]. Emamizadeh and Zivari-Rezapour investigated an optimization problem related to a linear elliptic boundary problem in [6]. Emamizadeh and Prajapat obtained the symmetry of the solution of an optimization problem in [7]. Marras used a different approach to prove the solvability of an optimization involving the p-Laplacian in [8]. Qiu et al introduced a new truncation trick to solve two optimization problems involving the p-Laplacian in [9]. Recent years, many authors were interested in the optimization problems involving the nonlocal operator, see for example in [10-16] and the references therein. Qiu et al considered an optimization problem related to the following fractional equation in [10]:

$$\begin{cases} -L_\theta^s u + h(x,u) = f(x), x \in \Omega, \\ u = 0, x \in R^N - \Omega \end{cases}$$

where $-L_\theta^s u$ is a fractional Laplace type operator defined as follows:

$$-L_\theta^s u(x) = \int_{R^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} \theta(y) dy$$

with $0 < s < 1$. For $h(x,u)$, it is assumed that $h(x,\cdot)$ is non-decreasing for almost all $x \in \Omega$. This assumption is essential in the arguments of proving the uniqueness of the fractional equation and the solvability of the corresponding optimization problems. In this paper, we are going to study a maximization problem related to the following fractional equation $(P_{\lambda,f})$:

$$\begin{cases} -L_\theta^s u - \lambda V(x)u = f(x), x \in \Omega, \\ u = 0, x \in R^N - \Omega \end{cases}$$

where $V(x) \in L^\infty(\Omega)$, $V(x) > 0$ and λ is a positive parameter. Apparently, here the term $-\lambda V(x)u$ has different monotonicity. Therefore, we need overcome more difficulties. Under suitable assumptions, we can show that the equation $(P_{\lambda,f})$ has a solution u_f which is in fact the global minimizer of the corresponding energy functional. Moreover, we can obtain the uniqueness of the solution by using reduction to absurdity. Then, we consider the following maximization problem:

$$(Max) \qquad \inf_{g \in R(f)} \Phi(g)$$

where $\Phi(g) = \int_\Omega g u_g dx$ and $R(f)$ is the set of all the rearrangements of f which is defined in details in section 2.

We organize the present paper as follows. In section 1, we give the brief introduction. In section 2, some preliminaries are prepared before the main results. Section 3 is devoted to show the solvability of the equation $(P_{\lambda,f})$ and the uniqueness of solution of the equation $(P_{\lambda,f})$. At last, in section 4, we show that the problem (Max) has a solution.

2. Preliminaries

2.1. Some Definitions

We give some definitions here. In the paper, we denote by

C a positive constant. Suppose $f : \Omega \mapsto R$ is a measurable function defined in a smooth bounded domain in R^N with $N \geq 3$. We say $g : \Omega \mapsto R$ is a rearrangement of $f : \Omega \mapsto R$ if and only if for any $t \in R$ it holds that:

$$meas\{x \in \Omega : f(x) \geq t\} = meas\{x \in \Omega : g(x) \geq t\}$$

where meas($\cdot$) denotes the Lebesgue measure of the set. The set of all the rearrangements of $f : \Omega \mapsto R$ is denoted by $R(f)$. $H^s(\Omega)(0 < s < 1)$ is the fractional Sobolev space defined by:

$$H^s(\Omega) = \{u \in L^2(R^N) : u = 0 \ in \ \Omega^c, \|u\| < \infty\}$$

where

$$\|u\| = \left(\int_{R^N} \int_{R^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}$$

is the norm of $H^s(\Omega)$. $H^s(\Omega)$ is Hilbert space with the inner product

$$\langle u, v \rangle = \int_{R^N} \int_{R^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} dxdy$$

For $1 \leq p < \infty$, we denote by $\|u\|_p$ the usual norm in $L^p(\Omega)$ and $\|u\|_\infty$ the norm in $L^\infty(\Omega)$. The energy functional $I_f : H^s(\Omega) \mapsto R$ corresponding to the equation $(P_{\lambda,f})$ is

$$I_f(u) = \frac{1}{2} \int_{R^{2N}} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \theta(x-y) dxdy - \lambda \int_\Omega Vu^2 dx - \int_\Omega fu dx$$

We say $u \in H^s(\Omega)$ is a solution of the equation $(P_{\lambda,f})$ if for any $v \in H^s(\Omega)$ it holds

$$\int_{R^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \theta(x-y) dxdy - \lambda \int_\Omega Vuvdx - \int_\Omega fvdx = 0$$

2.2. Some Lemmas

Lemma 1 ([17] Theorem 6.5) The embedding of $H^s(\Omega)$ into $L^r(R^N)$ is continuous for $1 \leq r \leq 2N/(N-2s)$ and compact for $1 \leq r < 2N/(N-2s)$.

Lemma 2 ([2] Lemma 2.1) For any $g \in R(f)$ where $f \in L^p(\Omega)$ for some $1 \leq p \leq \infty$ then $g \in L^p(\Omega)$ and $\|g\|_p = \|f\|_p$.

Lemma 3 ([7] Lemma 2.3) Let $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ where $1 \leq p \leq \infty$, $p' = p/(p-1)$, then there

exists $\hat{f} \in R(f)$ which maximizes the linear functional $\int_\Omega hgdx$ relative to $h \in \overline{R(f)}$ where $\overline{R(f)}$ is the weak closure of $R(f)$ in $L^p(\Omega)$.

Lemma 4 Suppose that $V(x) > 0$ and $V(x) \in L^\infty(\Omega)$ $\theta : R^N \to (0, \infty)$, $\theta(x) = \theta(-x), \forall x \in R^N \setminus \{0\}$; $\theta \in L^\infty(R^N)$ and there exists $\theta_0 \in R_+$ such that $\theta(x) \geq \theta_0, \forall x \in R^N$.

Define

$$\lambda_0 = \inf_{u \in H_V} \left\{ \int_{R^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \theta(x-y) dxdy \right\}$$

where $H_V = \{u \in H^s(\Omega) : \int_\Omega V \mid u \mid^2 dx = 1\}$, then

$$\lambda_0 > 0 .$$

Proof. Obviously, $0 \le \lambda_0 < \infty$. Choose $\{u_n\}$ as a minimize sequence, i.e.,

$$\int_\Omega V \mid u_n \mid^2 dx = 1, \int_{R^{2N}} \frac{\mid u_n(x) - u_n(y) \mid^2}{\mid x-y \mid^{N+2s}} \theta(x-y) dx dy \to \lambda_0 ,$$

Then there exists a subsequence (still denoted by $\{u_n\}$), such that u_n weakly converges to $\bar{u} \in H^s(\Omega)$. Since $H^s(\Omega)$ is compactly embedded into $L^2(\Omega)$, u_n converges strongly to $\bar{u} \in L^2(\Omega)$. Therefore,

$$\int_\Omega V \mid (\mid u_n \mid^2 - \mid \bar{u} \mid^2) \mid dx \le \parallel V \parallel_{L^\infty} \int_\Omega \mid (\mid u_n \mid^2 - \mid \bar{u} \mid^2) \mid dx \to 0$$

So that,

$$\int_\Omega V \mid u_n \mid^2 dx \to \int_\Omega V \mid \bar{u} \mid^2 dx = 1 .$$

Since $\theta(x) = \theta(-x), \forall x \in R^N \setminus \{0\}$; $\theta \in L^\infty(R^N)$ and there exists $\theta_0 \in R_+$ such that $\theta(x) \ge \theta_0, \forall x \in R^N$,

$$\left(\int_{R^{2N}} \frac{(u(x) - u(y))^2}{\mid x-y \mid^{N+2s}} \theta(x-y) dx dy \right)^{1/2}$$

is a norm in $H^s(\Omega)$ which is equivalent to $\parallel u \parallel$.

Combining with the weakly lower semi-continuity of the norm, we have

$$\lambda_0 \le \int_{R^{2N}} \frac{(\bar{u}(x) - \bar{u}(y))^2}{\mid x-y \mid^{N+2s}} \theta(x-y) dx dy \le \liminf_{n \to \infty} \int_{R^{2N}} \frac{(\bar{u}_n(x) - \bar{u}_n(y))^2}{\mid x-y \mid^{N+2s}} \theta(x-y) dx dy = \lambda_0$$

which implies that

$$\lambda_0 = \int_{R^{2N}} \frac{(\bar{u}(x) - \bar{u}(y))^2}{\mid x-y \mid^{N+2s}} \theta(x-y) dx dy > 0$$

3. Main Results

3.1. The Existence and Uniqueness of the Equation $(P_{\lambda,f})$

We use the following hypotheses on the function $\theta : R^N \to (0, \infty)$:

(H1) $\theta(x) = \theta(-x), \forall x \in R^N \setminus \{0\}$;

(H2) $\theta \in L^\infty(R^N)$ and there exists $\theta_0 \in R_+$ such that $\theta(x) \ge \theta_0, \forall x \in R^N$.

Proposition 1 Assume that (H1) and (H2) hold,

$$0 < V(x) \in L^\infty(\Omega) , \quad f \in L^q(\Omega) , \quad q > \frac{2N}{N+2s} , \quad 0 < \lambda < \lambda_0 ,$$

where λ_0 is defined in Lemma 4, then the equation $(P_{\lambda,f})$ has a unique solution $u_f \in H^s(\Omega)$ and

$$I_f(u_f) = \inf_{v \in H^s(\Omega)} I_f(v) .$$

Proof. Choose arbitrarily $u \in H^s(\Omega)$, by the Holder inequality and Lemma 1 we have

$$\left| \int_\Omega fu dx \right| \le \parallel f \parallel_{L^q} \parallel u \parallel_{L_{q'}} \le C \parallel u \parallel$$

where $1 < q' := q/(q-1) < 2N/(N-2s)$. From (H1) and (H2) we have

$$\int_{R^{2N}} \frac{(u(x) - u(y))^2}{\mid x-y \mid^{N+2s}} \theta(x-y) dx dy \ge \theta_0 \parallel u \parallel^2$$

Therefore, we can deduce that

$$I_f(u) \ge \frac{\theta_0}{2} \left(1 - \frac{\lambda}{\lambda_0} \right) \parallel u \parallel^2 - C \parallel u \parallel$$

which implies that I_f is a coercive functional. Since we can check by standard arguments that I_f is weakly lower

semi-continuous, I_f has a minimizer $u_f \in H^s(\Omega)$ with

$$I_f(u_f) = \inf_{v \in H^s(\Omega)} I_f(v).$$

By the standard arguments we can show that $I_f \in C^1(H^s(\Omega), R)$ which implies that $u_f \in H^s(\Omega)$ is a solution of the equation $(P_{\lambda,f})$, i.e.,

$$\int_{R^{2N}} \frac{(u_f(x) - u_f(y))(v(x) - v(y))}{|x-y|^{N+2s}} \theta(x-y)dxdy - \lambda \int_\Omega Vu_f v dx - \int_\Omega fv dx$$

holds for any $v \in H^s(\Omega)$. Assume that the equation $(P_{\lambda,f})$ has another solution w_f. Then we can easily deduce that

$$\int_{R^{2N}} \frac{(T_f(x) - T_f(y))(v(x) - v(y))}{|x-y|^{N+2s}} \theta(x-y)dxdy = \lambda \int_\Omega VT_f v dx$$

for any $v \in H^s(\Omega)$ where $T_f = u_f - w_f$. Take $v = T_f$ in the above equality and notice that $0 < \lambda < \lambda_0$ we have

$$\int_{R^{2N}} \frac{|T_f(x) - T_f(y)|^2}{|x-y|^{N+2s}} \theta(x-y)dxdy = \lambda \int_\Omega VT_f^2 dx$$

$$< \lambda_0 \int_\Omega VT_f^2 dx \le \int_{R^{2N}} \frac{|T_f(x) - T_f(y)|^2}{|x-y|^{N+2s}} \theta(x-y)dxdy$$

a contradiction. Therefore, the equation has a unique solution.

3.2. The Solvability of the Problem (Max)

Theorem 1 Assume that $0 < V(x) \in L^\infty(\Omega)$, $f \in L^q(\Omega)$, $q > 2N/(N+2s)$, $0 < \lambda < \lambda_0$, then the problem (Max) has a solution $\hat{f} \in R(f)$, i.e.,

$$\Phi(\hat{f}) = \sup_{g \in R(f)} \Phi(g).$$

Proof. By Proposition 1, there is a unique solution $u_f \in H^s(\Omega)$ of the equation $(P_{\lambda,f})$. Let $A = \inf_{g \in R(f)} I_g(u_g)$, and we can see that for each $g \in R(f)$, it holds

$$I_g(u_g) = \frac{1}{2} \int_{R^N} \int_{R^N} \frac{|u_g(x) - u_g(y)|^2}{|x-y|^{N+2s}} \theta(x-y)dxdy$$

$$- \frac{\lambda}{2} \int_\Omega V|u_g|^2 dx - \int_\Omega gu_g dx$$

$$\ge \frac{1}{2}(1 - \frac{\lambda}{\lambda_0})\theta_0 \|u_g\|^2 - C\|g\|_{L_q} \|u_g\|.$$

From Lemma 2, $\|g\|_{L_q} = \|f\|_{L_q}$, we can deduce from the above inequality that A is finite. Choose $\{g_i\}$ as a minimizing sequence, i.e., $g_i \in R(f)$, $\forall i \in \mathbf{N}$ and

$$A = \lim_{i \to \infty} I(u_i),$$

where $u_i = u_{g_i}, I(u_i) = I_{g_i}(u_{g_i})$. Then $\{u_i\}$ is bounded in $H^s(\Omega)$ and has a subsequence which still denoted by $\{u_i\}$ that converge weakly to $u \in H^s(\Omega)$ and strongly to $u \in L^{q'}(\Omega)$ for the compact embedding of $H^s(\Omega)$ into $L^{q'}(\Omega)(1 < q' = q/(q-1) < 2N/(N-2s))$. Notice that $\|g_i\|_{L_q} \equiv \|f\|_{L_q}$, then $\{g_i\}$ is bounded in $L^q(\Omega)$ which has a subsequence still denoted by $\{g_i\}$ converge weakly to $\bar{g} \in \overline{R(f)}^{q,w}$, where $\overline{R(f)}^{q,w}$ is denoted by the weak closure of $R(f)$ in $L^q(\Omega)$. Then,

$$\left| \int_\Omega (g_i - \bar{g})u dx \right| \to 0, i \to \infty.$$

Combining with the Holder inequality we have

$$\left| \int_\Omega (g_i u_i - \bar{g}u)dx \right| \le \left| \int_\Omega g_i(u_i - u)dx \right| + \left| \int_\Omega (g_i - \bar{g})u dx \right| \le \|g_i\|_{L^q} \|u_i - u\|_{L^{q'}} + \left| \int_\Omega (g_i - \bar{g})u dx \right| \to 0, i \to \infty$$

Since $H^s(\Omega)$ is compactly embedded into $L^2(\Omega)$, $\{u_i\}$ converge strongly to u in $L^2(\Omega)$. It holds then,

$$\lim_{i \to \infty} \int_\Omega V(|u_i|^2 - |u|^2)dx = 0$$

Therefore,

$$A = \lim_{i \to \infty} I(u_i) \ge \frac{1}{2} \int_{R^N} \int_{R^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \theta(x-y)dxdy - \frac{\lambda}{2} \int_\Omega V|u|^2 dx - \int_\Omega \bar{g}u dx \qquad (1)$$

By Lemma 3, linear functional

$$l : \overline{R(f)}^{q,w} \mapsto R, l(g) = \int_\Omega g u dx$$

has a maximizer $\hat{f} \in R(f)$,

$$\int_\Omega \overline{g} u dx \leq \int_\Omega \hat{f} u dx .$$

We can imply from (1) that

$$A \geq \frac{1}{2} \int_{R^N} \int_{R^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \theta(x-y) dx dy - \frac{\lambda}{2} \int_\Omega V |u|^2 \, dx - \int_\Omega \hat{f} u dx .$$

Since $\hat{u}$ is the global minimizer of the functional $I_{\hat{f}}$,

$$I_{\hat{f}}(\hat{u}) = \inf_{v \in W_0^{s,p}(\Omega)} \frac{1}{2} \int_{R^N} \int_{R^N} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} \theta(x-y) dx dy - \frac{\lambda}{2} \int_\Omega V |v|^2 \, dx - \int_\Omega \hat{f} v dx$$

$$\leq \frac{1}{2} \int_{R^N} \int_{R^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \theta(x-y) dx dy - \frac{\lambda}{2} \int_\Omega V |u|^2 \, dx - \int_\Omega \hat{f} u dx$$

From the above two inequalities, we have

$$I_{\hat{f}}(\hat{u}) \leq A .$$

Since $A = \inf_{g \in R(f)} I_g(u_g)$, $\hat{f} \in R(f)$,

$$I_{\hat{f}}(\hat{u}) \geq A .$$

So that,

$$I_{\hat{f}}(\hat{u}) = A .$$

Since $I_{\hat{f}}(\hat{u}) = -\frac{1}{2} \Phi(\hat{f})$, $\Phi(\hat{f}) = \sup_{g \in R(f)} \Phi(g)$. We complete the proof.

4. Conclusions

In this paper, we are focusing on the maximization problem:

$$(Max) \qquad \inf_{g \in R(f)} \Phi(g)$$

where $\Phi(g) = \int_\Omega g u_g dx$, and u_g is the unique solution of the equation $(P_{\lambda,f})$:

$$\begin{cases} -L_\theta^s u - \lambda V(x) u = f(x), x \in \Omega, \\ u = 0, x \in R^N - \Omega \end{cases}$$

with the right hand term f replaced by its rearrangement $g \in R(f)$.

Since the term $-\lambda V(x) u$ is no longer non-decreasing with respect to the second variable u, the arguments of the uniqueness of the solution of the equation $(P_{\lambda,f})$ in [10] are not valid here. By using reduction to absurdity and the tools of the rearrangement theory, we can obtain the uniqueness of the solution of the equation $(P_{\lambda,f})$ and prove that the maximization problem (Max) is solvable. To the best of our knowledge, the result obtained here is new.

Acknowledgements

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