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An Existence Result in α-norm for Impulsive Functional Differential Equations with Variable Times

Issa Zabsonre

Department of Mathematics, University of Joseph KI-ZERBO, Ouagadougou, Burkina Faso

Email address:
zabsonreissa@yahoo.fr

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Abstract: The dynamics of evolving processes is often subjected to abrupt changes such as shocks, harvesting, and natural disasters. Often these short-term perturbations are treated as having acted instantaneously or in the form of "impulses." In fact, there are many processes and phenomena in the real world, which are subjected during their development to the short-term external influences. Their duration is negligible compared with the total duration of the studied phenomena and processes. Impulsive differential equations take an important place in some area such that physics, chemical technology, population dynamics, biotechnology, and economics. The study of such equations is relatively less developed due to the difficulties created by the state-dependent impulses. In the case of impulses at variable times, a "beating phenomenon" may occur, that is to say, a solution of the differential equation may hit a given barrier several times (including infinitely many times). In this work, we study the existence of solutions for some partial impulsive functional differential equations with variable times in Banach spaces by using the fractional power of closed operators theory. We suppose that the undelayed part admits an analytic semigroup. The delayed part is assumed to be Lipschitz. We use Schaefer fixed-point Theorem to prove the existence of solutions for this first order equation with impulse in α-norm.

Keywords: α-norm, Analytic Semigroup, Delay Differential Equation, Impulsive Equation

1. Introduction

In this work, we study the existence of solutions for the initial value problems for first order functional differential equations with impulsive effects.

$$\begin{cases} u'(t) = -Au(t) + f(t, u_t) \text{ for } t \in J = [0, b],\ t \neq \tau_k(u(t)),\ k = 1, ..., m \\ u(t^+) = I_k(u(t)) \text{ for } t = \tau_k(u(t) - H(t, u_t)),\ k = 1, ..., m \\ u_0 = \varphi \in C_\alpha, \end{cases} \tag{1}$$

where $-A$ is the infinitesimal generator of compact analytic semigroup on $\mathbb{R}^n$, C and C_α are defined by

$$C = \left\{ \begin{array}{l} \psi : [-r, 0] \to \mathbb{R}^n,\ \ \psi \text{ is continuous everywhere except for a finite number} \\ \text{of points at which } \ \psi(s^-) \text{ and } \psi(s^+) \ \text{ exist and } \psi(s^-) \neq \psi(s^+) \end{array} \right\}$$

$$C_\alpha = \{\psi \in C : \ \psi(\theta) \in D(A^\alpha) \ \text{ for } \ \theta \in [-r, 0] \ \text{ and } \ A^\alpha \psi \in C\},$$

$0 < \alpha < 1$, A^α is the fractional α-power of A, this operator $(A^\alpha, D(A^\alpha))$ will be describe later. For $t \geq 0$, u_t denotes the history function of C_α defined by

$$u_t(\theta) = u(t+\theta) \text{ for } \; -r \leq \theta \leq 0,$$

$f : J \times C_\alpha \to \mathbb{R}^n$ is an appropriate function, $I_k : D(A^\alpha) \to D(A^\alpha)$ and $\tau_k : D(A^\alpha) \to \mathbb{R}, \; k : 1, 2, ..., m$, are given functions satisfying some assumptions that will be specified later.

For more details about impulsive differential equations, the readers can refer to Bainov and Simeonov [1], Lakshmikantham et al. [9], and Samoilenko and Perestyuk [14], Bajo and Liz [2] and Frigon and O'Regan [3] and the references therein and the recent book of Wang and Feckan [15].

This work generalize [8] where the author prove their result in $\mathbb{R}^n$. In [7], the authors built a special strictly ascending continuous delay for a class of system of impulsive differential equations. They prove that even though the dynamics of the system and the delay have ideal continuity properties, the right side may not even have limits at some points due to the impact of past impulses in the present. In [6], the authors investigate the unified theory for solutions of differential equations without impulses and with impulses, even at variable times, allowing the presence of beating phenomena, in the space of regulated functions. They give sufficient conditions to ensure that a regulated solution of an impulsive problem is globally defined. In [12], by employing a critical point theorem, the authors establish the existence of infinitely many solutions for fourth-order impulsive differential equations depending on two real parameters. In [10], the authors propose to model the unequal partitioning of the molecular content at cell division like to be a source of heterogeneity in a cell population by using impulsive differential equation (IDE). They consider a general autonomous IDE with fixed times of impulse and a specific form of impulse function and establish properties of the solutions of that equation, most of them obtained under the hypothesis that impulses occur periodically. They apply those results to an IDE describing the concentration of the protein Tbet in a CD8 T-cell, where impulses are associated to cell division, to study the effect of molecular partitioning at cell division on the effector/memory cell-fate decision in a CD8 T-cell lineage. In [5], the authors give sufficient conditions to established the existence of at least one positive periodic solution for a family of scalar periodic differential equations with infinite delay and nonlinear impulses.

As in [4], we use Schaefer fixed-point theorem to study the system (1) by using the fractional power of closed operators theory.

The organization of this work is as follows, in Section 2, we recall some preliminary results on the α-norm and Schaefer's theorem. In Section 3, we prove our main result.

2. Preliminary Results

Let $(X, \|.\|)$ be a Banach space and α be a constant such that $0 < \alpha < 1$ and $-A$ be the infinitesimal generator of a bounded analytic semigroup of linear operator $(T(t))_{t\geq 0}$ on X. We assume that $0 \in \rho(A)$. If $0 \notin \rho(A)$, we can substitute the operator A by the operator $(A - \sigma I)$ with σ large enough such that $0 \in \rho(A - \sigma I)$. We define the fractional power A^α for $0 < \alpha < 1$, as a closed linear invertible operator with domain $D(A^\alpha)$ dense in X. Since A^α is closed, then $D(A^\alpha)$, endowed with the graph norm of A^α, $|x| = \|x\| + \|A^\alpha x\|$, is a Banach space. Since A^α is invertible, its graph norm $|.|$ is equivalent to the norm $|x|_\alpha = \|A^\alpha x\|$. Consequently, $D(A^\alpha)$ endowed with the norm $|.|_\alpha$, is a Banach space, denoted by X_α. The space C_α is endowed with the uniform norm topology:

$$\|\psi\|_\alpha = \sup_{\theta \in [-r,0]} |\psi(\theta)|_\alpha.$$

For $0 < \beta \leq \alpha < 1$, the imbedding $X_\alpha \hookrightarrow X_\beta$ is compact if the resolvent operator of A is compact. Also, the following properties are well known.

Proposition 2.1. [11] Let $0 < \alpha < 1$. Assume that the operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on the Banach space X satisfying $0 \in \rho(A)$. Then we have

i) $T(t) : X \to D(A^\alpha)$ for every $t > 0$.

ii) $T(t)A^\alpha x = A^\alpha T(t)x$ for every $x \in D(A^\alpha)$ and $t \geq 0$.

iii) for every $t > 0$, $A^\alpha T(t)$ is bounded on X and there exist $M_\alpha > 0$ and $\omega > 0$ such that

$$\|A^\alpha T(t)\| \leq M_\alpha e^{-\omega t} t^{-\alpha} \text{ for } t > 0.$$

iv) If $0 < \alpha \leq \beta < 1$, $D(A^\beta) \hookrightarrow D(A^\alpha)$.

v) There exists $N_\alpha > 0$ such that

$$\|(T(t) - I)A^{-\alpha}\| \leq N_\alpha t^\alpha \text{ for } t > 0.$$

Recall that $A^{-\alpha}$ is given by the following formula

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} T(t) dt,$$

where the integral converges in the uniform operator topology for every $\alpha > 0$.

Consequently, if $T(t)$ is compact for each $t > 0$, then $A^{-\alpha}$ is compact.

Definition 2.1. A function $f : J \times C \to \mathbb{R}^n$ is said to be L^2-Carathéodory if:

i) $t \to f(t, u)$ is measurable for each $u \in C$;

ii) $t \to f(t, u)$ is continuous for almost all $t \in J$;

iii) for each $q > 0$, there exists $h_q \in L^2(J; \mathbb{R}^+)$, such that

$$\|f(t, u)\| \leq h_q(t) \text{ for all } |u| \leq q \text{ and for almost all } t \in J.$$

In what follows, we will assume that f is an L^2-Carathéodory function.

The main result of this paper is based on the following Schaefer's fixed-point theorem (cf. [13]).

Theorem 2.1. [13] Let X be a Banach space and $N : X \to X$ be a completely continuous map. If the set

$$\mathcal{E}(N) = \{u \in X :\ u = \sigma N(u) \text{ for some } 0 < \sigma < 1\}$$

is bounded, then N has a fixed point.

3. Mains Results

Let us start by defining what we mean by a solution of problem

Definition 3.1. A continuous function $u : [-r, b] \to \mathbb{R}^n_\alpha$ is a mild solution of equation (1) if

$$i) \quad u(t) = T(t)\varphi(0) + \int_0^t T(t-s)v(s)ds \ \text{ for } \ t \in J \ \text{ for } \ t \neq \tau_k(u(t)), \ \ k = 1, ..., m$$

$$ii) \quad u(t^+) = I_k(u(t)) \ \text{ for } \ t = \tau_k(u(t)), \ \ k = 1, ..., m$$

$$iii) \quad u(\theta) = \varphi(\theta) \ \text{ for } \ -r \leq \theta \leq 0$$

Now, we can prove our existence result under the following hypotheses.

(H_1) The operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on the Banach space $\mathbb{R}^n$, moreover, we assume that $0 \in \rho(A)$.

(H_2) The semigroup $((T(t))_{t\geq 0}$ is compact on $\mathbb{R}^n$ for $t > 0$.

(H_3) The functions $\tau_k \in C^1(\mathbb{R}^n_\alpha, \mathbb{R})$, for $k = 1, ..., m$. Moreover,

$$0 < \tau_1(x) < ... < \tau_m(x) \text{ for all } x \in \mathbb{R}^n_\alpha.$$

(H_4) There exist constants c_k, such that $|I_k(x)| \leq c_k,\ k = 1, ..., m$ for each $x \in \mathbb{R}^n_\alpha$

(H_5) There exist a continuous nondecreasing function $\psi : [0, +\infty[\to]0, +\infty[$ such that $p \in L^2([0, +\infty[)$ and

$$\sup_{s\in[0,t]} \|f(s, u)\| \leq p(s)\psi(\|u\|_\alpha)$$

for $t \in J$ and each $u \in C_\alpha$.

(H_6) For all $(t, x) \in [0, b] \times \mathbb{R}^n_\alpha$ and for all $u_t \in C_\alpha$, we have

$$\Big\langle \tau_k'(x), -Au(t) + f(t, u_t) \Big\rangle \neq 1 \ \text{ for } k = 1, ..., m,$$

where $\langle ., . \rangle$ denotes the scalar product in $\mathbb{R}^n$.

(H_7) For all $x \in \mathbb{R}^n_\alpha$

$$\tau_k(I_k(x)) \leq \tau_k(x) < \tau_{k+1}(I_k(x)) \ \text{ for } k = 1, ..., m.$$

In what follows, we choose α such that $0 < \alpha < \dfrac{1}{2}$.

Theorem 3.1. Assume that (H_1), (H_2), (H_3), (H_4), (H_5), (H_6) and (H_7) hold and et $\varphi \in C_\alpha$. Then, problem (1) has at least one solution $u(., \varphi)$ on $[-r, b]$.

Proof. The proof is done in several steps.

Step 1. Consider the following problem

$$\begin{cases} u'(t) = -Au(t) + f(t, u_t) \text{ for } t \in [0, b] \\ \\ u_0 = \varphi \in C_\alpha. \end{cases} \tag{2}$$

We transform equation (2) into a fixed-point problem.

Let $\mathcal{K} : C([-r, b]; \mathbb{R}^n_\alpha) \to C([-r, b]; \mathbb{R}^n_\alpha)$ be an operator defined by

$$\mathcal{K}(u)(t) = \begin{cases} T(t)\varphi(0) + \displaystyle\int_0^t T(t-s)f(s, u_s)ds \text{ for } t \in J \\ \\ \varphi(t) \ \text{ for } t \in [-r, 0]. \end{cases}$$

Claim 1. $\mathcal{K}$ is continuous.
Let $(u_n)_n$ be a sequence such that $u_n \to u$ in $C([-r,b];\mathbb{R}^n_\alpha)$. Then using Proposition (2.1), we have

$$\begin{aligned}
|\mathcal{K}(u)(t)-\mathcal{K}(u_n)(t)|_\alpha &= \Big|\int_0^t T(t-s)(f(s,u_s)-f(s,u_{n_s}))ds\Big|_\alpha \\
&\leq \int_0^t \|A^\alpha T(t-s)(f(s,u_s)-f(s,u_{n_s}))\|ds \\
&\leq M_\alpha \int_0^t \frac{\|f(s,u_s)-f(s,u_{n_s})\|e^{-\omega(t-s)}}{(t-s)^\alpha}ds \\
&\leq M_\alpha \Big(\int_0^t \|f(s,u_s)-f(s,u_{n_s})\|^2 ds\Big)^{\frac{1}{2}}\Big(\int_0^t \frac{e^{-2\omega(t-s)}}{(t-s)^{2\alpha}}ds\Big)^{\frac{1}{2}} \\
&\leq (2\omega)^{2\alpha-1}M_\alpha \Big(\int_0^t \|f(s,u_s)-f(s,u_{n_s})\|^2 ds\Big)^{\frac{1}{2}}\Big(\int_0^t e^{-s}s^{-2\alpha+1-1}ds\Big)^{\frac{1}{2}} \\
&\leq (2\omega)^{2\alpha-1}\Big[\Gamma(1-2\alpha)\Big]^{\frac{1}{2}} M_\alpha \Big(\int_0^t \|f(s,u_s)-f(s,u_{n_s})\|^2 ds\Big)^{\frac{1}{2}}.
\end{aligned}$$

Since f is an L^2-Carathéodory function, we have by the Lebesgue dominated convergence theorem

$$|\mathcal{K}(u)-\mathcal{K}(u_n)|_\alpha \leq (2\omega)^{2\alpha-1}\Big[\Gamma(1-2\alpha)\Big]^{\frac{1}{2}} M_\alpha \Big|f(s,u_s)-f(s,u_{n_s})\Big|_{L^2} \to 0 \text{ as } n\to+\infty$$

Claim 2. $\mathcal{K}$ maps bounded sets into bounded sets in $C([-r,b];\mathbb{R}^n_\alpha)$.
In order to apply Ascoli's Theorem we need to prove that the set $\{\mathcal{K}(u)(t) : u \in B_q\}$, where $B_q = \{u \in C([-r,b];\mathbb{R}^n_\alpha) : \|u\|_\infty \leq q\}$ is relatively compact for each $t \in]0,b]$, where

$$\|u\|_\infty = \sup_{t\in[-r,b]} |u(t)|_\alpha.$$

Let $t \in]0,b]$ be fixed, and $\gamma > 0$ be such that $\alpha < \gamma < \dfrac{1}{2}$. Since $(T(t))_{t>0}$ is compact, let us pose $M = \sup\{\|T(t)\| : \quad t \in [0,b]\}$, then using Proposition (2.1) and Definition (2.1), we have

$$\begin{aligned}
\|A^\gamma \mathcal{K}(u)(t)\| &\leq \|A^\gamma T(t)\varphi(0)\| + \Big\|\int_0^t A^\gamma T(t-s)f(s,u_s)ds\Big\| \\
&\leq M|\varphi(0)|_\gamma + M_\gamma \int_0^t \frac{\|f(s,u_s)\|e^{-\omega(t-s)}}{(t-s)^\gamma}ds \\
&\leq M|\varphi(0)|_\gamma + M_\gamma \int_0^t \frac{h_q(s)e^{-\omega(t-s)}}{(t-s)^\gamma}ds \\
&\leq M|\varphi(0)|_\gamma + (2\omega)^{2\gamma-1}\Big[\Gamma(1-2\gamma)\Big]^{\frac{1}{2}} M_\gamma \Big(\int_0^b h_q^2(s)ds\Big)^{\frac{1}{2}} < \infty.
\end{aligned}$$

Then for fixed $t \in]0,b]$, $\{A^\gamma \mathcal{K}(u)(t) : u \in B_q\}$ is bounded in $\mathbb{R}^n$. Using (H_2) and the definition of $A^{-\gamma}$, we deduce that $A^{-\gamma} : \mathbb{R}^n \to \mathbb{R}^n_\alpha$ is compact, it follows that $\{\mathcal{K}(u)(t) : u \in B_q\}$ is relatively compact set in $\mathbb{R}^n_\alpha$.
Claim 3. $\mathcal{K}$ maps bounded sets into equicontinuous sets of $C([-r,b];\mathbb{R}^n_\alpha)$.

Let $t_1, t_2 \geq 0,\ t_1 < t_2$, B_q be a bounded set of $C([-r, b]; \mathbb{R}^n_\alpha)$ as in Claim 2, and let $u \in B_q$. Then we have

$$\begin{aligned}\mathcal{K}(u)(t_2) - \mathcal{K}(u)(t_1) &= (T(t_2) - T(t_1))\varphi(0) + \int_{t_1}^{t_2} T(t_2 - s) f(s, u_s) ds \\ &\quad + \int_0^{t_1} (T(t_2 - s) - T(t_1 - s)) f(s, u_s) ds \\ &= (T(t_2) - T(t_1))\varphi(0) + \int_{t_1}^{t_2} T(t_2 - s) f(s, u_s) ds \\ &\quad + (T(t_2 - t_1) - I) \int_0^{t_1} T(t_1 - s) f(s, u_s) ds\end{aligned}$$

We obtain that

$$\begin{aligned}|\mathcal{K}(u)(t_2) - \mathcal{K}(u)(t_1)|_\alpha &\leq \|(T(t_2) - T(t_1))\varphi(0)\| \\ &\quad + (2\omega)^{2\alpha-1}\Big[\Gamma(1-2\alpha)\Big]^{\frac{1}{2}} M_\alpha \Big(\int_{t_1}^{t_2} h_q^2(s) ds\Big)^{\frac{1}{2}} \\ &\quad + \Big\|(T(t_2 - t_1) - I) \int_0^{t_1} A^\alpha T(t_1 - s) f(s, u_s) ds\Big\|.\end{aligned}$$

The first part converges to zero as $|t_2 - t_1| \to 0$. On the other hand, we have

$$\Big\|\int_0^{t_1} A^\alpha T(t_1 - s) f(s, u_s) ds\Big\| \leq (2\omega)^{2\alpha-1}\Big[\Gamma(1-2\alpha)\Big]^{\frac{1}{2}} M_\alpha \Big(\int_0^{t_1} h_q^2(s) ds\Big)^{\frac{1}{2}} < +\infty.$$

Consequently since $t_1 > 0$, then the set

$$\Big\{\int_0^{t_1} A^\alpha T(t_1 - s) f(s, x_s) ds, \quad u \in B_q\Big\}$$

is relatively compact in $\mathbb{R}^n$. There is a compact set Ω in $\mathbb{R}^n$ such that

$$\Big\{\int_0^{t_1} A^\alpha T(t_1 - s) f(s, u_s) ds, \quad u \in B_q\Big\} \subset \Omega.$$

By Banach-Steinhaus Theorem, we have

$$\Big\|(T(t_2 - t_1) - I) \int_0^{t_1} A^\alpha T(t_1 - s) f(s, u_s) ds\Big\| \to 0 \ \text{ as } \ t_1 \to t_2$$

uniformly in $u \in B_q$. Using similar argument for $0 \leq t_2 < t_1 \leq b$, we can conclude that $\{\mathcal{K}(u)(t) : u \in B_q\}$ is equicontinuous. As a consequence of Claims 1, 2 and 3, together with Arzelá-Ascoli's Theorem, it follows that $\mathcal{K}$ is completely continuous.

Claim 4. It remains to prove that the set

$$\mathcal{E}(\mathcal{K}) = \{u \in C([-r, b]; \mathbb{R}^n_\alpha) : \ u = \sigma \mathcal{K}(u) \text{ for some } 0 < \sigma < 1\}$$

is bounded.

Let $u \in \mathcal{E}(\mathcal{K})$. Then, $u = \sigma \mathcal{K}(u)$ for some $0 < \sigma < 1$. Thus, for each $t \in J$

$$u(t) = \sigma\Big(T(t)\varphi(0) + \int_0^t T(t - s) f(s, u_s) ds\Big).$$

Using (H_5), we can see that for each $t \in J$, we have

$$\begin{aligned}
|u(t)| &\leq M|\varphi(0)|_\alpha + \int_0^t |T(t-s)f(s,u_s))|_\alpha ds \\
&\leq M|\varphi(0)|_\alpha + M_\alpha \int_0^t \frac{\|f(s,u_s))\|e^{-\omega(t-s)}}{(t-s)^\alpha} ds \\
&\leq M|\varphi(0)|_\alpha + M_\alpha \int_0^t \frac{\|f(t-s,u_{t-s}))\|e^{-\omega s}}{s^\alpha} ds \\
&\leq M|\varphi(0)|_\alpha + M_\alpha \int_0^t \frac{(\sup_{s\in[0,t]} \|f(s,u_s))\|)e^{-\omega s}}{s^\alpha} ds \\
&\leq M|\varphi(0)|_\alpha + M_\alpha \int_0^t \frac{p(s)\psi(\|u_s\|_\alpha)e^{-\omega s}}{s^\alpha} ds.
\end{aligned}$$

Consider the function μ be defined by

$$\mu(t) = \sup\{|u(s)|_\alpha : \ -r \leq s \leq t\} \text{ for } t \in J.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = u(t^*)$. If $t^* \in [-r,b]$, by the previous inequality, we have

$$\mu(t) \quad \leq \quad M\|\varphi\|_\alpha + M_\alpha \int_0^t \frac{p(s)\psi(\mu(s))e^{-\omega s}}{s^\alpha} ds. \tag{3}$$

Define the function v on $\mathbb{R}^+$ by

$$v(t) = M\|\varphi\|_\alpha + M_\alpha \int_0^t \frac{p(s)\psi(\mu(s))e^{-\omega s}}{s^\alpha} ds.$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\varphi\|_\alpha$ and the inequality (3) holds. Then, we have

$$c = v(0) = M\|\varphi\|_\alpha \ \text{ and } \ \mu(t) \leq v(t) \ \text{ for } t \in J$$

Differentiating v, we obtain

$$v'(t) = \frac{M_\alpha p(t)\psi(\mu(t))e^{-\omega t}}{t^\alpha} \ \text{ for almost } t \in J.$$

Since ψ is a nondecreasing function, we get

$$v'(t) \leq \frac{M_\alpha p(t)\psi(v(t))e^{-\omega t}}{t^\alpha} \ \text{ for almost } t \in J.$$

Integrating previous equation, we obtain

$$\int_0^t \frac{v'(s)}{\psi(v(s))} ds \leq c_1\Big(\int_0^t p^2(s)ds\Big)^{\frac{1}{2}} < +\infty,$$

where

$$c_1 = (2\omega)^{2\alpha-1}\Big[\Gamma(1-2\alpha)\Big]^{\frac{1}{2}} M_\alpha$$

By a change of variables we get

$$\int_{v(0)}^{v(t)} \frac{ds}{\psi(s)} \leq \int_{c_0}^{\infty} \frac{ds}{\psi(s)} \leq c_2\Big(\int_0^{+\infty} p^2(s)ds\Big)^{\frac{1}{2}} < +\infty.$$

Thus, there exists a constant λ, such that $v(t) \leq \lambda$ for $t \in J$, hence $\mu(t) \leq \lambda$ for $t \in J$. Since for every $t \in J$, $\|u_t\|_\alpha \leq \mu(t)$, we have

$$\|u\|_\alpha \leq \lambda_0 = \max\{c_0, \lambda\},$$

where λ_0 depends on functions p and ψ. This shows that $\mathcal{E}(\mathcal{K})$ is bounded.

Set $X := C([-r,b], \mathbb{R}^n_\alpha)$. Using Schaefer's fixed point theorem (see [13, p. 29]), we deduce that $\mathcal{K}$ has a fixed-point u which is a solution to equation (2) denoted by u_1.

We define the function $r_{k,1}$ by

$$r_{k,1}(t) = \tau_k(u_1(t)) - t \quad \text{for } t \in J.$$

Then by (H_3) we have

$$r_{k,1}(0) \neq 0 \text{ for } t \in J \text{ and } k = 1, ..., m.$$

If

$$r_{k,1}(t) \neq 0 \text{ on } [0,b] \text{ and for } k = 1, ..., m,$$

then

$$t \neq \tau_k(u_1(t)) \text{ on } [0,b] \text{ and for } k = 1, ..., m,$$

consequently u_1 is a solution of equation (1). Finally, consider the case when $r_{1,1}(t) = 0$ for some $t \in J$. Now, since $r_{1,1}(0) \neq 0$ and $r_{1,1}$ is continuous, there exists $t_1 > 0$ such that

$$r_{1,1}(t_1) = 0 \ \text{ and } \ r_{1,1}(t) \neq 0 \ \text{ for all } \ t \in [0, t_1[.$$

Thus, by (H_3), we have

$$r_{k,1}(t) \neq 0 \text{ for all } \ t \in [0, t_1[\ \text{ and } k = 1, ..., m.$$

Step 2. Consider now the following problem

$$\begin{cases} u(t) = u_1(t) \text{ for } t \in [t_1 - r, t_1] \\ u'(t) = -Au(t) + f(t, u_t) \text{ for almost } [t_1, b] \\ u(t_1^+) = I_1(u_1(t_1)). \end{cases} \tag{4}$$

Transform equation (4) into a fixed-point problem. Consider the operator $\mathcal{H}_1 : C([t_1 - r, b]; \mathbb{R}^n_\alpha) \to C([t_1 - r, b]; \mathbb{R}^n_\alpha)$ defined by

$$\mathcal{H}_1(u)(t) = \begin{cases} I(u(t_1)) + T(t)\varphi(0) + \displaystyle\int_{t_1}^{t} T(t-s) f(s, u_s) ds \text{ for } t \in [t_1, b] \\ u_1(t) \ \text{ for } t \in [t_1 - r, t_1]. \end{cases}$$

As in Step 1, we can show that $\mathcal{K}_1$ is completely continuous and the set

$$\mathcal{E}(\mathcal{K}_1) = \{u \in C([t_1 - r, b]; \mathbb{R}^n_\alpha) : \ u = \sigma \mathcal{H}_1(u) \text{ for some } 0 < \sigma < 1\}$$

is bounded.

Set $X := C([t_1 - r, b], \mathbb{R}^n_\alpha)$. By Schaefer's theorem, we deduce that $\mathcal{K}_1$ has a fixed-point u which is a solution to equation (4). Denote this solution by u_2.

We define the function

$$r_{k,2}(t) = \tau_k(u_2(t)) - t \text{ for } t \geq t_1.$$

(H_3) implies that

$$r_{k,2}(0) \neq 0 \text{ for } t \in]t_1, b] \text{ and } k = 1, ..., m.$$

If

$$r_{k,2}(t) \neq 0 \text{ for } t \in]t_1, b] \ \text{ and } k = 1, ..., m,$$

then

$$u(t) = \begin{cases} u(t_1) \text{ if } t \in [0, t_1] \\ u_2(t) \ \text{ if } t \in [t_1, b] \end{cases}$$

is a solution of equation (1). It remains to consider the case when the case when $r_{2,2}(t) = 0$ for some $t \in J$. By (H_7), we have

$$\begin{aligned} r_{2,2}(t_1^+) &= \tau_2(u_2(t_1^+)) - t_1 \\ &= \tau_2(I_1(u_1(t_1))) - t_1 \\ &> \tau_1(u_1(t_1)) - t_1 \\ &> r_{1,1}(t_1) = 0. \end{aligned}$$

Since $r_{2,2}$ is continuous, there exists $t_2 > t_1$ such that

$$r_{2,2}(t_2) = 0 \ \text{ and } \ r_{2,2}(t) \neq 0 \ \text{ for all } \ t \in]t_1, t_2[.$$

By (H_3), it follows that

$$r_{k,2}(t) \neq 0 \text{ for } t \in]t_1, t_2[\ \text{ and } k = 2, ..., m.$$

Suppose now, that there is $\overline{s} \in]t_1, t_2[$, such that $r_{1,2}(\overline{s}) = 0$.

From (H_7), it follows that

$$\begin{aligned} r_{1,2}(t^+) &= \tau_1(u_2(t_1^+)) - t_1 \\ &= \tau_1(I(u_1(t_1))) - t_1 \\ &\leq \tau_1(u_1(t_1)) - t_1 \\ &\leq r_{1,1}(t_1) = 0. \end{aligned}$$

Thus, the function $r_{1,2}$ attains a nonnegative maximum at some point $s_1 \in]t_1, b]$. Since

$$u_2'(t) = -Au_2(t) + f(t, u_{2_t})$$

and

$$r_{1,2}'(s_1) = \tau_1'(u_2(s_1))u_2'(s_1) - 1,$$

then

$$\left\langle \tau_1'(u_2(s_1)), -Au(s_1) + f(s_1, u_{2s_1}) \right\rangle = 1$$

which is a contradiction by (H_6).

Step 3. We continue this process and taking into account that $u_{m+1} := u_{|[t_m,b]}$ is a solution to the problem

$$\begin{cases} u(t) = u_m(t) \text{ for } t \in [t_m - r, t_m] \\ u'(t) = -Au(t) + f(t, u_t) \text{ for almost } t \in]t_m, b[\\ u({t_m}^+) = I_m(u_{m-1}(t_m)). \end{cases}$$

The solution u of equation (1) is then defined by the problem

$$u(t) = \begin{cases} u_1(t) \text{ if } t \in [-r, t_1] \\ u_2(t) \text{ if } t \in]t_1, t_2] \\ \vdots \\ u_{m+1}(t) \text{ if } t \in]t_m, b]. \end{cases}$$

4. Conclusion

In this work, we have used Schaefer fixed-point Theorem to prove the existence of solutions for this first order equation with impulse in α-norm under some assumptions. The proof is done in several step by bulding a function which is contiuous, maps bounded sets into bounded and equicontinuous sets. The controllability in the α-norm of impulsive systems with nonlocal conditions with variable finite delay will be presented in next works.

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