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Numerical Simulation of the Heat Equation Using RBF Collocation Method

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Abstract: For a very long time, finite volume, finite element, or finite difference methods have been used to solve partial differential equations (PDEs) numerically. These techniques have been used by researchers for centuries to solve a wide range of mathematical, physical, or chemical problems. The complexity of these numerical approaches, for the resolution of the PDEs in space dimensions equal to two or higher, can come from the coding, the management, and the good choice of the triangulation or the mesh of the domain in which one wishes to locate the solution. The radial basis function collocation method is a meshless technique used to numerically solve some partial differential equations and is based on the nodes of the domain and a radial basis function is a real-valued function whose value only depends on the separation of its input parameter x from another fixed point, sometimes known as the function's origin or center. This method was introduced by KANSA in the 1990s. In this study, the numerical simulation of the one-dimensional heat equation was carried out using the RBF Collocation Method and particularly the Gaussian function. This model was used to test the accuracy and efficiency of this method by comparing numerical and analytical solutions on rectangular geometry with collocation nodes. The results show that the RBF collocation approximate solution and the exact solution coincided in test case problems 2, 3 and 4.

Keywords: RBF Collocation Method, Theta Scheme, RBF, Heat Equation

1. Introduction

Partial differential equations model various real-world problems, then the mathematical model of the heat equation $\frac{\partial u}{\partial t}+K\frac{\partial^2 u}{\partial x^2}=f$, where K is the heat conduction coefficient and f is the source term, is a parabolic partial differential equation introduced in 1811 by the French mathematician and physicist Joseph FOURIER that describes the physical phenomenon of heat conduction [9]. For centuries, researchers have developed various numerical methods to solve PDEs, such as finite volume, finite difference, or finite element methods, which always involve mesh, or an assemblage of elements, that facilitate the interpolation. In a complicated solution geometry, the effort involved in constructing such mesh and its connectivity data, that is, how each node is associated with other nodes in an interpolation scheme, and how each element shares the common nodes with other elements, is not trivial. The RBF methodology was first introduced by Hardy in 1971 in connection with a topological application on quadric surfaces, so he introduced the multiquadric approximation scheme [5]. Edward Kansa in 1990, first use the multiquadric, a globally supported interpolant to solve a PDE known as the Kansa method [6, 7, 15].

There exist several meshless methods which can be found in the literature [12] and also a comparison study between the finite element method, finite difference method, and meshless method [14].

Therefore, we presented the radial basis function collocation methods developed in the research [1, 10, 11], which is a numerical method based on the quasi-interpolation and on the approximation by the basic radial functions to solve the parabolic heat equation. This method does not require a subdivision of the method or a mesh as in the case of the finite difference method, finite element method, or the finite volumes method. The quasi-interpolation method is based on the following principle:

Given the partial differential equation, we first seek to quasi-interpolate the forced termination of the equation by using the basic radial functions. A very exact approximation of the solution can then be obtained by solving the corresponding fundamental.

We solve the system of equations relating to the initial condition or the boundary condition.

Nawaz et al (2020) [2], used the global meshless scheme and RK4 to solve the inverse heat source problem. In this paper, we used the RBF collocation method to solve and model the heat distribution in a rectangular area, and a comparative study was conducted under numerical and exact solutions to test the effectiveness and accuracy of this meshless method and also to check the error made in the resolution [1, 4].

2. Mathematical Model

The problem we are interested in is a one-dimensional heat equation with some Cauchy-Dirichlet boundary conditions given by

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,t) + K\dfrac{\partial u}{\partial t}(x,t) = f(x,t), \forall (x,t) \in \Omega =]0,1[\times]0,T[\\ u(x,0) = u_0(x), \forall x \in [0,1] \\ u(x,t) = g(x,t), \forall (x,t) \in \partial\Omega \times]0,T] \end{cases} \tag{1}$$

Let L and Q be linear operators defined on $\mathcal{C}^\infty(\bar{\Omega},\mathbb{R})$, more precisely, $Qu = u_{|\partial\Omega}$ is the trace of $u \in \mathcal{C}^\infty(\bar{\Omega},\mathbb{R})$ on $\partial\Omega$. u is the solution to the problem (1) and f is a given function of the space $\mathcal{C}^\infty(\bar{\Omega},\mathbb{R})$ called source term.

As u is sought in $\mathcal{C}^\infty(\bar{\Omega},\mathbb{R})$, we have $Q : \mathcal{C}^\infty(\bar{\Omega},\mathbb{R}) \to \mathcal{C}^\infty(\bar{\Omega},\mathbb{R})$, $u \to u_{|\partial\Omega}$. We place ourselves in the case where $L : \mathcal{C}^\infty(\bar{\Omega},\mathbb{R}) \to \mathcal{C}^\infty(\bar{\Omega},\mathbb{R})$, $u \to \dfrac{1}{K}\dfrac{\partial^2 u}{\partial x^2}$ where $K \in \mathbb{R}^*$.

Then the problem (1) can be written as:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = -Lu(x,t) + F(x,t), \forall (x,t) \in \Omega =]0,1[\times]0,T[\\ u(x,0) = u_0(x), \forall x \in [0,1] \\ u(x,t) = g(x,t), \forall (x,t) \in \partial\Omega, \times]0,T] \end{cases} \tag{2}$$

Where $F = \dfrac{1}{K} f \in \mathcal{C}^\infty(\bar{\Omega};\mathbb{R})$

3. Radial Basis Function Collocation Method

Definition 3.1. A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is radial if there exists a one variable function $\varphi : [0,\infty) \to \mathbb{R}$ such that $\phi(\vec{x}) = \varphi(\|\vec{x}\|)$, where $\|\bullet\|$ is the Euclidean norm. We then refer to $\vec{x} \to \varphi(\|\vec{x} - \overrightarrow{x_k}\|_2)$, $\vec{x} \in \mathbb{R}^d, k \in \{1,\ldots,n\}$ as radial basis functions centered at $\overrightarrow{x_k}$.

3.1. Most Commonly Used Radial Basis Function [13]

We denote $r = \|\vec{x} - \overrightarrow{x_k}\|_2$ and c is a shape parameter.

1. Gaussian function: $\rho(r) = \mathrm{e}^{(-cr)^2}, c > 0.$
2. Multiquadric function: $\varphi(r) = \sqrt{1 + (cr^2)}, c > 0.$
3. Inverse quadratic function: $\varphi(r) = \dfrac{1}{1 + (cr)^2}, c > 0.$
4. Inverse multiquadric: $\varphi(r) = \dfrac{1}{\sqrt{1 + (cr)^2}}, c > 0.$
5. Polyharmonic spline function: $\varphi(r) = r^k, k = 1,3,5,\ldots$ $\varphi(r) = r^k \ln(r), k = 2,4,6,\ldots$
6. Thin plate spline function: $\varphi(r) = r^2 \ln(r)$.

RBF collocation methods have been actively developed over the years from global to local approximation and then to hybrid methods. RBF methods have been applied to various diverse fields like image processing, geo-modeling, pricing option and neural network, etc. [6, 8, 15]. The mathematical formulation of different RBF methods is discussed for better understanding.

3.2. The General Algorithm: Collocation with RBFs

We consider the following partial derivatives equations.

$$Lu(x) = f(x), \quad \forall x \in \Omega \subset \mathbb{R}^d, \tag{3}$$

$$Qu(x) = g(x), \quad \forall x \in \partial\Omega \tag{4}$$

Let $(x_j)_{1\leqslant j\leqslant n} \subset \Omega$ et $(x_j)_{n+1\leqslant j\leqslant n+m} \subset \partial\Omega$

The RBF collocation method seeks an approximate solution u^*, of problem (3) and (4) in the form

$$u^*(x) = \sum_{j=1}^{n+m} \lambda_j \phi_j(x), \tag{5}$$

Where $\phi_j(x) = \phi(\| x - x_j \|_2)$, ϕ is the radial function. L and Q are linear operators, substituting (5) into (3) and (4), we obtain

$$\begin{cases} Lu^*(x_i) = \sum_{j=1}^{n+m} \lambda_j L\phi_j(x_i) = f(x_i), & 1 \leqslant i \leqslant n \\ Qu^*(x_i) = \sum_{j=1}^{n+m} \lambda_j Q\phi_j(x_i) = g(x_i), & n+1 \leqslant i \leqslant n+m \end{cases} \tag{6}$$

Therefore, we solve the system of equations of the unknowns $(\lambda_j)_{1\leqslant j\leqslant n+m}$,

$$\begin{cases} \sum_{j=1}^{n+m} \lambda_j L\phi_j(x_i) = f(x_i), & 1 \leqslant i \leqslant n \\ \sum_{j=1}^{n+m} \lambda_j Q\phi_j(x_i) = g(x_i), & n+1 \leqslant i \leqslant n+m \end{cases} \tag{7}$$

and the approximate solution of the problem (3) and (4) is given by (5). For an evolution problem in dimension d, for instance of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} + Lu = f & \text{on} \quad \Omega\times[0,T], \\ Bu = g & \text{on} \quad \partial\Omega, \\ u(x,t_0) = h(x) \end{cases} \tag{8}$$

We use the $\theta-$ scheme to discretize the problem, it's given by

$$\frac{u^{n+1}(x) - u^n(x)}{\Delta t} = \theta(-Lu^{n+1}(x) + f(x,t_{n+1})) + (1-\theta)(-Lu^n(x) + f(x,t_n), \forall (x,t_n) \in \Omega\times[0,T] \tag{9}$$

where $\Delta t = t_{n+1} - t_n$ is the time step, $u^n (n = 0,1,2,...)$ is the solution at the time step $t_n = n\Delta t$ and $\theta \in [0,1]$. For $\theta = \frac{1}{2}$, we have the Crank-Nicolson scheme. In this case, the approximate solution at time t_n by the RBF collocation method is written

$$u^n(x) \simeq u(x,t_n) = \sum_{j=1}^{n+m} \lambda_j(t_n)\phi_j(x). \tag{10}$$

4. Numerical Schemes

We consider problem (2) for numerical resolution, using a mixed θ-scheme and RBF collocation method, in this study, the Crank-Nicolson scheme and the Gaussian basis radial function have been considered.

Let $u^n = (u_i^n)_{0\leqslant i\leqslant M}$ and $F^n = (F_i^n)_{0\leqslant i\leqslant M}$ respectively the approximation of the solution u and the value of the function F at the points (x_i,t_n) with $0 \leqslant i \leqslant M$ and $0 \leqslant n \leqslant N$, with $x_i = ih, 0 \leqslant i \leqslant M$, $t_n = n\Delta t, 0 \leqslant n \leqslant N$, we obtain the following θ-scheme:

$$u_i^{n+1} - \theta\Delta t\Big[-Lu_i^{n+1} + F(x_i,t_{n+1})\Big] = u_i^n + (1-\theta)\Delta t\Big[-Lu_i^n + F(x_i,t_n)\Big]. \tag{11}$$

Suppose that

$$u(x,t)=\sum_{j=0}^{M}\lambda_j(t)\Phi_j(x),\forall(x,t)\in[0,1]\times[0,T]. \quad (12)$$

Where $\Phi_j(x)=\Phi(|x-x_j|),\forall x\in[0,1],\forall j\in\{0,1,...,M\}$$, the radial function Gaussian basis has been chosen in this study [3, 4].

Substituting equation (12) into (11) yields

$$\sum_{j=0}^{M}\lambda_j(t_{n+1})\Phi_j(x_i)-\theta\Delta t\Big[-\sum_{j=0}^{M}\lambda_j(t_{n+1})L\Phi_j(x_i)+F(x_i,t_{n+1})\Big]=\sum_{j=0}^{M}\lambda_j(t_n)\Phi_j(x_i)+(1-\theta)\Delta t\Big[-\sum_{j=0}^{M}\lambda_j(t_n)L\Phi_j(x_i)+F(x_i,t_n)\Big]. \quad (13)$$

Making use of the $\Phi(r)=e^{-cr^2}$, i.e., $\Phi_j(x)=\Phi(|x-x_j|)=e^{-c(x-x_j)^2}$ the equation (13) becomes

$$\begin{aligned}&\sum_{j=0}^{M}\lambda_j^{n+1}\Big(1-\frac{2c\theta\Delta t}{K}\big(1-2c(i-j)^2h^2\big)\Big)e^{-c(i-j)^2h^2}-\frac{\theta\Delta t(1+K)}{K}e^{-ih+(n+1)\Delta t}=\\&\sum_{j=0}^{M}\lambda_j^{n}\Big(1+\frac{2c(1-\theta)\Delta t}{K}\big(1-2c(i-j)^2h^2\big)\Big)e^{-c(i-j)^2h^2}+\frac{(1-\theta)(1+K)\Delta t}{K}e^{-ih+n\Delta t}.\end{aligned} \quad (14)$$

Let us denote by

$$A_{ij}=\Big(1-\frac{2c\theta\Delta t}{K}\big(1-2c(i-j)^2h^2\big)\Big)e^{-c(i-j)^2h^2},0\leqslant i,j\leqslant M, \quad (15)$$

$$B_{ij}=\Big(1+\frac{2c(1-\theta)\Delta t}{K}\big(1-2c(i-j)^2h^2\big)\Big)e^{-c(i-j)^2h^2},0\leqslant i,j\leqslant M, \quad (16)$$

$$\lambda^{n+1}=\big(\lambda_j^{n+1}\big)_{0\leqslant j\leqslant M},\forall n\in\{0,1,...,N\}, \quad (17)$$

$$\eta^{n+1}=(\eta_i^{n+1})_{0\leqslant i\leqslant M}=\Big(\frac{\theta\Delta t(1+K)}{K}e^{-ih+(n+1)\Delta t}\Big)_{0\leqslant i\leqslant M}, \quad 18)$$

$$\rho^{n}=(\rho_i^{n})_{0\leqslant i\leqslant M}=\Big(\frac{(1-\theta)(1+K)\Delta t}{K}e^{-ih+n\Delta t}\Big)_{0\leqslant i\leqslant M}, \quad (19)$$

The equation (13) can be now written in a matrix form

$$A\lambda^{n+1}-\eta^{n+1}=B\lambda^{n}+\rho^{n} \quad (20)$$

This is,

$$\lambda^{n+1}=A^{-1}B\lambda^{n}+A^{-1}\big(\eta^{n+1}+\rho^{n}\big). \quad (21)$$

The obtained results of (21) is substituted into (12) , then,

$$u_*^{n+1}=\Big(\sum_{j=0}^{M}\lambda_j^{n+1}\Phi_j(x_i)\Big)_{0\leqslant i\leqslant M},\forall n\in\{0,1,...,N\}. \quad (22)$$

5. Results and Discussions

The comparative tables had been addressed in this section between the exact and numerical solutions, the parameters M,N,c,K,h and θ are fixed, the root mean square REMS is the error norm in order to overcome with the comparison results given by the following equation

$$RMSE=\sqrt{\sum_{i=0}^{M-1}\Big(\frac{u_*^{n+1}-u^{n+1}(x_i,t_{n+1})}{u^{n+1}(x_i,t_{n+1})}\Big)} \quad (23)$$

The test problem considered in this study are for $f=(1+K)e^{-x+t}$ and $K\neq -1$, the exact solution is given by the following relationships [2]:

$$u(x,t)=e^{-x+t},\forall(x,t)\in]0,1[\times]0,T[$$
$$u(x,0)=u_0(x)=e^{-x},\forall x\in[0,1]$$
$$g(x,t)=e^{-x+t},\forall(x,t)\in\partial\Omega\times]0,T]$$

The justification of the existence and uniqueness of the solution of our problem can be found in the literature related to that research field [3, 4].

Table 1. *For* $M = N = 11, \Delta t = 1, c = 1, K = 6, h = 0.001$ *and* $\theta = 0.5$.

u_*^{n+1}	$u(x_i, t_{n+1})$
32.655313	7.389056
32.658250	7.381671
32.661122	7.374293
32.663928	7.366922
32.666669	7.359559
32.669345	7.352203
32.671955	7.344854
32.674500	7.337513
32.676980	7.330179
32.679394	7.389056

$RMSE = 1.038252$, shows that there is a divergence between the solutions for the case of the Crank-Nicolson scheme, thus the analysis of the result point of view proof that the exact and numerical solutions are totally divergence as shown in the Figure 1.

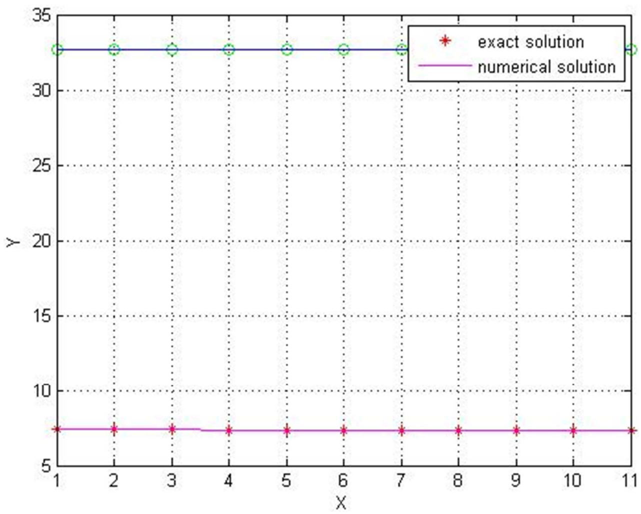

Figure 1. *Numerical Solution Using* $M = N = 11, \Delta t = 1, c = 1, K = 6, h = 0.001$ *and* $\theta = 0.5$.

Table 2. *For* $M = N = 11, \Delta t = 0.05, c = 781, K = 10000, h = 0.1$ *and* $\theta = 0.5$.

u_*^{n+1}	$u(x_i, t_{n+1})$
1.122037	1.105171
1.003573	1.000000
0.908058	0.904837
0.821645	0.818731
0.743455	0.740818
0.672706	0.670320
0.608690	0.606531
0.550765	0.548812
0.498353	0.496585
0.450928	0.449329

$RMSE = 0.001701$, shows that there is a convergence between the solutions for the case θ-scheme, thus it is clear that from (1, 2) the two curves are slightly not coincided from analysis of the result point as shown in the Figure 2.

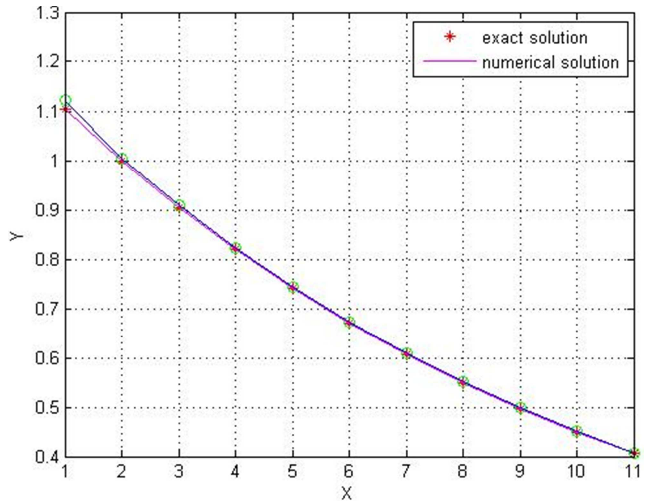

Figure 2. *Numerical Solution Using* $M = N = 11, \Delta t = 0.05, c = 781, K = 10000, h = 0.1$ *and* $\theta = 0.5$.

Table 3. *For* $M = N = 11, \Delta t = 0.05, c = 781, K = 10000, h = 0.1$ *and* $\theta = 0.4$.

u_*^{n+1}	$u(x_i, t_{n+1})$
1.121413	1.105171
1.006045	1.000000
0.910298	0.904837
0.823671	0.818731
0.745289	0.740818
0.674365	0.670320
0.610191	0.606531
0.552123	0.548812
0.499582	0.496585
0.452040	0.449329

$RMSE = 0.002155$, shows that there is a convergence between the solutions for the case θ-scheme, thus it is clear that from (1, 3) the two curves are slightly not coincided from analysis of the result point as shown in the Figure 3.

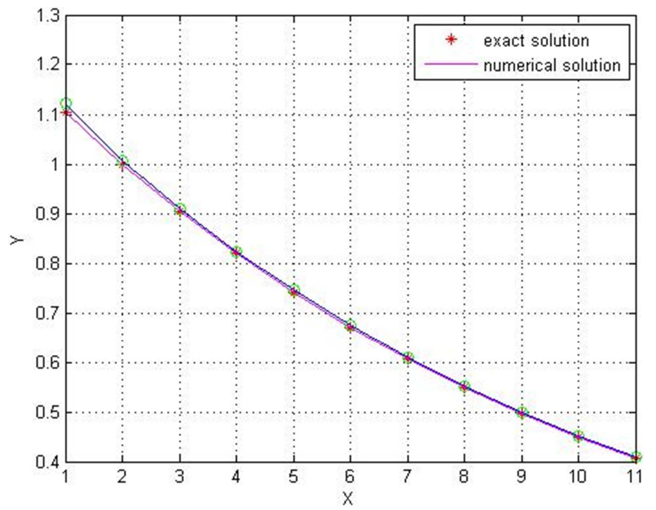

Figure 3. *Numerical Solution Using* $M = N = 11, \Delta t = 0.05, c = 781, K = 10000, h = 0.1$ *and* $\theta = 0.4$.

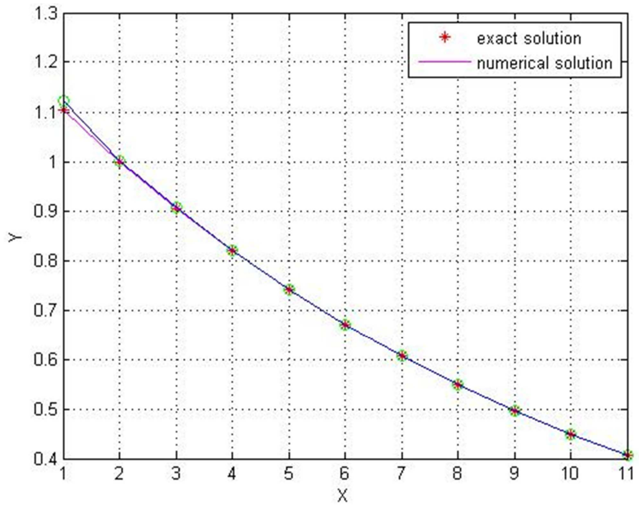

Figure 4. *Numerical solution using* $M = N = 11, \Delta t = 0.05, c = 781, K = 10000, h = 0.1$ *and* $\theta = 0.6$.

Table 4. *For* $M = N = 11, \Delta t = 0.05, c = 781, K = 10000, h = 0.1$ *and* $\theta = 0.6$.

u_*^{n+1}	$u(x_i, t_{n+1})$
1.122663	1.105171
1.001108	1.000000
0.905824	0.904837
0.819623	0.818731
0.741626	0.740818
0.671051	0.670320
0.607192	0.606531
0.549410	0.548812
0.497127	0.496585
0.449818	0.449329

$RMSE = 0.001471$, shows that there is a convergence between the solutions for the case θ-scheme, thus it is clear that from (1, 2) the two curves are slightly not coincided from analysis of the result point as shown in the Figure 4.

6. Conclusion

The main purpose of this study has been to seek the numerical solution of a parabolic partial differential equation more precisely the linear heat equation. Such a study being motivated by the fact that many contemporary studies of the resolution of partial differential equations are focused on the mesh method, we firstly sought an exact solution, secondly the problem was set to Cauchy problem, finally the mixed-scheme and RBF collocation method was used to solve numerically the problem and the results was compared to the exaction. The tests problems 2, 3 and 4 given respectively in tables 2, 3 and 4, shown that the solution converge and diverge in the test problem 1.

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