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On a Mixed Problem for a Parabolic Type Equation with General form Constant Coefficients Under Inhomogeneous Boundary Conditions

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Abstract: In this work, we study a one-dimensional mixed problem for an inhomogeneous parabolic equation with constant coefficients of general form, under non-local and non-self-conjugate boundary conditions. The considered mixed problem consist of two parts. The first problem is a mixed problem with a regular boundary condition, and the uniqueness of the solution is proved through the deduction operator. Then the existence of a solution to the mixed problem is shown, and an exact formula for the solution is found. A second mixed problem is the time delay in the boundary conditions. Since the spectral problem obtained after the integral transformation is not homogeneous, the considered problem is again divided into two problems. Under the minimum conditions at the initial data, by combining the deduction method and the contour integral method, the existence and uniqueness of the solution to the mixed problem is proved, where an explicit analytic representation for it is obtained.

Keywords: Residual Method, Time Shift, Mixed Problem

1. Introduction

The paper studies a one-dimensional mixed problem for a parabolic type equation with general form constant coefficients under inhomogeneous boundary conditions. It should be especially noted that the boundary conditions have time advance.

In the study conducted by Mammadov Y. A et al., mixed problems were considered for the heat equation which have in boundary conditions time advance of a more general form [1]. The existence and uniqueness of the considered problem is proved and the solution is presented in the form of a contour integral.

The mixed problems for the heat equation under inhomogeneous boundary conditions, which have time advance in the boundary conditions also considered in Ref. [2]. Under certain specific conditions on the data, the unique solvability of the problem under consideration is proved and an explicit analytical representation for the solution is obtained. The mixed problems for the heat equation for a partially determined boundary regime are studied in work [3]. Under certain specific conditions on the data, the existence of the solution is proved.

Should be noted that a problem for a parabolic-hyperbolic equation with heat conduction operators and strings in a rectangular domain, and the problem with the nonlocal Samorskii-Ionkin boundary condition was studied in this study [4]. A criterion for the uniqueness of solutions is established by the method of spectral expansions.

A boundary value problem for a mixed-type equation with the Lavrentev-Bitsadze operator with advanced retarded arguments and a closed change line in the main part was studied in the research [5]. A uniqueness theorem is proved under a restriction on the value of deviation of the arguments, and an explicit integral representation is found for its solutions.

We can note some more results that have general and qualitative questions of problems for partial differential equations, including parabolic ones with deviating arguments and for differential-difference leading-retarded equations, uniqueness and existence theorems have been proved [6-9].

2. Problem Solution

Let

$$L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)u(x,t)=u_t-au_{xx}(x,t)-bu_x(x,t)-cu(x,t)$$

$$l_ju(x,t)=u(x,t+(1-j)\omega)+\alpha_ju(1-x,t+j\omega),(j=0,1)$$

$$l_ju(x,t)=a_{j-2}u_x^{(j-2)}(x,t)+b_{j-2}u_x^{(j-2)}(1-x,t),(j=2,3)$$

where $a,b,c,\omega,\alpha_j,a_j,b_j$ $(j=0,1)$ are real constants, $a>0$, $\omega>0$, $\alpha_0\alpha_1\neq 0$.

In the half-strip $\Pi=\{(x,t):0<x<1,t>0\}$ we consider the mixed problem

$$L\,u(x,t)=f(x,t),(x,t)\in\Pi, \tag{1}$$

$$u(x,0)=\varphi(x),0<x<1, \tag{2}$$

$$l_ju(x,t)\big|_{x=0}=\psi_j(t),t>0,j=0,1, \tag{3}$$

$$l_ju(x,t)\big|_{x=0}=\psi_j(t),0<t\leq\omega,j=2,3, \tag{4}$$

where $f(x,t),\varphi(x),\psi_k(t)$ $(k=0,1,2,3)$ are known functions, while $u(x,t)$ is a desired function.

By the solution of problem (1)-(4) we mean the function $u(x,t)$ which

1) $u(x,t)\in C^{2,1}(\Pi)\cap C(0<x<1,t\geq 0)$; $\int_0^t u(x,\tau)d\tau\in C(0\leq x\leq 1,t\geq 0)$,

2) $l_ju(x,t)\in C(0\leq x\leq 1,t>0),(j=0,1)$;

3) $l_ju(x,t)\in C(0\leq x<1,0\leq t\leq\omega),(j=2,3)$;

4) $u(x,t)$ satisfies equalities (1)-(4) in the usual sense.

3. Uniqueness of the Solution

It is known [10] that with the help of the substitution

$$\vartheta(x,t)=\vartheta_0(t)+x\vartheta_1(t) \tag{5}$$

boundary conditions (4) can be reduced to a homogeneous boundary condition. Indeed, substituting (5) into (4), one easily obtains

$$\vartheta_0(t)=\frac{\psi_2(t)}{a_0+b_0}-\frac{b_0\psi_3(t)}{(a_0+b_0)(a_1+b_1)}, \tag{6}$$

$$\vartheta_1(t)=\frac{\psi_3(t)}{a_1+b_1}, \tag{7}$$

$$\vartheta(x,t)=\frac{\psi_2(t)}{a_0+b_0}-\frac{b_0\psi_3(t)}{(a_0+b_0)(a_1+b_1)}+\frac{x\psi_3(t)}{a_1+b_1}. \tag{8}$$

Using the following substitution

$$u(x,t)=\vartheta(x,t)+w(x,t) \tag{9}$$

we obtain mixed problems of the following form

$$w_t=aw_{xx}+bw_x+c\,w+\phi(x,t),0<x<1,0<t\leq\omega \tag{10}$$

$$w(x,0)=\tilde{\varphi}(x),0<x<1, \tag{11}$$

$$l_jw\big|_{x=0}=0,0<t\leq\omega, \tag{12}$$

here, $l_jw(x,t)=a_{j-2}w_x^{(j-2)}(x,t)+b_{j-2}w_x^{(j-2)}(1-x,t),(j=2,3)$,

$$\phi(x,t)=f(x,t)+\frac{b\psi_3(t)}{a_1+b_1}+\frac{c\psi_2(t)}{a_0+b_0}-\frac{bc\psi_3(t)}{(a_0+b_0)(a_1+b_1)}++\frac{cx\psi_3(t)}{a_1+b_1}-\frac{\psi_2'(t)}{a_0+b_0}+\frac{b_0\psi_3'(t)}{(a_0+b_0)(a_1+b_1)}-\frac{x\psi_3'(t)}{a_1+b_1}, \tag{13}$$

$$\tilde{\varphi}(x)=\varphi(x)-\frac{\psi_2(0)}{a_0+b_0}+\frac{b_0\psi_3(0)}{(a_0+b_0)(a_1+b_1)}-\frac{x\psi_3(0)}{a_1+b_1}. \tag{14}$$

According to the general scheme of the residue method, of M. L. Rasulov [11], at first, two problems with a complex parameter μ are compared to the mixed problem (10)–(12).

The Cauchy problem.

$$\frac{dz}{dt} - \mu^2 z = \phi(x,t), 0 < t \le \omega, \tag{15}$$

$$z(x,0) = \tilde{\varphi}(x), 0 < x < 1, \tag{16}$$

B. The spectral problem.

$$L\left(\frac{d}{dx}, \mu^2\right) y(x,\mu) = h(x,\mu), 0 < x < 1, \tag{17}$$

$$l_j y\big|_{x=0} = 0, (j = 2,3). \tag{18}$$

It is known [12] that if $a_0 b_1 + a_1 b_0 \ne 0$, then for all complex values of μ not belonging to the set $S = \{\mu_\nu, \nu = 1,2,\dots\}$ there exists a Green's function $G_1(x,\xi,\mu)$ of the spectral problem B (we will call it the first spectral problem) analytic in μ everywhere, except for the points of the set S, which are its poles and have the asymptotic representation

$$\mu_\nu = \sqrt{a}\,\pi\,\nu\, i + \frac{(-1)^\nu (a_0 a_1 + b_0 b_1)}{2(a_0 b_1 + a_1 b_0)} + O\left(\frac{1}{\nu}\right), \nu \to \infty.$$

Renumbering points from S in ascending order of their moduli, taking into account their multiplicity, we denote $S = \{\mu_\nu, \nu = 1,2,\dots\}$, $|\mu_1| \le |\mu_2| \le \cdots$, μ_ν has multiplicity χ_ν, where $\chi_\nu = 1$ or $\chi_\nu = 2$. It is clear that $|\mu_\nu| \to \infty$ $(\nu \to \infty)$, there exist such $h > 0$, $\delta > 0$ that

$$-h < \operatorname{Re}\mu_\nu < h, |\mu_{\nu+1} - \mu_\nu| > 2\delta \ (\nu = 1,2,\dots). \tag{19}$$

Outside the δ -neighborhoods of the points μ_ν, the following estimates hold:

$$\left|\frac{\partial^k G_1(x,\xi,\mu)}{\partial x^k}\right| \le c|\mu|^{k-1}, c > 0, k = 0,1,2. \tag{20}$$

For any function $f(x)$ from the domain of definition of the operator of the first spectral problem, we have the following representation

$$\int_0^1 G_1(x,\xi,\mu) f(\xi) d\xi = \frac{f(x)}{\mu^2} + \frac{c}{\mu^2}\int_0^1 G_1(x,\xi,\mu) f(\xi) d\xi + \frac{1}{\mu^2}\int_0^1 G_1(x,\xi,\mu)(af''(\xi) + bf'(\xi))d\xi. \tag{21}$$

Now let's take a number of notations that we will use later: let $c > 0, r > 0$ be some numbers z be a complex variable, $L_c = \{z: \operatorname{Re} z^2 = c\}$ be a hyperbola with branches $L_c^\pm = \{z: \operatorname{Re} z^2 = c, \pm\operatorname{Re} z > 0\}$, $\Omega_r = \{z: |z| = r\}$, $\Omega_r(\theta_1,\theta_2)$ be arc of the circle Ω_r, enclosed between the rays $z = \sigma\, e^{i\theta j}$ $(0 \le \sigma < \infty, i = \sqrt{-1}, j = 1,2)$.

Note that the arcs $\{z: |z| = r, \operatorname{Re} z^2 \ge c, \operatorname{Re} z < 0\}$ и $\{z: |z| = r, \operatorname{Re} z^2 \le c, \operatorname{Im} z < 0\}$ connecting the branches and sides of the hyperbola L_c in our notation will be, respectively

$$\Omega_r(-\theta_{c,r}, \theta_{c,r}), \Omega_r(\theta_{c,r}, -\theta_{c,r} + \pi), \Omega_r(-\theta_{c,r} + \pi, \theta_{c,r} + \pi), \Omega(\theta_{c,r} + \pi, -\theta_{c,r} + 2\pi),$$

where $\theta_{c,r} = \operatorname{arctg}\sqrt{\frac{r^2 - c}{r^2 + c}}$.

Let's introduce the contours

$$\hat{L}_c = \hat{L}_c^+ \cup \hat{L}_c^-,$$

$$\hat{L}_c^\pm = \left\{z: \pm z = \sigma\, e^{-\frac{3\pi}{8}i}, \sigma \in \left(2c\sqrt{1+\sqrt{2}}, \infty\right]\right\} \cup$$

$$\left\{z: \pm z = c(1 + i\eta, \eta \in \left[-1-\sqrt{2}, 1+\sqrt{2}\right]\right\} \cup$$

$$\left\{z: \pm z = \sigma\, e^{\frac{3\pi}{8}i}, \sigma \in \left[2c\sqrt{1+\sqrt{2}}, \infty\right]\right\}.$$

Part of the contours L_c, $L_c^\pm$, $\hat{L}_c$, $\hat{L}_c^\pm$, enclosed inside the circle Ω_r, will be denote by $L_{c,r}$, $L_{c,r}^\pm$, $\hat{L}_{c,r}$, $\hat{L}_{c,r}^\pm$ respectively. Finally, we denote by $\Gamma_{c,r}$, $\Gamma_{c,r}^\pm$, $\hat{\Gamma}_{c,r}$, $\hat{\Gamma}_{c,r}^\pm$ as $r \ge 2c\sqrt{1 + \frac{\sqrt{2}}{2}}$ the closed loops

$$\Gamma_{c,r} = \Omega_r(\theta_{c,r} + \pi, -\theta_{c,r} + 2\pi) \cup L_{c,r}^+ \cup \Omega_r(\theta_{c,r}, -\theta_{c,r} + \pi) \cup L_{c,r}^-,$$

$$\Gamma_{c,r}^{+} = L_{c,r}^{+} \cup \left(-\theta_{c,r},\ \theta_{c,r}\right), \hat{\Gamma}_{c,r}^{+} = \hat{L}_{c,r}^{+} \cup \Omega_r\left(-\frac{3\pi}{8},\frac{3\pi}{8}\right),$$

$$\hat{\Gamma}_{c,r} = \Omega_r\left(-\frac{5\pi}{8},-\frac{3\pi}{8}\right) \cup \hat{L}_{c,r}^{+} \cup \Omega_r\left(\frac{3\pi}{8},\frac{5\pi}{8}\right) \cup \hat{L}_{c,r}^{-}.$$

The positive direction on all these and subsequent lines will be considered the direction counterclockwise.

Let $\{r_n\}$ - be a sequence of numbers such that

$$0 < r_1 < r_2 < \cdots < r_n < \cdots, \lim_{n\to\infty} r_n = \infty,$$

the circles Ω_{r_n} do not intersect δ- neighborhoods (δ is sufficiently small, fixed) of points $\mu_\nu \in S$. In view of the structure of S the existence of such a number δ and such a sequence $\{r_n\}$ is beyond doubt. The number of points μ_ν lying inside $\hat{\Gamma}_{c,r_n}$ from the domain of definition we denote by m_n. It can be seen from (21) that for any function $f(x)$ from the domain of definition of the operators, of problem B, i.e. $f(x) \in c^2[0,1]$, $l_j f\big|_{x=0} = 0$ $(j = 2,3)$ the following representation holds:

$$\frac{1}{2\pi i}\int_{\hat{\Gamma}_{h,r_n}} \mu d\mu \int_0^1 G_1(x,\xi,\mu) f(\xi) d\xi = \frac{1}{2\pi i}\int_{\hat{\Gamma}_{h,r_n}} \mu^{-1} f(x) d\mu + +\frac{1}{2\pi i}\int_{\hat{\Gamma}_{h,r_n}} \mu^{-1} d\mu \int_0^1 G_1(x,\xi,\mu) f(\xi) d\xi + \frac{1}{2\pi i}\int_{\hat{\Gamma}_{h,r_n}} \mu^{-1} d\mu \times$$

$$\times \int_0^1 G_1(x,\xi,\mu)[af''(\xi) + bf'(\xi) + cf(\xi)]d\xi,$$

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_{\hat{\Gamma}_{h,r_n}} \mu^{-1} f(x) dx = \frac{f(x)}{2\pi i}\lim_{n\to\infty}\int_{\Omega_{r_n}} \frac{d\mu}{\mu} = \frac{f(x)}{2\pi i}\lim_{n\to\infty}\int_0^{2\pi} \frac{i r_n e^{i\varphi} d\varphi}{r_n e^{i\varphi}} = \frac{f(x)}{2\pi i}\cdot 2\pi i = f(x),$$

$$\lim_{n\to\infty}\int_{\hat{\Gamma}_{h,r_n}} \mu^{-1} d\mu \int_0^1 G_1(x,\xi,\mu) f(\xi) d\xi = \lim_{n\to\infty}\int_{\Omega_{r_n}} \mu^{-1} d\mu \int_0^1 G_1(x,\xi,\mu) f(\xi) d\xi,$$

$$\left|\int_{\Omega_{r_n}} \mu^{-1} d\mu \int_0^1 G_1(x,\xi,\mu) f(\xi) d\xi\right| \le \int_0^{2\pi}\left|\frac{d\mu}{\mu}\right|\cdot\frac{c}{|\mu|}\cdot c_0 \le c_1\int_0^{2\pi}\left|\frac{d\mu}{\mu^2}\right| = c_1\int_0^{2\pi}\left|\frac{i r_n e^{i\varphi} d\varphi}{r_n^2 e^{2i\varphi}}\right| = c_1\int_0^{2\pi}\frac{1}{r_n} d\varphi \to 0\ n \to\ \infty.$$

In a similar way, it is proved that

$$\lim_{n\to\infty}\int_{\hat{\Gamma}_{h,r_n}} \mu^{-1} d\mu \int_0^1 G_1(x,\xi,\mu)(af''(\xi) + bf'(\xi)) d\xi = 0,$$

consequently

$$f(x) = \lim_{n\to\infty}\frac{1}{2\pi i}\int_{\hat{\Gamma}_{h,r_n}} \mu d\mu \int_0^1 G_1(x,\xi,\mu) f(\xi) d\xi\ = \sum_{\nu=1}^{\infty} \operatorname{res}_{\mu\nu} \mu \int_0^1 G_1(x,\xi,\mu) f(\xi) d\xi \tag{22}$$

uniformly in $x \in [0,1]$.

The solution to problem A has the form:

$$z(x,t,\mu) = \tilde{\varphi}(x) e^{\mu^{2t}} + \int_0^t e^{\mu^2(t-\tau)} \phi(x,\tau) d\tau. \tag{23}$$

Using the scheme of the residue method [11] for an ordinary problem (a problem without time deviation) (10)-(12), it is proved that if it has a solution, then this solution is represented by the formula

$$w(x,t) = \sum_{\nu=1}^{\infty} \operatorname{res}_{\mu_\nu} \mu \int_0^1 G_1(x,\xi,\mu) z(\xi,t,\mu) d\xi, \tag{24}$$

and by transforming and justifying this formula allowing for (9) for problem (1), (2), (4) (in the rectangle $0 < x < 1, 0 < t \le \omega$) we prove

Theorem 1. Let $a_0 b_1 + a_1 b_0 \ne 0$, $\varphi(x) \in c^2[0,1]$ и $l_j\varphi\big|_{x=0} = 0$ $(j = 2,3)$, $f(x,t) \in C^{2,0}([0,1]\times[0,\omega])$, $l_j f\big|_{x=0} = 0$, $(j = 2,3)$, $\psi_k(t) \in C^1[0,\omega]$ $(k = 2,3)$ and $(a_0 + b_0)(a_1 + b_1) \ne 0$. Then problem (1), (2), (4) has a solution and it is represented by the formula

$$u(x,t) = \frac{\psi_2(t)}{\alpha_0+\beta_0} - \frac{\beta_0\psi_3(t)}{(\alpha_0+\beta_0)(\alpha_1+\beta_1)} + \frac{x\psi_3(t)}{\alpha_1+\beta_1} + +\sum_{\nu=1}^{\infty} \operatorname{res}_{\mu_\nu} \mu\ e^{\mu^2 t}\left[\int_0^1 G_1(x,\xi,\mu)\left(\varphi(\xi) - \frac{\psi_2(0)}{a_0+b_0} + \frac{b_0\psi_3(0)}{(a_0+b_0)(a_1+b_1)} -\right.\right.$$

$$\left.\left. - \frac{\xi\psi_3(0)}{a_1+b_1}\right) d\xi + \int_0^t e^{-\mu^2\tau} d\tau \int_0^1 G_1(x,\xi,\mu)\phi(\xi,\tau) d\xi\right], \tag{25}$$

for $0 \le x \le 1, 0 \le t \le \omega$.

Formula (25) allows us to draw some more conclusions that we will need in what follows. Under the conditions of Theorem 1, from (25), allowing for (21), we have

$$\sum_{\nu=1}^{\infty} \operatorname{res}_{\mu_\nu} \mu\, e^{\mu^2 t} \int_0^1 G_1(x,\xi,\mu)\varphi(\xi)d\xi = \lim_{n\to\infty} \frac{1}{2\pi i} \times \int_{\hat{\Gamma}_{h,r_n}} \mu e^{\mu^2 t} d\mu \int_0^1 G_1(x,\xi,\mu)\varphi(\xi)d\xi = \varphi(x) + \lim_{n\to\infty} \frac{c}{2\pi i}$$

$$\times \int_{\hat{\Gamma}_{h,r_n}} \mu^{-1} e^{\mu^2 t} d\mu \int_0^1 G_1(x,\xi,\mu)\varphi(\xi)d\xi + \lim_{n\to\infty} \frac{1}{2\pi i} \times \int_{\hat{\Gamma}_{h,r_n}} \mu^{-1} e^{\mu^2 t} \int_0^1 G_1(x,\xi,\mu)[a\varphi''(\xi) + b\varphi'(\xi)]d\xi d\mu$$

from estimate (20) and the fact that on the arcs $\Omega_{r_n}\left(-\frac{5\pi}{8} + j\pi, -\frac{3\pi}{8} + j\pi\right)$ $(j = 0,1)$ $\operatorname{Re}\mu^2 \le -\frac{\sqrt{2}}{2}|\mu|^2$ it follows

$$\lim_{n\to\infty} \int_{\Omega_{r_n}\left(-\frac{5\pi}{8}+j\pi, -\frac{3\pi}{8}+j\pi\right)} \mu^{-1} e^{\mu^2 t} d\mu \int_0^1 G_1(x,\xi,\mu)[a\varphi''(\xi) + b\varphi'(\xi) + c\varphi(\xi)]d\xi = 0, (j = 0,1),$$

consequently,

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{\hat{\Gamma}_{h,r_n}} \mu e^{\mu^2 t} d\mu \int_0^1 G(x,\xi,\mu)\varphi(\xi)d\xi = \varphi(x) + \frac{1}{2\pi i} \int_{\hat{\mathrm{L}}_h} \mu^{-1} e^{\mu^2 t} d\mu \int_0^1 G_1(x,\xi,\mu)[a\varphi''(\xi) + b\varphi'(\xi) + c\varphi(\xi)]d\xi$$

and finally, using $G_1(x,\xi,\mu) = G_1(x,\xi,-\mu)$, we obtain

$$u(x,t) = \frac{\psi_2(t)}{\alpha_0+\beta_0} - \frac{\beta_0\psi_3(t)}{(\alpha_0+\beta_0)(\alpha_1+\beta_1)} + \frac{x\psi_3(t)}{\alpha_1+\beta_1} + \varphi(x) - \frac{\psi_2(0)}{\alpha_0+b_0} + \frac{b_0\psi_3(0)}{(\alpha_0+b_0)(\alpha_1+b_1)} - \frac{x\psi_3(0)}{a_1+b_1} +$$

$$+\frac{1}{\pi i}\int_{\hat{\mathrm{L}}_h} \mu^{-1} e^{\mu^2 t} d\mu \int_0^1 G_1(x,\xi,\mu)[a\varphi''(\xi) + b\varphi'(\xi) + c\varphi(\xi)]d\xi + \int_0^t e^{-\mu\tau}d\tau \left[\left(f(x,\tau) + \frac{b\psi_3(\tau)}{a_1+b_1} + \frac{c\psi_3(\tau)}{a_1+b_1} - \frac{bc\psi_3(\tau)}{(a_0+b_0)(a_1+b_1)} + \right.\right.$$

$$\left.\left. + \frac{cx\psi_3(\tau)}{a_1+b_1} - \frac{\psi_2'(\tau)}{a_0+b_0} + \frac{b\psi_3'(\tau)}{(a_0+b_0)(a_1+b_1)} - \frac{x\psi_3'(\tau)}{a_1+b_1}\right) + \frac{1}{\pi i}\int_{\hat{\mathrm{L}}_h} \mu^{-1} e^{\mu^2 t} d\mu \int_0^1 G_1(x,\xi,\mu)\left[af_{\xi\xi}''(\xi,t) + bf_\xi'(\xi,t) + cf(\xi,t)\right]d\xi\right], \quad (26)$$

for $0 \le x \le 1, 0 \le t \le \omega$, where h is a number from (19). From this theorem we have the validity of the following statement:

Theorem 2. Under the conditions of Theorem 1, problem (1)-(4) can have at most one solution.

Indeed, if problem (1)-(4) had two solutions $u_1(x,t), u_2(x,t)$, then their difference $v = u_1(x,t) - u_2(x,t)$ would be the solution of homogeneous problem (1)-(4) with $\varphi(x) = 0, \psi_k(t) = 0$ $(k = 0,1,2,3)$, $f(x,t) = 0$, and by the same token of the homogenous problem (1), (2), (4) in $\{(x,t): 0 \le x \le 1, 0 \le t \le \omega\}$.

Then by virtue of (26) $v(x,t) \equiv 0$ for $0 \le x \le 1, 0 \le t \le \omega$, and from conditions (2) it follows that $v(0,t) = v(1,t) = 0$ for $t \ge 0$. In connection with these and condition 1) it is easy to see that the function

$$w(x,t) = \int_0^t v(x,\tau)d\tau$$

is the solution to the homogeneous problem $u_t = au_{xx} + bu_x + c\,u$ $(0 < x < 1, t \ge \omega)$, $v(x,\omega) = 0$ $(0 \le x \le 1)$, where, $v(0,t) = v(1,t) = 0$ continuous in $\{0 \le x \le 1, t \ge \omega\}$. , whence taking into account the maximum principle [13], [14], we conclude that $w(x,t) \equiv 0$, $(0 \le x < 1, t \ge \omega)$ hence, $v(x,t) \equiv 0$ $(0 \le x \le 1, t \ge 0)$.

4. Investigation of the Existence of a Solution to the Main Mixed Problem

Applying the integral transformation $A(f) = \int_0^\infty e^{-\lambda^2 t} f(t)dt$ (see [15]) to equation (1) and boundary condition (3), we obtain the following spectral problem with a complex parameter λ (we will call it the second spectral problem):

$$L\left(\frac{d}{dx}, \lambda^2\right) z(x,\lambda) = F(x,\lambda), \quad (27)$$

$$\begin{cases} e^{\lambda^2\omega} z(0,\lambda) + \alpha_0 z(1,\lambda) = A(\lambda), \\ z(0,\lambda) + \alpha_1 e^{\lambda^2\omega} z(1,\lambda) = B(\lambda), \end{cases} \quad (28)$$

$$L\left(\frac{d}{dx}, \lambda^2\right) z(x,\lambda) = az''(x,\lambda) + bz'(x,\lambda) + (c - \lambda^2)z(x,\lambda),$$

$$F(x,\lambda) = -\varphi(x) - \tilde{f}(x,\lambda), \quad (29)$$

$$\tilde{f}(x,\lambda) = \int_0^\infty e^{-\lambda^2 t} f(x,t)dt, \quad (30)$$

$$z(x,\lambda) = \int_0^\infty e^{-\lambda^2 t} u(x,t)dt, \quad (31)$$

$$A(\lambda) = \tilde{\psi}_1(\lambda) + e^{\lambda^2\omega}\int_0^\omega e^{-\lambda^2 t} u(0,t)dt,$$

$$B(\lambda) = \tilde{\psi}_2(\lambda) + \alpha_1 e^{\lambda^2\omega}\int_0^\omega e^{-\lambda^2 t} u(1,t)dt, \quad (32)$$

$$\tilde{\psi}_k(\lambda) = \int_0^\infty e^{-\lambda^2 t}\psi_k(t)dt, (k = 0,1),$$

$u(s,t)$ $(s=0,1)$ are boundary values of the solution on the parts of the side boundary of the domain $\{(x;t): 0<x<1, t>0\}$ which are determined from (26).

Boundary conditions (28) can be reduced to the following form

$$z(0,\lambda)=m(\lambda),\, z(1,\lambda)=n(\lambda), \tag{33}$$

where

$$m(\lambda)=z_0(\lambda)=\left[\alpha_1 e^{2\lambda^2\omega}-\alpha_0\right]^{-1}\left(A(\lambda)\alpha_1 e^{2\lambda^2\omega}-\alpha_0 B(\lambda)\right)$$

$$n(\lambda)=z_1(\lambda)=\left[\alpha_1 e^{2\lambda^2\omega}-\alpha_0\right]^{-1}\left(B(\lambda)e^{\lambda^2\omega}-A(\lambda)\right). \tag{34}$$

The solution of problem (27), (33) can be represented as the sum of solutions of two problems:

A. $L\left(\frac{d}{dx},\lambda^2\right)z(x,\lambda)=0,\, z(0,\lambda)=m(\lambda),\, z(1,\lambda)=n(\lambda)$,

B. $L\left(\frac{d}{dx},\lambda^2\right)z(x,\lambda)=F(x,\lambda),\, z(0,\lambda)=0,\, z(1,\lambda)=0.$

The solution to problem A is represented by the formula

$$Q(x,\lambda,m,n)=\left[e^{-\left(\frac{b}{2a}+\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)}-e^{\left(\frac{b}{2a}-\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)}\right]^{-1}\times\left[\left(m(\lambda)e^{-\left(\frac{b}{2a}+\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)}-n(\lambda)\right)e^{-\left(\frac{b}{2a}-\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)x}\right.$$
$$\left.+\left(n(\lambda)-m(\lambda)e^{-\left(\frac{b}{2a}-\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)}\right)e^{-\left(\frac{b}{2a}-\frac{\lambda}{\sqrt{a}}+O\left(\frac{1}{\lambda}\right)\right)x}\right], \tag{35}$$

where $m(\lambda)$ and $n(\lambda)$ is determined by formula 1 (34).

If $m=z_0(\lambda)$, $n=z_1(\lambda)$, then the function $Q(x,\lambda,m,n)$, analytic in λ everywhere except for the points $\lambda_\nu=\sqrt{a}\pi\nu\, i+O\left(\frac{1}{\nu}\right)$ $(\nu=0,\pm1,\pm2,...)$ and the points $\lambda_m^{\pm}=\pm\left[\frac{1}{2\omega}\ln\left|\frac{\alpha_0}{\alpha_1}+2\pi\, m\, i\right|\right]^{1/2}$ $(m=0,\pm1,\pm2,...)$ are its poles.

Obviously, at all the points λ, where $Q(x,\lambda,m,n)$ exists, the identities

$$L\left(\frac{d}{dx},\lambda^2\right)Q(x,\lambda,m,n)=0,\, Q(0,\lambda,m,n)=m,\, Q(1,\lambda,m,n)=n \tag{36}$$

are valid.

The solution of problem B is constructed using its Green's function. Denote the Green's function of Problem B by $G_2(x,\xi,\lambda)$ and this function is analytic in λ everywhere except for the points $\lambda_\nu=\sqrt{a}\pi\nu\, i+O\left(\frac{1}{\nu}\right)$ which are its simple poles.

We note some facts [11], [12] about the Green function $G_2(x,\xi,\lambda)$: there exists such $\delta>0$ that on the λ -plane outside the set $\bigcup_{\nu=1}^{\infty}\{\lambda: |\lambda-\lambda_\nu|<\delta\}$ the estimate $\left|\frac{\partial^k G_2(x,\xi,\lambda)}{\partial x^k}\right|\le C_0|\lambda|^{k-1}, C_0>0, k=0,1,2,$ is valid for all $x,\xi\in[0,1]$; for $\lambda\ne\lambda_\nu$ $(\nu=0,\pm1,\pm2,...)$

$$L\left(\frac{d}{dx},\lambda^2\right)\int_0^1 G_2(x,\xi,\lambda)\varphi(\xi)d\xi=-\varphi(x),\, G_2(0,\xi,\lambda)=G_2(1,\xi,\lambda)=0.$$

The solution of the second spectral problem is represented by the sum of two solutions (problem A and problem B)

$$z(x,\lambda)=-\int_0^1 G_2(x,\xi,\lambda)F(\xi,\lambda)d\xi+Q\big(x,\lambda,m(\lambda),n(\lambda)\big). \tag{37}$$

For any function $\varphi(x)$ from the domain of definition of the operator of the second spectral problem, the following equality holds:

$$\int_0^1 G_2(x,\xi,\lambda)\varphi(\xi)d\xi=-\frac{\varphi(x)}{\lambda^2}+\frac{1}{\lambda^2}\int_0^1 G_2(x,\xi,\lambda)[a\varphi''(\xi)+b\varphi'(\xi)+c\varphi(\xi)]d\xi+\frac{Q\big(x,\lambda,\varphi(0),\varphi(1)\big)}{\lambda^2}, \tag{38}$$

Using (38), solution (37) can be represented as

$$z(x,\lambda)=\frac{\varphi(x)}{\lambda^2}-\frac{1}{\lambda^2}\int_0^1 G_2(x,\xi,\lambda)[a\varphi''(\xi)+b\varphi'(\xi)+c\varphi(\xi)]d\xi-\frac{1}{\lambda^2}Q\big(x,\lambda,m(\lambda),n(\lambda)\big), \tag{39}$$

we fix a number $c_1>\max\left(0,\ln\left|\frac{\alpha}{\beta}\right|\right)$.

Theorem 3. Let $a_0b_1+a_1b_0\ne0$, $\varphi(x)\in c^2[0,1]$, $l_j\varphi\big|_{x=0}=0$ $(j=0,1)$; $f(x,t)\in C^{2,0}([0,1]\times[0,\infty])$, $l_jf\big|_{x=0}=0$, $(j=0,1)$, $\psi_k(t)\in C^1[0,\infty]$ $(k=0,1)$ and $(a_0+b_0)(a_1+b_1)\ne0$. Then problem (1)-(4) has a classical solution, and it is represented by the formula

$$u(x,t) = \varphi(x) + \int_0^t f(x,\tau)d\tau - \frac{1}{\pi i}\int_{\hat{L}_{c_1}^+} \lambda^{-1} e^{\lambda^2 t} d\lambda \times \Big[\int_0^1 G_2(x,\xi,\lambda)(a\varphi''(\xi) + b\varphi'(\xi) + c\varphi(\xi) +$$

$$+ \int_0^t e^{-\lambda^2\tau}\left[af''_{\xi\xi}(\xi,\tau) + bf'_{\xi}(\xi,\tau) + cf(\xi,\tau)\right]d\tau\Big)d\xi - Q\big(x,\lambda,\varphi(0),\varphi(1)\big)\Big] + \frac{1}{\pi i}\int_{L_{c_1}^+} \lambda e^{\lambda^2 t} Q\big(x,\lambda,m(\lambda),n(\lambda)\big)d\lambda. \quad (40)$$

The three integrals in (35) are studied in a similar way. For example, consider the second integral

$$u_2(x,t) = \frac{1}{\pi i}\int_{\hat{L}_{c_1}^+} \lambda^{-1} e^{\lambda^2 t} Q\big(x,\lambda,\varphi(0),\varphi(1)\big)d\lambda. \quad (41)$$

On the distant parts of the contour $\hat{L}_{c_1}^+(\operatorname{Re}\lambda > c_1)$

$$\left|e^{\lambda^2 t}\right| = e^{t\operatorname{Re}\lambda^2} = e^{t|\lambda|^2\cos 2\arg\lambda} = e^{t|\lambda|^2\cos\left(\pm\frac{3\pi}{4}\right)} = e^{-\frac{\sqrt{2}}{2}t|\lambda|^2}. \quad (42)$$

Further, from formula (35), the function $Q(x,\lambda,\varphi(0),\varphi(1))$ is analytic in the domain $\operatorname{Re}\lambda > c_1$, and the estimates

$$\left|\frac{\partial^k Q(x,\lambda,\varphi(0),\varphi(1))}{\partial x^k}\right| \le c|\lambda|^k + \frac{c_0}{|\lambda|^k}, (k = 0,1,2) \quad (43)$$

are valid for all $x \in [0,1]$.

On the distant parts of the contour $\hat{L}_{c_1}^+(\operatorname{Re}\lambda > c_1)$ and on the arcs $\Omega_r\left(-\frac{3\pi}{8},\frac{3\pi}{8}\right)$ $(r > 2c_1\sqrt{1+\sqrt{r}}$ we have the estimate

$$|Q(x,\lambda,\varphi(0),\varphi(1))| \le c_1 e^{-\left|\frac{\lambda}{\sqrt{a}}\right|(1-x)\cos\frac{3\pi}{8}} + c_2 e^{-\left|\frac{\lambda}{\sqrt{a}}\right|x\cos\frac{3\pi}{8}} + \frac{c_3}{|\lambda|}. \quad (44)$$

from (41) and (42) it follows

$$u_2(x,t) \in c^{2,1}(0 \le x \le 1, t > 0), \quad (45)$$

in (40) for $t > 0$ the operations $L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)$ as $x \to 0$, $x \to 1$ can be transferred under the integral sign. Then, taking into account (36|), we obtain

$$L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)u_2(x,t) = 0,$$

$$u_2(0,t) = \frac{\varphi(0)}{\pi i}\int_{\hat{L}_{c_1}^+}\lambda^{-1}e^{\lambda^2 t}d\lambda = \frac{\varphi(0)}{2\pi i}\lim_{r\to\infty}\int_{\hat{L}_{c_1,r}^+}\lambda^{-1}e^{\lambda^2 t}d\lambda = \frac{\varphi(0)}{2\pi i}\cdot 2\pi i = \varphi(0)$$

$$u_2(1,t) = \frac{\varphi(1)}{\pi i}\int_{\hat{L}_{c_1}^+}\lambda^{-1}e^{\lambda^2 t}d\lambda = \frac{\varphi(1)}{2\pi i}\lim_{r\to\infty}\int_{\hat{L}_{c_1,r}^+}\lambda^{-1}e^{\lambda^2 t}d\lambda = \frac{\varphi(1)}{2\pi i}\cdot 2\pi i = \varphi(1)$$

From equality (43) it can be seen that for x, belonging to any segment $[x_1,x_2] \in (0,1)$ integral (40) converges uniformly in $t \ge 0$. Then

$$u_2(x,t) \in c(0 < x < 1, t \ge 0), \quad (46)$$

and for $x \in [x_1,x_2]$

$$u_2(x,0) = \frac{1}{\pi i}\int_{\hat{L}_{c_1}^+}\lambda^{-1}Q\big(x,\lambda,\varphi(0),\varphi(1)\big)d\lambda = \frac{1}{\pi i}\lim_{r\to\infty}\Big[\int_{\hat{L}_{c_1,r}^+}\lambda^{-1}Q\big(x,\lambda,\varphi(0),\varphi(1)\big)d\lambda +$$

$$+ \int_{\Omega_r\left(-\frac{3\pi}{8},\frac{3\pi}{8}\right)}\lambda^{-1}Q\big(x,\lambda,\varphi(0),\varphi(1)\big)d\lambda\Big] = 0, \quad (47)$$

by the analyticity of $Q(x,\lambda,\varphi(0),\varphi(1))$ inside the closed contour $\hat{\Gamma}_{c_1,r}^+$.

Combining Theorems 1 and 2, we arrive at the following final statement:

Theorem. Let all the conditions of Theorems 1 and 2 be satisfied. Then problem (1)-(4) has a unique solution represented by formula (40).

5. Conclusion

In this study, we analyzed a one-dimensional mixed problem for an inhomogeneous parabolic equation with constant coefficients in general form under unconventional boundary conditions. A unified method of Rasulov contour integral and the residue method were presented to solve the problem. First, important properties of related spectral problems were analyzed and the results obtained were applied to the mixed problem. Under some algebraic conditions on the given data, the existence and uniqueness of the solution of the mixed problem under study and an explicit analytical notation for it were obtained.

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