**American Journal of Applied Mathematics**
2022; 10(4): 125-133
http://www.sciencepublishinggroup.com/j/ajam
doi: 10.11648/j.ajam.20221004.12
ISSN: 2330-0043 (Print); ISSN: 2330-006X (Online)

# Existence and Multiplicity of Solutions for a Class of Quasilinear Schrödinger Equations ◇

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## To cite this article:

Ziqing Yuan, Shen Liu. Existence and Multiplicity of Solutions for a Class of Quasilinear Schrödinger Equations ◇. *American Journal of Applied Mathematics*. Vol. 10, No. 4, 2022, pp. 125-133. doi: 10.11648/j.ajam.20221004.12

**Received:** June 22, 2022; **Accepted:** July 8, 2022; **Published:** July 26, 2022

**Abstract:** Quasilinear Schrödinger equations appear in several differential physical phenomena. We consider the quasilinear Schrödinger equation $-\Delta u + V(x)u + \frac{\lambda}{2}[|\nabla u|^2]\Delta u = f(x,u)$ in $\mathbb{R}^N$, where $V$ and $f$ are periodic in $x_1, ..., x_N$ and $f$ is odd in $u$ and subcritical. By employing the genus theory and variational method, we only need $f$ is continuous, which is allowed to have weaker asymptotic growth than usually assumed, and obtain infinitely many geometrically distinct solutions for $\lambda > 0$.

**Keywords:** Quasilinear Schrodinger Equation, Multiplicity of Solutions, Genus Theory

## 1. Introduction

This paper concerns the following quasilinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u + \dfrac{\lambda}{2}[\Delta |u|^2]u = f(x,u) \text{ a.e. in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1}$$

where $\lambda$ is a positive parameter, $V : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Such problems are related to standing wave solutions to the so-called modified Schrödinger equation

$$i\psi_t = -\Delta\psi + W(x)\psi - \rho(|\psi|^2)\psi + \frac{\lambda}{2}[\Delta|z|^2]z, \quad x \in \mathbb{R}^N, \tag{2}$$

where $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential and $\rho : \mathbb{R} \to \mathbb{R}$ is a real function. Quasilinear Schrödinger equations like (2) appear in several differential physical phenomena, such as in plasma physics, in superfluid films and in condensed matter theory, see [1, 2, 3, 4]. For the case where $\lambda = 0$, problem (1) becomes into the semilinear equation and it has been widely studied by various conditions, see e.g. [6, 7, 8]. When $\lambda < 0$, equation (1) has been introduced in [9, 10] to deal with a model of self-trapped electrons in hexagonal or quadratic lattices and it has caused much attention.

There exist lots of works on the existence and multiplicity of solutions to problem (1) via different methods for $\lambda < 0$. For example, [5, 11] for a change of variables, [12, 19] for a constrained minimization argument, [20] for Nehari manifold methods.

However, most of these results are based on the fact $\lambda < 0$. There exist few results on dealing with the case $\lambda > 0$.

From the variational point of view, the first difficulty is to find a suitable Sobolev space as (5) is not well defined in $H^1(\mathbb{R}^N)$. The other difficulty is to ensure the positiveness of the principal part, i.e., $1 - \lambda u^2 > 0$. We should mention that for $\lambda > 0$, Lange et al. [21] considered the Cauchy problem for quasilinear Schrödinger equation (2) with $\rho = 0$ and $W = 0$. When $N = 1$ and $\psi(0,x) = \phi(x)$, they derived $L^2$-solutions for (2) with $\lambda|\phi(x)| \le \delta < 1$. Furthermore, for $2\lambda\|\phi\|_{W^{1,\infty}} < 1$, they also derived the existence of $H^2$-solution for arbitrary space dimension. Alves et. al., combining variational methods with perturbation arguments, obtained the existence of nontrivial solutions for problem (1) by replacing $f(x,u)$ with $|u|^{q-1}u$ and $\left[1 - \frac{1}{(1+|u|^2)^3}\right]u$ respectively[13].

Here we want to show that problem (1) has infinitely many pairs $\pm u$ of geometrically distinct solutions. To the best of our knowledge, there exist few results on the existence of infinitely many solutions to problem (1) for $\lambda > 0$. According to [13], we make a change of variables, and then use the method developed in [14] to obtain multiple results.

Let $F(x,u) := \int_0^u f(x,t)dt$ and $G(u) = \int_0^t g(s)ds$, $g$ is defined in (6). We assume that $V$ and $g$ satisfy the following hypotheses:

$(H1)$ $V$ is continuous, 1-periodic in $x_i$ for $1 \le i \le N$, and there is $m_0 > 0$ such that $V(x) \ge m_0$ for all $x \in \mathbb{R}^N$.

$(H2)$ $f$ is continuous, 1-periodic in $x_i$ for $1 \le i \le N$, and $|f(x,u)| \le c(1+|u|^{p-1})$ for some $c > 0$ and $2 < p < 2^*$, where $2^* = \frac{2N}{N-2}$ if $N \ge 3$, $2^* := \infty$ if $N = 1$ or 2.

$(H3)$ $f(x,u) = o(u)$ uniformly in $x$ as $u \to 0$.

$(H4)$ $\frac{f(x,u)-V(x)u}{G(u)g(u)}$ is strictly increasing for $u \in (0,+\infty)$.

$(H5)$ $\frac{F(x,u)}{u^2} \to \infty$ as $|u| \to \infty$.

*Remark* 1.1. (i) $(H3)$ and $(H4)$ mean that $uf(x,u) > 2F(x,u) > 0$ for $n \ne 0$.

(ii) $(H2)$ and $(H3)$ imply that for $\forall \epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|f(x,u)| \le \epsilon|u| + C_\epsilon|u|^{p-1} \quad for \ \ all \ \ u \in \mathbb{R}. \tag{3}$$

(iii) $(H2)$, $(H3)$ and $(H5)$ are standard assumptions in this context, we drop the well-known (AR) condition. Our condition $(H5)$ is weaker than (AR) condition.

Set $*$ denote the action of $\mathbb{Z}^n$ on $H^1(\mathbb{R}^N)$ given by

$$(k * u)(x) := u(x-k), \;\; k \in \mathbb{Z}^N. \tag{4}$$

$(H1)$ and $(H2)$ imply that if $u_0$ is a solution of problem (1), then so is $k * u_0$ for all $k \in \mathbb{Z}^N$. Define

$$\mathcal{O}(u_0) := \{k * u_0 : k \in \mathbb{Z}^N\}.$$

$\mathcal{O}(u_0)$ is called the orbit for a energy functional $J$ if $u_0$ is a critical point of $J$ and $J$ is $\mathbb{Z}^N$-invariant, i.e., $J(k*u) = J(u)$ for all $k \in \mathbb{Z}^N$ and all $u$. Two solutions $u_1$ and $u_2$ of (2) are said to be geometrically distinct if $\mathcal{O}(u_1) \ne \mathcal{O}(u_2)$.

We now give our main result.

*Theorem* 1.1. If hypotheses $(H1) - (H5)$ hold and $f$ is odd in $u$, then problem (1) has infinitely many pairs $\pm u$ of geometrically distinct solutions.

This paper is organized as follows. In Section 2, we present an auxiliary problem and some necessary preliminary knowledge. We prove our main result in Section 3.

# 2. Preliminary Results

From hypothesis $(H1)$ we will discuss problem (1) in the space $H^1(\mathbb{R}^N)$ endowed with the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2)\right)^{\frac{1}{2}},$$

which is an equivalent norm in $H^1(\mathbb{R}^N)$. $S$ is the unit sphere in $H^1(\mathbb{R}^N)$. $C_1, C_2, c_1, c_2, ...$ denote different positive constants whose exact values may be different. $|\Omega|$ is the Lebesgue measure of a measurable set $\Omega \subset \mathbb{R}^N$. $B_r(y) := \{x \in \mathbb{R}^N : |x-y| < r\}$. The usual norm of the Lebesgue space $L^p(\Omega)$ is denoted by $|u|_{p,\Omega}$, and by $|u|_p$ if $\Omega = \mathbb{R}^N$. For a energy functional $J$ we set $J^c := \{u : J(u) \le c\}$, $J_c := \{u : J(u) \ge c\}$, $J_{c_1}^{c_2} := \{u : c_1 \le J(u) \le c_2\}$.

Note that (1) is the Euler-Lagrange equation associated with the energy functional

$$\varphi(u) = \frac{1}{2}\int_{\mathbb{R}^N}(1-\lambda u^2)|\nabla u|^2dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2dx - \int_{\mathbb{R}^N} F(x,u)dx. \tag{5}$$

In order to seek solutions of $\varphi(u)$, define $g : [0,+\infty) \to \mathbb{R}$ by

$$g(t) = \begin{cases} \sqrt{1-\lambda t^2} & if\ 0 \le t < \frac{1}{\sqrt{3\lambda}}, \\ \frac{1}{3\sqrt{2\lambda}t} + \frac{1}{\sqrt{6}}, & if\ \frac{1}{\sqrt{3\lambda}} \le t. \end{cases} \tag{6}$$

Set $g(t) = g(-t)$ for all $t \le 0$. It is easy to see that $g \in C^1\left(\mathbb{R}, \left(\frac{1}{\sqrt{6}}, 1\right]\right)$, and $g$ is an even function, which is decreasing in $[0,+\infty)$ and increasing in $(-\infty, 0)$. Then $\varphi(u)$ becomes into

$$\varphi(u) = \frac{1}{2}\int_{\mathbb{R}^N} g^2(u)|\nabla u|^2dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2dx - \int_{\mathbb{R}^N} F(x,u)dx. \tag{7}$$

Let $G(t) = \int_0^t g(s)ds$. From a simple computation, we have that the inverse function $G^{-1}(t)$ exists, $G^{-1}, G \in C^2(\mathbb{R})$, and it is an old function. The following lemma comes from [13], which will be used later.

*Lemma* 2.1. (1) $\lim_{t\to 0} \frac{G^{-1}(t)}{t} = 1$;

(2) $\lim_{t\to\infty} \frac{G^{-1}(t)}{t} = \sqrt{6}$;

(3) $t \leq G^{-1}(t) \leq \sqrt{6}t \;\; for \;\; all \;\; t \geq 0$;
(4) $-\frac{1}{2} \leq \frac{t}{g(t)}g'(t) \leq 0 \;\; for \;\; all \;\; t \geq 0$.

In order to find the critical points of $\varphi(u)$, we make a change variable $v = G(u) = \int_0^u g(s)ds$. Thus the functional $\varphi(u)$ can be written in the following form:

$$\psi(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx - \int_{\mathbb{R}^N} F(x, G^{-1}(v))dx.$$

Then $\psi$ is well defined in $H := H^1(\mathbb{R}^N)$ and $\psi \in C^1(H, \mathbb{R})$ under the hypotheses $(H1) - (H3)$. Note that $(H1)$ and $(H2)$ mean that $\psi$ is invariant with respect to the action of $\mathbb{Z}^N$ given by (4). It is easy to see that

$$\langle \psi(v), w\rangle = \int_{\mathbb{R}^N} \left[\nabla v \nabla w + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}w - \frac{f(x, G^{-1}(v))}{g(G^{-1}(v))}w\right] dx \tag{8}$$

for all $v, w \in H$, and the critical points of $\psi$ are weak solutions of the following problem:

$$\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = \frac{f(x, G^{-1}(v))}{g(G^{-1}(v))}, \quad \forall v \in H.$$

It has been proved that in [25] if $v \in H$ is a critical point of the functional $\psi$, then $u = G^{-1}(v) \in H$ and $u$ is a solution of problem (1).

# 3. Proof of the Main Result

Let

$$\mathscr{M} := \{v \in H \setminus \{0\} : \langle \psi'(v), v\rangle = 0\},$$

where $\mathscr{M}$ is called the Nehari manifold. Since we don't know whether $\mathscr{M}$ is of class $C^1$ under our hypotheses, we can't employ minimax theory directly on $\mathscr{M}$. In order to overcome this difficulty, we use the method developed in [14].

For $t > 0$, set

$$I(t) = \psi(tv) = \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(tv)|^2 dx - \int_{\mathbb{R}^N} F(x, G^{-1}(tv))dx.$$

*Lemma* 3.1. For all $u \neq 0$ there exists a unique $t_u > 0$ such that $I'(t) > 0$ for $0 < t < t_u$ and $I'(t) < 0$ for $t > t_u$. Furthermore, $t_u \in \mathscr{M}$ if and only if $t = t_u$.

*Proof* By virtue of (3), $(H1)$ and Lemma 2.1(3) , for $\epsilon$ sufficiently small we have

$$\begin{aligned} I(t) &\geq \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(tv)|^2 dx - \frac{\epsilon}{2}\int_{\mathbb{R}^N} |G^{-1}(tv)|^2 dx - \frac{C_\epsilon}{p}\int_{\mathbb{R}^N} |G^{-1}(tv)|^p dx \\ &\geq \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{6^{\frac{p}{2}}}{p}C_\epsilon t^p \int_{\mathbb{R}^N} |v|^p dx. \end{aligned}$$

Since $p > 2$ and $V$ is not a constant, the above inequality deduces that $I(t) > 0$ when $t > 0$ is sufficiently small. From Lemma 2.1-(3), one has

$$\begin{aligned} I(t) &\leq \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + 3t^2\int_{\mathbb{R}^N} V(x)v^2 dx - \int_{\mathbb{R}^N} F(x, G^{-1}(tv))dx \\ &\leq \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + 3t^2\int_{\mathbb{R}^N} V(x)v^2 dx - t^2\int_{v\neq 0} \frac{F(x, G^{-1}(tv))}{(G^{-1}(tv))^2} \cdot \frac{(G^{-1}(tv))^2}{(tv)^2}v^2 dx. \end{aligned}$$

Lemma 2.1(2), $(H5)$, and Fatou's lemma deduce that

$$\int_{v\neq 0} \frac{F(x, G^{-1}(tv))}{(G^{-1}(tv))^2} \cdot \frac{(G^{-1}(tv))^2}{(tv)^2}v^2 dx \to \infty \;\; as \;\; t \to \infty,$$

which infers that $I(t) \to -\infty$ as $t \to \infty$. Then $I$ has a positive maximum.

$I'(t) = 0$ is equivalent to

$$\int_{\mathbb{R}^N} |\nabla v|^2 dx = \int_{v\neq 0} \left[ \frac{f(x, G^{-1}(tv))}{(g(G^{-1}(tv))} \frac{1}{tv} - V(x) \frac{G^{-1}(tv)}{g(G^{-1}(tv))} \frac{1}{tv} \right] v^2 dx.$$

Set

$$h(s) = \frac{f(x, G^{-1}(s))}{sg(G^{-1}(s))} - \frac{V(x)G^{-1}(s)}{sg(G^{-1}(s))}.$$

Hypothesis $(H4)$ infers that $s \mapsto h(s)$ is strictly increasing for $s > 0$. Thus there is a unique $t_u > 0$ such that $I'(t_u) = 0$ and the first conclusion derives. The second conclusion can be obtained by the fact that $I'(t) = t^{-1}\langle \psi'(tu), tu\rangle$. □

(1) There exists $r > 0$ such that $c := \inf_{\mathscr{M}} \psi \geq \inf_{S_r} \psi$, where $S_r := \{u \in E : \|u\| = r\}$.

(2) $\|u\|^2 \geq \frac{1}{3}c$ for all $u \in \mathscr{M}$.

*Proof* (1) If this is not true, then for $\forall n \in \mathbb{Z}^+$ there is $v_n \neq 0$ such that $v_n \to 0$ in $H$ and

$$\int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\right) dx \leq \frac{1}{n}\|v_n\|^2.$$

Set $w_n := \frac{v_n}{\|v_n\|}$. Then

$$\int_{\mathbb{R}^N} (|\nabla w_n|^2 + V(x)w_n^2)dx + \int_{\mathbb{R}^N} V(x) \left( \frac{|G^{-1}(v_n)|^2}{v_n^2} - 1 \right) w_n^2 dx \leq \frac{1}{n}.$$

Pass to a subsequence if necessary, then $v_n \to 0$ a.e. in $\mathbb{R}^N$. Since $v_n \to 0$ in $L^2(\mathbb{R}^N)$, for $\forall \epsilon > 0$ the measure $|\{x \in \mathbb{R}^N : |v_n(x)| > \epsilon\}| \to 0$ as $n \to \infty$. Thus from the Hölder inequality

$$\int_{|v_n|>\epsilon} w_n^2 dx \leq |\{x \in \mathbb{R}^N : |w_n(x)| > \epsilon\}|^{\frac{q-2}{q}} \|w_n\|_q^2 \to 0, \quad as\ n \to \infty, \tag{9}$$

where $q = 2^*$ if $N \geq 3$ and $q > 2$ if $N = 1$ or 2. This, combining Lemma 2.1-(1), infers that $\|w_n\| = 1$ and $w_n \to 0$ in $H$, a contradiction. By (3) and Lemma 2.1-(3), we have

$$\begin{aligned}
\int_{\mathbb{R}^N} F(x, G^{-1}(v))dx &\leq \frac{\epsilon}{2} \int_{\mathbb{R}^N} |G^{-1}(v)|^2 dx + \frac{C_\epsilon}{p} \int_{\mathbb{R}^N} |G^{-1}(v)|^p dx \\
&\leq 3\epsilon \int_{\mathbb{R}^N} |v|^2 dx + \frac{6^{\frac{p}{2}}}{p} C_\epsilon \int_{\mathbb{R}^N} |v|^p dx \\
&\leq C_1\epsilon\|v\|^2 + C_2\|v\|^p.
\end{aligned}$$

Letting $\epsilon$ sufficiently small, we derive

$$\psi(v) \geq C_3\|v\|^2 - C_4\|v\|^p$$

and $\inf_{S_r} \psi > 0$ for sufficiently small $r$. The inequality $\inf_{\mathscr{M}} \psi \geq \inf_{S_r} \psi$ is a consequence of Lemma 3.1 since for each $v \in \mathscr{M}$ there exists $\tilde{t} > 0$ such that $\tilde{v} \in S_r(\psi(t_u v) \geq \psi(\tilde{t}v))$.

(2) For $v \in \mathscr{M}$,

$$\begin{aligned}
c &\leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx - \int_{\mathbb{R}^N} F(x, G^{-1}(v))dx \\
&\leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + 3 \int_{\mathbb{R}^N} V(x)|v|^2 dx \\
&\leq 3\|u\|^2.
\end{aligned}$$

Thus the proof is completed. □

*Lemma* 3.2. $\psi$ is coercive on $\mathscr{M}$, i.e., $\psi(v) \to \infty$ as $\|v\| \to \infty,\ v \in \mathscr{M}$.

*Proof* We proceed by contradiction. Let $\{v_n\} \subset \mathscr{M}$ be a sequence such that $\|v_n\| \to \infty$ and $\psi(v_n) \leq d$ for some $d$. Set $z_n := \frac{v_n}{\|v_n\|}$. Then passing to a subsequence if necessary, we have $z_n \rightharpoonup z$ in $E$ and $z_n(x) \to z(x)$ a.e. in $\mathbb{R}^N$. Choose $y_n \in \mathbb{R}^N$ such that

$$\int_{B_1(y_n)} z_n^2 dx = \max_{y\in\mathbb{R}^N} \int_{B_1(y)} z_n^2 dx. \tag{10}$$

Because $\psi$ and $\mathscr{M}$ are invariant with respect to the action of $\mathbb{Z}^N$ given by (4), we can suppose that $\{y_n\}$ is bounded in $\mathbb{R}^N$. If

$$\int_{B_1(y_n)} z_n^2 dx \to 0 \ \ as \ \ n \to \infty, \tag{11}$$

then $z_n \to 0$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2^*$. Employing (3) and Lemma 2.1 we have that $\int_{\mathbb{R}^N} F(x, G^{-1}(sz_n))dx \to 0$ for all $s \in \mathbb{R}$. Then

$$\begin{aligned} d &\geq \psi(v_n) \geq \psi(sz_n) \\ &= \frac{s^2}{2}\int_{\mathbb{R}^N} |\nabla z_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(sz_n)|^2 dx - \int_{\mathbb{R}^N} F(x, G^{-1}(sz_n))dx \\ &\geq \frac{s^2}{2}\int_{\mathbb{R}^N} |\nabla z_n|^2 dx + \frac{s^2}{2}\int_{\mathbb{R}^N} V(x)|z_n|^2 dx - \int_{\mathbb{R}^N} F(x, G^{-1}(sz_n))dx \\ &\to \frac{s^2}{2}. \end{aligned}$$

Choosing sufficiently large $s$ we derive a contradiction. Thus (11) can not hold. Note that $|v_n| \to \infty$ if $z(x) \neq 0$, we have

$$\int_{\mathbb{R}^N} \frac{F(x, G^{-1}(v_n))}{\|v_n\|^2} dx = \int_{\mathbb{R}^N} \frac{F(x, G^{-1}(v_n))}{(G^{-1}(v_n))^2} \frac{(G^{-1}(v_n))^2}{v_n^2} z_n^2 dx \to \infty.$$

Consequently,

$$0 \leq \frac{\Psi(v_n)}{\|v_n\|^2} \leq 3 - \int_{\mathbb{R}^N} \frac{F(x, G^{-1}(v_n))}{\|v_n\|^2} dx \to -\infty,$$

which is a contradiction. The proof is completed. □

*Lemma* 3.3. If $A$ is compact subset of $H \setminus \{0\}$, then there exists $R > 0$ such that $\psi \leq 0$ on $(\mathbb{R}^+ A) \setminus B_R(0)$.

*Proof* Without loss of generality we suppose that $A \subset S$. Proceeding by contradiction, assume that there exists $v_n \in A$ and $w_n = t_n v_n$ such that $\psi(w_n) \geq 0$ and $t_n \to \infty$. Passing to a subsequence if necessary, we may suppose that $v_n \to v \in S$. Because of $|w_n(x)| \to \infty$ if $v(x) \neq 0$, it follows from $(H5)$, Lemma 2.1-(2), and Fatou's lemma that

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{F(x, G^{-1}(w_n))}{t_n^2} dx &= \int_{\mathbb{R}^N} \frac{F(x, G^{-1}(w_n))}{w_n^2} v_n^2 dx \\ &= \int_{\mathbb{R}^N} \frac{F(x, G^{-1}(w_n))}{(G^{-1}(w_n))^2} \frac{(G^{-1}(w_n))^2}{w_n^2} v_n^2 dx \\ &\to \infty. \end{aligned}$$

From Lemma 2.1-(3)

$$\begin{aligned} 0 &\leq \frac{\psi(w_n)}{t_n^2} \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\frac{\int_{\mathbb{R}^N} V(x)|G^{-1}(w_n)|^2 dx}{t_n^2} - \frac{\int_{\mathbb{R}^N} F(x, G^{-1}(w_n)|^2 dx}{t_n^2} \\ &\leq 3 - \frac{\int_{\mathbb{R}^N} F(x, G^{-1}(w_n)|^2 dx}{t_n^2} \\ &\to -\infty, \end{aligned}$$

which is a contradiction. Thus the proof is completed. □

Remember that $S$ is the unit sphere in $H$ and define the mapping $m : S \to \mathscr{M}$ by $m(w) := t_w w$, where $t_w$ is as in Lemma 3.1. We consider the functional $\Psi : S \to \mathbb{R}$ defined by $\Psi(w) := \psi(m(w))$.

*Lemma* 3.4. ([14]) The mapping $m$ is a homeomorphism between $S$ and $\mathscr{M}$, and the inverse of $m$ is given by $m^{-1}(v) = \frac{v}{\|v\|}$.

*Lemma* 3.5. ([14]) The application defined above satisfies:

(i) $\Psi \in C^1(S, \mathbb{R})$ and

$$\langle \Psi'(w), z\rangle = \|m(w)\| \langle \psi'(w), z\rangle \ \ for \ \ all \ \ z \in T_w(S),$$

where $T_w(s)$ denotes the tangent space of $S$ at $w$.

(ii) If $(w_n)$ is a Palais-Smale sequence for $\Psi$ then $m(w_n)$ is a Palais-Smale sequence for $\psi$. If $(v_n) \subset \mathscr{M}$ is a bounded Palais-Smale sequence for $\psi$, then $m^{-1}(v_n)$ is a Palais-Smale sequence for $\Psi$.

(iii) $w \in \mathcal{S}$ is a critical point of $\Psi$ if, and only if $m(w)$ is a (nonzero) critical point of $\psi$. Moreover,

$$\inf_{\mathcal{S}} \Psi = \inf_{\mathcal{M}} \psi.$$

(v) If $\psi$ is even, then so is $\Psi$.

*Lemma* 3.6. ([14]) The mapping $m^{-1}$ defined in Lemma 3.4 is Lipschitz continuous.

For the convenience we give some notations

$$K := \{w \in S : \Psi'(w) = 0\}, \quad K_d := \{w \in K : \Psi(w) = d\}.$$

Choose a subset $\mathcal{F}$ of $K$ such that $\mathcal{F} = -\mathcal{F}$ and each orbit $\mathcal{O}(w) \subset K$ has a unique representative in $\mathcal{F}$. Then we need to prove that the set $\mathcal{F}$ is infinite. We assume that

$$\mathcal{F} \ is \ a \ finite \ set. \tag{12}$$

Since we will prove that Palais-Smale sequences have a certain discreteness property later, we give some preparations.

*Lemma* 3.7. ([14]) $\kappa := \inf\{\|v - w\| : v, \ w \in K, \ \ v \neq w\} > 0$.

*Lemma* 3.8. For each $c > 0$ there exists $\delta > 0$ such that

$$\frac{d}{dv}\left(\frac{G^{-1}(v)}{g(G^{-1}(v))}\right) \geq \sigma > 0 \ \ as \ \ |v| \leq C.$$

*Proof* It follows from Lemma 2.1-(4)

$$\frac{d}{dv}\left(\frac{G^{-1}(v)}{g(G^{-1}(v))}\right) = \frac{1 - \frac{G^{-1}(v)g'(G^{-1}(v))}{g'(G^{-1}(v))}}{g^2(G^{-1}(v))} \geq \frac{1}{g^2(G^{-1}(v))}.$$

Since $|v| \leq C$, we have that $g^2(G^{-1}(v))$ is bounded. Hence, the conclusion is proved. □

*Lemma* 3.9. If $\{v_n^1\}$ and $\{v_n^2\}$ are bounded in $H$, then there exists $C > 0$, depending only on $\|v_n^1\|$ and $\|v_n^2\|$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla(v_n^1 - v_n^2)|^2 dx + \int_{\mathbb{R}^N} V(x)\left[\frac{G^{-1}(v_n^1)}{g(G^{-1}(v_n^1))} - \frac{G^{-1}(v_n^2)}{g(G^{-1}(v_n^2))}\right](v_n^1 - v_n^2)dx \\
&\geq C\int_{\mathbb{R}^N} (|\nabla(v_n^1 - v_n^2)|^2 + V(x)(v_n^1 - v_n^2)^2)dx \\
&= C\|v_n^1 - v_n^2\|.
\end{aligned}$$

*Proof* We assume that $v_n^1 \neq v_n^2$, otherwise the result is trivial. Let

$$z_n := \frac{v_n^1 - v_n^2}{\|v_n^1 - v_n^2\|} \ \ and \ \ h_n := \frac{\frac{G^{-1}(v_n^1)}{g(G^{-1}(v_n^1))} - \frac{G^{-1}(v_n^2)}{g(G^{-1}(v_n^2))}}{v_n^1 - v_n^2}.$$

We proceed by contradiction and suppose that $v_n^1$ and $v_n^2$ satisfy

$$\int_{\mathbb{R}^N} \left(|\nabla z_n|^2 + V(x)h_n(x)z_n^2\right) dx \to 0. \tag{13}$$

From a direct computation, we derive $h(x) = \frac{G^{-1}(v)}{g(G^{-1}(v))}$ is an increasing function. Then $h_n(x)$ is positive if $z_n(x) \neq 0$. Thus

$$\int_{\mathbb{R}^N} |\nabla z_n|^2 dx \to 0 \ \ and \ \ \int_{\mathbb{R}^N} V(x)z_n^2 dx \to 1. \tag{14}$$

For a given $c > 0$, set $\Omega_n := \{x \in \mathbb{R}^N : |v_n^1| \geq c \ \ or \ \ |v_n^2| \geq c\}$, $Z_n := \mathbb{R}^N \setminus \Omega_n$. Hence, for any $\epsilon > 0$, $c$ may be chosen such that $|\Omega_n| \leq \epsilon$. By Lemma 3.8, (13) and the Mean value theorem, one has

$$\sigma\int_{Z_n} V(x)z_n^2 dx \leq \int_{Z_n} V(x)h_n(x)z_n^2 dx \to 0. \tag{15}$$

Letting $\epsilon$ sufficiently small and arguing as in (9)(with the same $q$) we derive

$$\int_{\Omega_n} V(x)z_n^2dx \le C_5\epsilon^{\frac{q-2}{q}} \le \frac{1}{2},$$

where $C_5$ is independent of $\epsilon$. This together with (15) contradict (14). □

*Lemma* 3.10. If $\{v_n^1\}$ and $\{v_n^2\}$ are bounded in $H$, then for any $\epsilon > 0$ there exists $C_\epsilon > 0$, depending only on the bound of $\|v_n^1\|$ and $\|v_n^2\|$ such that

$$\left|\int_{\mathbb{R}^N} l_n(x)(v_n^1 - v_n^2)\right| dx \le \epsilon\|v_n^1 - v_n^2\| + C_\epsilon|v_n^1 - v_n^2|_p,$$

where

$$l_n(x) := \frac{f(x, G^{-1}(v_n^1))}{g(G^{-1}(v_n^1))} - \frac{f(x, G^{-1}(v_n^2))}{g(G^{-1}(v_n^2))} = l_n^1(x) - l_n^2(x).$$

*Proof* From (3) and Lemma 2.1 we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^N} l_n^1(v_n^1 - v_n^2)dx\right| &\le \int_{\mathbb{R}^N} \frac{\epsilon|G^{-1}(v_n^1)| + C_\epsilon|G^{-1}(v_n^2)|^{p-1}}{g(|G^{-1}(v_n^1)|)}|v_n^1 - v_n^2|dx \\ &\le \int_{\mathbb{R}^N}(C_6\epsilon|v_n^1||v_n^1 - v_n^2| + C_7C_\epsilon|v_n^2|^{p-1}|v_n^1 - v_n^2|)dx \\ &\le C_6\epsilon|v_n^1|_2|v_n^1 - v_n^2|_2 + C_7C_\epsilon|v_n^2|_p^{p-1}|v_n^1 - v_n^2|_p \\ &\le C\epsilon\|v_n^1 - v_n^2\| + CC_\epsilon|v_n^1 - v_n^2|_p.\end{aligned}$$

Using the same way we have

$$\left|\int_{\mathbb{R}^N} l_n^2(v_n^1 - v_n^2)dx\right| \le C\epsilon\|v_n^1 - v_n^2\| + CC_\epsilon|v_n^1 - v_n^2|_p,$$

Ehere $C$ only depends on the bound of $\|v_n^1\|$ and $\|v_n^2\|$ but is independent of $\epsilon$ and the choice of $v_n^1$ and $v_n^2$, we can replace $C\epsilon$ by $\epsilon/2$ and $CC_\epsilon$ by $C_\epsilon/2$. Thus the proof is completed. □

*Lemma* 3.11. Let $d \ge c$. If $\{z_n^1\}$, $\{z_n^2\} \subset \Psi^d$ are two (PS) sequences for $\Psi$, then either $\|z_n^1 - z_n^2\| \to 0$ as $n \to \infty$ or $\limsup_{n\to\infty}\|z_n^1 - z_n^2\| \ge \gamma(d) > 0$, where $\gamma(d)$ depends on $d$ but not on the particular choice of (PS) sequences.

*Proof* Let $v_n^1 := m(z_n^1)$ and $v_n^2 := m(z_n^2)$. Then $\{v_n^1\}$ and $\{v_n^2\}$ are (PS) sequences for $\Psi$ and these sequences are bounded as $\{v_n^1\}$, $\{v_n^2\} \subset \Psi^d$. We discuss two cases.

*case 1.* $|v_n^1 - v_n^2|_p \to 0$ as $n \to \infty$. Lemmas 3.9 and 3.10 mean that for any $\epsilon > 0$ and sufficiently large $n$,

$$\begin{aligned}C\|v_n^1 - v_n^2\| &\le \int_{\mathbb{R}^N}(|\nabla(v_n^1 - v_n^2)|^2 + V(x)h_n(x)(v_n^1 - v_n^2)^2)dx \\ &= \langle\Psi'(v_n^1), v_n^1 - v_n^2\rangle - \langle\Psi'(v_n^2), v_n^1 - v_n^2\rangle + \int_{\mathbb{R}^N} l_n(v_n^1 - v_n^2)dx \\ &\le 2\epsilon\|v_n^1 - v_n^2\| + C_\epsilon|v_n^1 - v_n^2|_p,\end{aligned}$$

where $h_n$ and $l_n$ are defined in Lemmas 3.9 and 3.10.

Consequently, $\|v_n^1 - v_n^2\| \to 0$ and Lemma 3.6 means that $\|z_n^1 - z_n^2\| = \|m^{-1}(v_n^1) - m^{-1}(v_n^2)\| \to 0$.

*case 2.* $|v_n^1 - v_n^2| \not\to 0$ as $n \to \infty$. From Lemma 1.21 in [16] there exist $\epsilon > 0$ and $y_n \in \mathbb{R}^N$ such that

$$\int_{B_1(y_n)}(v_n^1 - v_n^2)^2dx = \max_{y\in\mathbb{R}^N}\int_{B_1(y)}(v_n^1 - v_n^2)^2dx \ge \epsilon \ \ for \ \ all \ \ n \tag{16}$$

after passing to a subsequence if necessary. Because $m$, $m^{-1}$, $\Psi'$, and $\psi'$ are equivalent with respect to the action of $\mathbb{Z}^n$ defined by (4), we may suppose that the sequence $\{y_n\}$ is bounded in $\mathbb{R}^N$. Passing to a subsequence if necessary there exist $v^1$, $v^2$, $a^1$ and $a^2$ such that

$$v_n^1 \rightharpoonup v^1, \ \ v_n^2 \rightharpoonup v^2, \ \ \|v_n^1\| \to a^1, \ \ \|v_n^1\| \to a^2, \ \ and \ \ \Psi'(v^1) = \Psi'(v^2) = 0.$$

By virtue of (16) we have $v^1 \ne v^2$. This together with Lemma 3 deduce that

$$\frac{1}{\sqrt{3c}} \le a^i \le \nu(d) < \infty,$$

where $\nu(d) = \sup\{\|v\| : v \in \Psi^d \cap \mathscr{M}\}$ and $i = 1, 2$ ($\nu(d) < \infty$ is a consequence of Lemma 3.2). Assume that $v^1, v^2 \neq 0$. Then $v^1,\ v^2 \in \mathscr{M}$ and $z^1 := m^{-1}(v^1) \in K$, $z^2 := m^{-1}(v^2) \in K$ , $v^1 \neq v^2$. Thus

$$\liminf_{n\to\infty} \|z_n^1 - z_n^2\| = \liminf_{n\to\infty} \left\| \frac{v^1}{\|v_n^1\|} - \frac{v^2}{\|v_n^2\|} \right\| \geq \left\| \frac{v^1}{a^1} - \frac{v^2}{a^2} \right\| = \|z^1 - z^2\|,$$

where $b_1 := \frac{\|v^1\|}{a^1} \geq \frac{\sqrt{\frac{c}{3}}}{\nu(d)}$ and $b_2 := \frac{\|v^2\|}{a^2} \geq \frac{\sqrt{\frac{c}{3}}}{\nu(d)}$.

As $\|z^1\| = \|z^2\| = 1$, it is obvious that from the above inequalities

$$\liminf_{n\to\infty} \|z_n^1 - z_n^2\| \geq \|b_1 z^1 - b_2 z^2\| \geq \min\{b_1, b_2\} \|z^1 - z^2\| \geq \frac{\kappa\sqrt{\frac{c}{3}}}{\nu(d)}, \tag{17}$$

where $\kappa$ is the constant in Lemma 3.7. Therefore (17) means that $\liminf_{n\to\infty} \|z^1 - z^2\| \geq \gamma(d) > 0$, where $\gamma(d)$ depends only on $d$(via $\nu(d)$).

If $v^2 = 0$, then $v^1 \neq 0$ and

$$\liminf_{n\to\infty} \|z_n^1 - z_n^2\| = \liminf_{n\to\infty} \left\| \frac{v_n^1}{\|v_n^1\|} - \frac{v_n^2}{\|v_n^2\|} \right\| \geq \frac{\|v^1\|}{a^1} \geq \frac{\kappa\sqrt{\frac{c}{3}}}{\nu(d)}$$

The case $v^1 = 0$ is similar. □

As is well known that $\Psi$ has a pseudo-gradient vector field $L : S \setminus K \to TS$. Furthermore, as $\Psi$ is even, we may assume that $L$ is odd. Set $\eta : G \to S \setminus K$ be the flow given by

$$\begin{cases} \dfrac{d}{dt}\eta(t, w) = -L(\eta(t, w)), \\ \eta(0, w) = w, \end{cases} \tag{18}$$

where $G_S := \{(t, w) : w \in S \setminus K, T^-(w) < t < T^+(w)\}$ and $(T^-(w), T^+(w))$ is the maximal existence time for the trajectory $t \mapsto \eta(t, w)$. Recall that $\eta$ is odd in $w$ from $L$ and $\Psi(\eta(t, w))$ is strictly decreasing by the properties of a pseudogradient.

Set $P \subset S,\ \delta > 0$ and define

$$U_\delta(P) := \{w \in S : dist(w, P) < \delta\}.$$

We now give some properties of $\Psi$ and $\eta$, which comes from [14], will be used in the proof of Theorem 1.1.

*Lemma* 3.12. Let $d \geq c$. Then for all $\delta > 0$ there is $\epsilon = \epsilon(\delta) > 0$ such that

(i) $\Psi_{d-\epsilon}^{d+\epsilon} \cap K = K_d$,

(ii) $\lim_{t\to T^+(w)} \Psi(\eta(t, w)) < d - \epsilon$ for $w \in \Psi^{d+\epsilon} \setminus U_\delta(K_d)$.

*Proof of Theorem 1.1* Set

$$\Sigma := \{\Omega \subset S : \Omega = \bar{\Omega},\ \ \Omega = -\Omega\}.$$

Note that for $\Omega \subset \Sigma$, the Krasnoselskii genus $\gamma(\Omega)$ is the smallest integer $k$ from [17, 18]. Then there exists an old mapping $\Omega \to \mathbb{R}^k \setminus \{0\}$. If there is no such mapping for any $k$, then $\gamma(\Omega) := +\infty$. Furthermore, $\gamma(\emptyset) := 0$. Define

$$C_k := \inf\{d \in \mathbb{R} : \gamma(\Psi^d) \geq k\},\ \ k \geq 1.$$

Hence $C_k$ is the number at which the set $\Psi^d$ changes genus and it is obvious to see that $C_k \leq C_{k+1}$. Set $k \geq 1$ and $d := C_k$. From Lemma 3.7, $K_d$ is either empty or a discrete set, so $\gamma(K_d) = 0$ or 1. Due to the continuity property of the genus there exists $\delta > 0$ such that $\gamma(\bar{u}) = \gamma(K_d)$, where $U := U_\delta(K_d)$ and $\delta < \frac{k}{2}$. For such $\delta$, choose $\epsilon > 0$ such that the results of Lemma 3.12 hold. Then for all $w \in \Psi^{d+\epsilon} \setminus U$ there is $t \in [0, T^+(w))$ such that $\Psi(\eta(t, w)) < d - \epsilon$. Set $e = e(w)$ be the infimum of the time for which $\Psi(\eta(t, w)) < d - \epsilon$. Recalling that $d - \epsilon$ is not a critical value of $\Psi$, it is easy to see that $e$ is a continuous mapping by the implicit function theorem. Because $\Psi$ is even, we have $e(-w) = e(w)$. Define a mapping $l : \Psi^{d+\epsilon} \setminus U \to \Psi^{d-\epsilon}$ by setting $l(w) := \eta(ew, w)$. Then $l$ is odd and continuous. Consequently, it follows from properties of the genus and the definition of $c_k$ that

$$\gamma(\Psi^{d+\epsilon}) \leq \gamma(\bar{U}) + \gamma(\Psi^{d-\epsilon}) \leq \gamma(\bar{U}) + k - 1 = \gamma(K_d) + k - 1.$$

If $\gamma(K_d) = 0$, then $\gamma(\Psi^{d+\epsilon}) \leq k - 1$, which contradicts to the definition of $c_k$. So $\gamma(K_d) = 1$ and $K_d \neq 1$ and $K_d \neq \emptyset$. If $c_{k+1} = c_k = d$, then $\gamma(K_d) > 1$. While this is impossible, we must have $c_{k+1} > c_k$ and $K_{c_k} \neq \emptyset$ for all $k \geq 1$, a contrary to (12). Thus the proof is completed. □


# Funding

Research are supported by the National Natural Science Foundation of China (Grant No. 11901126), the Scientific Research fund of Hunan provincial Education Department (Grant No. 20B524) and the Guizhou technology plan project(Grant No. [2020]1Y004).


# Competing Interests

The authors declare that they have no competing interests.

# Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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