New Theorems and Formulas to Solve Fourth Degree Polynomial Equation in General Forms by Calculating the Four Roots Nearly Simultaneously

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Abstract: This paper presents new formulary solutions for fourth degree polynomial equations in general forms, where we present four solutions for any fourth-degree equation with real coefficients, and thereby having the possibility to calculate the four roots of any quartic equation nearly simultaneously. In this paper, the used logic to determine the solutions of a fourth-degree polynomial equation enables to deduce if the polynomial accepts complex roots with imaginary parts different from zero and how many of them there are. As a result, we are proposing six new theorems, where two among them are allowing to calculate the four roots nearly simultaneously for any fourth-degree polynomial equation in simple forms and complete forms, whereas the other four theorems are allowing to deduce the number of complex roots with imaginary parts different from zero before conducting further fetching for the values. Furthermore, the proposed formulas in this paper are building the ground to concretize precise solutions for polynomial equations with degrees higher than four while relying on radical expressions. Each proposed theorem in this paper is presented along with a detailed proof in a scaling manner starting from propositions based on precise formulas whereas building on progressive logic of calculation and deduction. Each formulary solution in proposed theorems is based on a distributed group of radical expressions designed to be neutralized when they are multiplied by each other, which allow the elimination of complexity while reducing degrees of terms. All presented theorems are developed according to a specific logic where we engineer the structure of solutions before forwarding calculations to express the precis formulas of roots.

Keywords: Fourth Degree Polynomial, Nearly Simultaneous Calculations, New Four Solutions, New Theorems, Solving Quartic Equation

1. Introduction

Polynomial equations has been center of focus in research for centuries, where complexity of roots is dependent on degrees of polynomials and implicated coefficients, whereas defining one solution for a polynomial equation was always sufficient as a first step toward forwarding calculations and finding the rest of solutions. However, defining all the formulary roots for a polynomial equation in order to calculate them simultaneously or nearly simultaneously were never before a principal axe of concern.

Solving n^{th} degree polynomial equations has been an enigmatic problem over hundreds of years, where many attempts concluded to the impossibility of elaborating universal solution formulas for polynomial equations with degrees equal or higher than fifth degree by using radicals. However, finding breakthroughs is a living hope for mathematicians who seek to solve quantic equations and above by using radical expressions.

Even though solving polynomial equations of fifth degree and higher by using radicals is center of focus, defining specific formulary solutions for fourth degree polynomial equations is still open to find different algebraic expressions, which may determine all four roots of any fourth degree polynomial in general form or bring different characteristics and new theorems.

This article presents the results of a research in mathematic science, where we present four formulary solutions for any fourth degree polynomial equation in general form with real coefficients. We attack the problem of solving polynomial equations of nth degree, where ($n = 4$), with a clear manner by using extendable logic, which enables us to elaborate different new theorems.

This paper is an introduction of a large research in mathematics to solve n^{th} degree polynomial equations with new direct concepts and methods basing on projecting the presented work in this paper on other polynomial equations with degrees higher than four, which will be presented in other articles.

Lodovico Ferrari is attributed the credit of discovering a principal solution for fourth degree polynomial equation in 1540, but since his root expression required having a solution for cubic equation, which wasn't published yet by Gerolamo Cardano, Lodovico Ferrari couldn't publish his discovery immediately. Ferrari's mentor, Gerolamo Cardano, in the book Ars Magna [1], published this discovered quartic solution along with the cubic solution.

The discovered cubic solution by Cardano [2] for third degree equations under the form $x^3 + px + q = 0$, helps Ferrari's solution to solve quartic equations by reducing them from the fourth degree to the second degree, but it doesn't directly help to properly define the four roots of any quartic equation. Cardano's method was also the base to solve particular forms of nth degree equations; such as by giving radical expressions under the form $\sqrt{na + \sqrt{b}} + \sqrt{na - \sqrt{b}}$ for $n = 2,3,4, …,$ etc. [3].

There are also other elaborated methods and proposed solutions for quartic equations such as Euler's solution [4], Galois's method [5], Descartes's method [6], Lagrange's method [7] and algebraic geometry [8].

Cubic solution was always essential as a base for further research to solve polynomial equations of fourth degree and above [9, 10], such as trying to prove that quantic equations do not accept quadratic expressions as solutions, which is discussed in [11-14].

A solution for third degree equation was first found in 1515 by Scipione del Ferro (1465–1526), for some specific cases defined by the values of coefficients. However, the official form of cubic solution, which is recognized as the base of further *historical* research on solving quartic equations and other specific forms of polynomial equations, is the published solution by Cardano.

There are also other recent research dedicated to solve quartic equations where the used methods are based on expression reduction [15, 17], whereas other research is relying on algorithms and numerical analysis to find the roots of polynomial equations with degrees higher than four [17, 18].

The advantage of this paper is presenting, with details, the expressions of four new formulary solutions for any form of fourth degree polynomial equations with real coefficients, and thereby having the possibility to calculate the four roots of any quartic equation nearly simultaneously. The used logic

of presented solutions in this paper offers the possibility to deduce, directly, if a polynomial equation accepts complex solutions with imaginary parts different from zero and how many of them there are. As a result, this paper presents six theorems on solving fourth degree polynomial equations in simple forms and complete forms.

Furthermore, the presented results in this paper are the ground for an advanced work to solve polynomial equations of fifth degree [19] and sixth degree [20] in general forms by using radicals, which is enabled by reducing these equations to a quartic form. As a result, by using the proposed formulas in this paper, we are capable of solving polynomial equations higher than fourth degree.

Because the content of this paper is original, and there are many new proposed formulas, mathematical expressions and theorems related in a scaling manner basing on extendable logic; every formula will be proved mathematically and used to build the rest of content, and we will go through them by logical analysis and deduction basing on structured development.

This paper is structured as follow: Section 2 where we present new solutions and theorems for fourth degree polynomial equations in simple form, to solve these equations and predetermine if they accept roots with imaginary parts different from zero. Section 3, where we present new solutions and theorems for fourth degree polynomial equations in complete form, to solve these polynomial equations and predetermine how many solutions with imaginary parts different from zero they may accept. Finally, Section 4 for discussion.

2. New Four Solutions for Fourth Degree Polynomial Equation in Simple Form

This section presents new unified formulary solutions for fourth degree polynomial equations in simple form (eq.1) along with their proof.

2.1. First Proposed Theorem

In this subsection we propose a new theorem to solve fourth degree polynomial equations that may be presented as shown in (eq.1).

The expressions of proposed solutions are dependent on the value of d.

We are proposing four solutions for $d < 0$, four solutions for $d > 0$ and four solutions for $d = 0$.

The proposed solutions for $d < 0$, $d > 0$ and for $d = 0$ are expressed by using $x_{0,1}$ in (eq.12), c in (eq.6), d in (eq.7) and (eq.8).

The proof of this theorem is presented in an independent subsection, because it integrates long mathematical expressions.

Theorem 1

A fourth degree polynomial equation under the expression (eq.1), where coefficients belong to the group of numbers $\mathbb{R}$, has four solutions:

$$x^4 + cx^2 + dx + e = 0 \qquad (1)$$

If $d < 0$, and by using the expressions of $x_{0,1}$ in (eq.12), c in (eq.6) and d in (eq.7):

Solution 1: $S_{1,1}$ in expression (eq.35);
Solution 2: $S_{1,2}$ in expression (eq.36);
Solution 3: $S_{1,3}$ in expression (eq.37);
Solution 4: $S_{1,4}$ in expression (eq.38).

If $d > 0$, and by using the expressions of $x_{0,1}$ in (eq.12), c in (eq.6) and d in (eq.8):

Solution 1: $S_{2,1}$ in expression (eq.39);
Solution 2: $S_{2,2}$ in expression (eq.40);
Solution 3: $S_{2,3}$ in expression (eq.41);
Solution 4: $S_{2,4}$ in expression (eq.42).

If $d = 0$, and by using the expressions of $x_{0,1}$ in (eq.24) and c in (eq.6):

Solution 1: $S_{3,1}$ in expression (eq.43);
Solution 2: $S_{3,2}$ in expression (eq.44);
Solution 3: $S_{3,3}$ in expression (eq.45);
Solution 4: $S_{3,4}$ in expression (eq.46).

2.2. Proof and Mathematical Expressions of Theorem 1

To solve the polynomial equation (eq.1), we propose new expressions for the variable x; expressions (eq.2) and (eq.3):

$$\text{For } d \le 0: x = \sqrt{x_0} + \sqrt{x_1} + \sqrt{x_2} \qquad (2)$$

$$\text{For } d \ge 0: x = -\sqrt{x_0} - \sqrt{x_1} - \sqrt{x_2} \qquad (3)$$

We propose the expressions (eq.4) and (eq.5) for x_1 and x_2 successively, in condition of $x_0 \ne 0$.

$$x_1 = -\frac{\frac{c}{2}+x_0}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_0}{2}\right)^2 - \frac{d^2}{64x_0}} \qquad (4)$$

$$x_2 = -\frac{\frac{c}{2}+x_0}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_0}{2}\right)^2 - \frac{d^2}{64x_0}} \qquad (5)$$

To reduce the expression of equation (eq.1) and find a way to solve it, we propose the following expressions for the coefficients c and d:

$$-2\left[\sqrt{x_0}^2 + \sqrt{x_1}^2 + \sqrt{x_0}^2\right] = c \qquad (6)$$

$$\text{For } d \le 0: -8\sqrt{x_0}\sqrt{x_1}\sqrt{x_2} = d \qquad (7)$$

$$\text{For } d \ge 0: 8\sqrt{x_0}\sqrt{x_1}\sqrt{x_2} = d \qquad (8)$$

In the following calculation, we replace the variable x with the expression (eq.2), where we suppose $d < 0$:

$$x^4 + cx^2 + dx + e = -\left[\sqrt{x_0}^4 + \sqrt{x_1}^4 + \sqrt{x_2}^4\right] + 2\left[\sqrt{x_0}^2\sqrt{x_1}^2 + \sqrt{x_0}^2\sqrt{x_2}^2 + \sqrt{x_1}^2\sqrt{x_2}^2\right] + e$$

$$= -\left[\sqrt{x_0}^2 + \sqrt{x_1}^2 + \sqrt{x_2}^2\right]^2 + 4\left[\sqrt{x_0}^2\sqrt{x_1}^2 + \sqrt{x_0}^2\sqrt{x_2}^2 + \sqrt{x_1}^2\sqrt{x_2}^2\right] + e$$

$$= 4\left[-x_0\left(\frac{c}{2} + x_0\right) + \frac{d^2}{64x_0}\right] + e - \frac{c^2}{4} = 0$$

$$-\left[\sqrt{x_0}^2 + \sqrt{x_1}^2 + \sqrt{x_2}^2\right]^2 + 4\left[\sqrt{x_0}^2\sqrt{x_1}^2 + \sqrt{x_0}^2\sqrt{x_2}^2 + \sqrt{x_1}^2\sqrt{x_2}^2\right] + e = 0 \Rightarrow x_0^3 + \frac{c}{2}x_0^2 + \frac{c^2-4e}{16}x_0 - \frac{d^2}{64} = 0 \quad (9)$$

To solve the resulted expression in (eq.9), we use Cardan's solution for third degree polynomial equations.

For $y^3 + cy + d = 0$, Cardan's solution is as follow:

$$y = \sqrt[3]{\frac{-d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3}} + \sqrt[3]{\frac{-d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3}} \quad (10)$$

For $x^3 + bx^2 + cx + d = 0$, we use the form $x = \frac{-b+y}{3}$, and we suppose $D = 27d + 2b^3 - 9cb$ and $C = 9c - 3b^2$ to express the cubic solution as follow:

$$x = \frac{-b}{3} + \frac{1}{3}\sqrt[3]{-\frac{D}{2} + \sqrt{\left(\frac{D}{2}\right)^2 + \left(\frac{C}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D}{2} - \sqrt{\left(\frac{D}{2}\right)^2 + \left(\frac{C}{3}\right)^3}} \qquad (11)$$

By using the expression (eq.11), the solutions of third degree polynomial equation expressed in (eq.9) are $x_{0,1}, x_{0,2}$ and $x_{0,3}$ where $B = \frac{c}{2}$, $D^. = \frac{-27d^2-2c^3+72ce}{64}$ and $C^. = -\frac{3c^2+36e}{16}$:

$$x_{0,1} = \frac{-B}{3} + \frac{1}{3}\sqrt[3]{-\frac{D^.}{2} + \sqrt{\left(\frac{D^.}{2}\right)^2 + \left(\frac{C^.}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D^.}{2} - \sqrt{\left(\frac{D^.}{2}\right)^2 + \left(\frac{C^.}{3}\right)^3}} \qquad (12)$$

In condition of $x_{0,1} \ne 0$, $x_{0,2}$ and $x_{0,3}$ are expressed as shown in (eq.13) and (eq.14).

$$x_{0,2} = -\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}} \qquad (13)$$

$$x_{0,3} = -\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}} \qquad (14)$$

We deduce that when x_0 takes the value $x_{0,1}$, the value of $x_{0,2}$ is equal to the shown value of x_1 in (eq.4), and the value of $x_{0,3}$ is equal to the shown value of x_2 in (eq.5).

There are three other possible expressions for x which respect the proposition $-8\sqrt{x_0}\sqrt{x_1}\sqrt{x_2} = d$ when $d \leq 0$, and they give the same results of calculations toward having the shown third degree polynomial in (eq.9). Thereby, they give the same values for roots $x_{0,1}, x_{0,2}$ and $x_{0,3}$. These three expressions are $x = -\sqrt{x_0} - \sqrt{x_1} + \sqrt{x_2}$, $x = -\sqrt{x_0} + \sqrt{x_1} - \sqrt{x_2}$ and $x = \sqrt{x_0} - \sqrt{x_1} - \sqrt{x_2}$.

$$x^4 + cx^2 + dx + e = 0 \ with \ d < 0 \qquad (15)$$

As a result, there are four solutions for the polynomial equation form expressed in (eq.15), and they are as shown in (eq.16), (eq.17), (eq.18) and (eq.19).

$$\text{Solution 1: } S_{1,1} = \sqrt{x_{0,1}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}}} \qquad (16)$$

$$\text{Solution 2: } S_{1,2} = -\sqrt{x_{0,1}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}}} \qquad (17)$$

$$\text{Solution 3: } S_{1,3} = -\sqrt{x_{0,1}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}}} \qquad (18)$$

$$\text{Solution 4: } S_{1,4} = \sqrt{x_{0,1}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}}} \qquad (19)$$

There are three other possible expressions for x which respect the proposition $8\sqrt{x_0}\sqrt{x_1}\sqrt{x_2} = d$ when $d \geq 0$, and they give the same third degree polynomial shown in (eq.9) after calculations. These three expressions are $x = -\sqrt{x_0} + \sqrt{x_1} + \sqrt{x_2}$, $x = \sqrt{x_0} - \sqrt{x_1} + \sqrt{x_2}$ and $x = \sqrt{x_0} + \sqrt{x_1} - \sqrt{x_2}$.

As a result, the proposed solutions for fourth degree polynomial equation shown in (eq.1) when $d > 0$ are as expressed in (eq.20), (eq.21), (eq.22) and (eq.23).

$$\text{Solution 1: } S_{2,1} = -\sqrt{x_{0,1}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}}} \qquad (20)$$

$$\text{Solution 2: } S_{2,2} = -\sqrt{x_{0,1}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}}} \qquad (21)$$

$$\text{Solution 3: } S_{2,3} = \sqrt{x_{0,1}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}}} \qquad (22)$$

$$\text{Solution 4: } S_{2,4} = \sqrt{x_{0,1}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}}} \qquad (23)$$

Concerning d=0:

The expression of $x_{0,1}$ is as shown in (eq.24) where $B = \frac{c}{2}$, $D^{\cdot} = \frac{-2c^3+72ce}{64}$ and $C^{\cdot} = -\frac{3c^2+36e}{16}$, whereas $x_{0,2}$ and $x_{0,3}$ are as shown in (eq.25) and (eq.26).

$$x_{0,1} = \frac{-B}{3} + \frac{1}{3}\sqrt[3]{-\frac{D;}{2} + \sqrt{\left(\frac{D;}{2}\right)^2 + \left(\frac{C;}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D;}{2} - \sqrt{\left(\frac{D;}{2}\right)^2 + \left(\frac{C;}{3}\right)^3}} \qquad (24)$$

$$x_{0,2} = -\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}} = 0 \; or \; x_{0,2} = -\left(\frac{c}{2} + x_{0,1}\right) \qquad (25)$$

$$x_{0,3} = -\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}} = -\left(\frac{c}{2} + x_{0,1}\right) \; or \; x_{0,3} = 0 \qquad (26)$$

Because of having either of the expressions $x_{0,2} = 0$ or $x_{0,3} = 0$, and having the intersection between the shown forms in (eq.7) and (eq.8) for d=0 ($x = \sqrt{x_0} + \sqrt{x_1} + \sqrt{x_2}$ for $d \leq 0$ and $x = -\sqrt{x_0} - \sqrt{x_1} - \sqrt{x_2}$ for $d \geq 0$); there are four solutions for the polynomial equation shown in (eq.1) when $d = 0$ and they are as expressed in (eq.27), (eq.28), (eq.29) and (eq.30).

$$\text{Solution 1: } S_{3,1} = \sqrt{x_{0,1}} + \sqrt{-\left(\frac{c}{2} + x_{0,1}\right)} \qquad (27)$$

$$\text{Solution 2: } S_{3,2} = -\sqrt{x_{0,1}} - \sqrt{-\left(\frac{c}{2} + x_{0,1}\right)} \qquad (28)$$

$$\text{Solution 3: } S_{3,3} = -\sqrt{x_{0,1}} + \sqrt{-\left(\frac{c}{2} + x_{0,1}\right)} \qquad (29)$$

$$\text{Solution 4: } S_{3,4} = \sqrt{x_{0,1}} - \sqrt{-\left(\frac{c}{2} + x_{0,1}\right)} \qquad (30)$$

When we give the value $x_{0,1}$ in (eq.12) to x_0, the values of $x_{0,2}$ in (eq.13) and $x_{0,3}$ in (eq.14) become equal to the shown values of x_1 in (eq.4) and x_2 in (eq.5) respectively. Thereby, even when we replace the value of $x_{0,1}$ in the expressions of proposed solutions by the values of $x_{0,2}$ or $x_{0,3}$, the results are only redundancies of proposed solutions, because the used value of c in these solutions is expressed as follows: $\frac{c}{2} =$

$-\left(\sqrt{x_0^2} + \sqrt{x_1^2} + \sqrt{x_2^2}\right) = -\left(\sqrt{x_{0,1}^2} + \sqrt{x_{0,2}^2} + \sqrt{x_{0,3}^2}\right)$.

2.3. Second Proposed Theorem

In this subsection, we present a second theorem, which we rely on to deduce if the polynomial equation in (eq.1) accepts complex solutions with imaginary parts different from zero and how many of them there are. This proposed theorem is based on the proposed expression in (eq.6).

In the proof, we refer to real part of numbers by $Re()$.

Theorem 2

Considering the fourth degree polynomial equation $x^4 + cx^2 + dx + e = 0$ where all coefficients belong to the group of numbers $\mathbb{R}$. If $e \neq 0$ and $c > 0$; then, this fourth degree polynomial equation accepts at least two complex solutions with imaginary parts different from zero.

2.4. Proof of Theorem 2

To reduce the expression of the shown form in (eq.1) and find a way to solve it, we proposed the next expression for the coefficient c: $-2\left[\sqrt{x_0^2} + \sqrt{x_1^2} + \sqrt{x_2^2}\right] = c$. We considered all coefficients of quartic equation belong to the group of numbers $\mathbb{R}$.

If $e \neq 0$ and $c > 0$; then, the fourth degree polynomial equation $x^4 + cx^2 + dx + e = 0$ accepts at least two complex solutions with imaginary parts different from zero, because:

$$c > 0 \Rightarrow -2\left[\sqrt{x_0^2} + \sqrt{x_1^2} + \sqrt{x_2^2}\right] > 0 \Rightarrow x_0 + x_1 + x_2 < 0$$

$$\Rightarrow Re(x_0) < 0 \; or \; Re(x_1) < 0 \; or \; Re(x_2) < 0$$

$$\Rightarrow \left(\sqrt{x_0} \in (\mathbb{C}\backslash\mathbb{R})\right) or \left(\sqrt{x_1} \in (\mathbb{C}\backslash\mathbb{R})\right) or \left(\sqrt{x_2} \in (\mathbb{C}\backslash\mathbb{R})\right)$$

$$\Rightarrow \exists \, x \in (\mathbb{C}\backslash\mathbb{R}) \mid x^4 + cx^2 + dx + e = 0 \; (Considering \; the \; expressions \; of \; proposed \; solutions \; in \; Theorem \; 1.)$$

$$\Rightarrow \exists \, \{x, \overline{x}\} \in (\mathbb{C}\backslash\mathbb{R}) * (\mathbb{C}\backslash\mathbb{R}) \mid x^4 + cx^2 + dx + e = 0 \; and \; \overline{x}^4 + c\overline{x}^2 + d\overline{x} + e = 0$$

3. Solutions and Theorems for Fourth Degree Polynomial Equations in Complete Form

In this section, we solve fourth degree polynomial equation in complete form, which is expressed in (eq.31). The proposed theorems in this section are extending to presented theorems in previous section (section 2).

3.1. Third Proposed Theorem

In this subsection we propose the third theorem, which is an extending to first proposed theorem by reducing the form of fourth degree polynomial equation from expression (eq.31) to expression (eq.35). We rely on replacing x with $\frac{-\frac{b}{a}+y}{4}$.

The expressions of proposed solutions are dependent on the value of $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right)$.

We are proposing four solutions for $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) <$ 0 , four solutions for $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) > 0$ and four solutions for $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) = 0$.

The proposed solutions for $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) <$ $0, \left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) > 0$ and for $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) =$ 0 are expressed by using $y_{0,1}$ in (eq.38), P in (eq.36) and Q in (eq.36).

The proof of this theorem is presented in an independent subsection, because it integrates long mathematical expressions.

Theorem 3

A fourth degree polynomial equation under the expressed form in (eq.31), where coefficients belong to the group of numbers $\mathbb{R}$ and $a \neq 0$, has four solutions.

$$ax^4 + bx^3 + cx^2 + dx + e = 0 \; with \; a \neq 0 \quad (31)$$

If $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) < 0$, and by using $y_{0,1}$ in (eq.38), P in (eq.36) and Q in (eq.36):

Solution 1: $S_{1,1}$ in expression (eq.53);
Solution 2: $S_{1,2}$ in expression (eq.54);
Solution 3: $S_{1,3}$ in expression (eq.55);
Solution 4: $S_{1,4}$ in expression (eq.56).

If $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) > 0$, and by using $y_{0,1}$ in (eq.38), P in (eq.36) and Q in (eq.36):

Solution 1: $S_{2,1}$ in expression (eq.57);
Solution 2: $S_{2,2}$ in expression (eq.58);
Solution 3: $S_{2,3}$ in expression (eq.59);
Solution 4: $S_{2,4}$ in expression (eq.60).

If $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) = 0$, and by using $y_{0,1}$ in (eq.38), P in (eq.36) and Q in (eq.36):

Solution 1: $S_{3,1}$ in expression (eq.61);
Solution 2: $S_{3,2}$ in expression (eq.62);
Solution 3: $S_{3,3}$ in expression (eq.63);
Solution 4: $S_{3,4}$ in expression (eq.64).

3.2. Proof of Theorem 3

By dividing the equation (eq.31) on the coefficient a, we have the next form:

$$x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0 \; with \; a \neq 0 \quad (32)$$

We suppose that x is expressed as shown in (eq.33):

$$x = \frac{\frac{-b}{a} + y}{4} \quad (33)$$

We replace x with supposed expression in (eq.33) to reduce the form of presented polynomial equation in (eq.32). Thereby, we have the presented expression in (eq.34).

$$y^4 + y^2 \left[-6\left(\frac{b}{a}\right)^2 + \frac{16c}{a}\right] + y\left[8\left(\frac{b}{a}\right)^3 - \frac{32cb}{a^2} + \frac{64d}{a}\right] -$$
$$3\left(\frac{b}{a}\right)^4 + \frac{16cb^2}{a^3} - \frac{64db}{a^2} + \frac{256e}{a} = 0 \quad (34)$$

To simplify the expression of polynomial equation in (eq.34), we replace the expression of used coefficients as shown in (eq.35) where the values of those coefficients are defined in (eq.36).

$$y^4 + Py^2 + Qy + R = 0 \quad (35)$$

$$P = -6\left(\frac{b}{a}\right)^2 + \frac{16c}{a}; Q = 8\left(\frac{b}{a}\right)^3 - \frac{32cb}{a^2} + \frac{64d}{a}; R =$$
$$-3\left(\frac{b}{a}\right)^4 + \frac{16cb^2}{a^3} - \frac{64db}{a^2} + \frac{256e}{a} \quad (36)$$

As a next step, we project the developed results to solve fourth degree polynomial equation with simplified form shown in previous section (section 2). Thereby, we have $y = \sqrt{y_0} + \sqrt{y_1} + \sqrt{y_2}$ for $Q \leq 0$, $y = -\sqrt{y_0} - \sqrt{y_1} - \sqrt{y_2}$ for $Q \geq 0$ and we have the third degree equation in (eq.37) where we search for the expressions of the solutions $y_{0,1}$, $y_{0,2}$ and $y_{0,3}$.

$$y_0^3 + \frac{P}{2}y_0^2 + \frac{P^2 - 4R}{16}y_0 - \frac{Q^2}{64} = 0 \quad (37)$$

By projecting the expressions of shown solutions in (eq.12), (eq.13) and (eq.14), the resulted roots for third degree polynomial in (eq.37) are $y_{0,1}$ in (eq.38), $y_{0,2}$ in (eq.39) and $y_{0,3}$ in (eq.40), where $P^i = \frac{P}{2}$, $R^i = \frac{-27Q^2 - 2P^3 + 72PR}{64}$ and $Q^i = -\frac{3P^2 + 36R}{16}$.

$$y_{0,1} = -\frac{P^i}{3} + \frac{1}{3}\sqrt[3]{-\frac{R^i}{2} + \sqrt{\left(\frac{R^i}{2}\right)^2 + \left(\frac{Q^i}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{R^i}{2} + \sqrt{\left(\frac{R^i}{2}\right)^2 + \left(\frac{Q^i}{3}\right)^3}} \quad (38)$$

In condition of $y_{0,1} \neq 0$, $y_{0,2}$ and $y_{0,3}$ are as follow:

$$y_{0,2} = -\frac{\frac{P}{2} + y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2} + y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} \quad (39)$$

$$y_{0,3} = -\frac{\frac{P}{2} + y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2} + y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} \quad (40)$$

By using the shown expressions in (eq.36) and (eq.38), the solutions of expressed quartic equation in (eq.35) when $\left(8\left(\frac{b}{a}\right)^3 - \frac{32cb}{a^2} + \frac{64d}{a}\right) < 0$ are as shown in (eq.41), (eq.42), (eq.43) and (eq.44).

$$\text{Solution 1: } s_{1,1} = \sqrt{y_{0,1}} + \sqrt{-\frac{\frac{P}{2} + y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2} + y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} + \sqrt{-\frac{\frac{P}{2} + y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2} + y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (41)$$

$$\text{Solution 2: } s_{1,2} = -\sqrt{y_{0,1}} - \sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} + \sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} } \qquad (42)$$

$$\text{Solution 3: } s_{1,3} = -\sqrt{y_{0,1}} + \sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} - \sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} } \qquad (43)$$

$$\text{Solution 4: } s_{1,4} = \sqrt{y_{0,1}} - \sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} - \sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} } \qquad (44)$$

By using the shown expressions in (eq.36) and (eq.38), the solutions of expressed quartic equation in (eq.35) when $\left(8\left(\frac{b}{a}\right)^3 - \frac{32cb}{a^2} + \frac{64d}{a}\right) > 0$ are as shown in (eq.45), (eq.46), (eq.47) and (eq.48).

$$\text{Solution 1: } s_{2,1} = -\sqrt{y_{0,1}} - \sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} - \sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} } \qquad (45)$$

$$\text{Solution 2: } s_{2,2} = -\sqrt{y_{0,1}} + \sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} + \sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} } \qquad (46)$$

$$\text{Solution 3: } s_{2,3} = \sqrt{y_{0,1}} - \sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} + \sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} } \qquad (47)$$

$$\text{Solution 4: } s_{2,4} = \sqrt{y_{0,1}} + \sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} - \sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} } \qquad (48)$$

By using the shown expressions in (eq.36) and (eq.38), the solutions of expressed quartic equation in (eq.35) when $\left(8\left(\frac{b}{a}\right)^3 - \frac{32cb}{a^2} + \frac{64d}{a}\right) = 0$ are as shown in (eq.49), (eq.50), (eq.51) and (eq.52).

$$\text{Solution 1: } s_{3,1} = \sqrt{y_{0,1}} + \sqrt{-\left(\frac{P}{2} + y_{0,1}\right)} \qquad (49)$$

$$\text{Solution 2: } s_{3,2} = -\sqrt{y_{0,1}} - \sqrt{-\left(\frac{P}{2} + y_{0,1}\right)} \qquad (50)$$

$$\text{Solution 3: } s_{3,3} = -\sqrt{y_{0,1}} + \sqrt{-\left(\frac{P}{2} + y_{0,1}\right)} \qquad (51)$$

$$\text{Solution 4: } s_{3,4} = \sqrt{y_{0,1}} - \sqrt{-\left(\frac{P}{2} + y_{0,1}\right)} \qquad (52)$$

In order to solve the polynomial equation shown in (eq.32), we use the expression $x = \frac{-\frac{b}{a}+y}{4}$ where y is the unknown variable in polynomial equation (eq.34). By using expressions (eq.36) and (eq.38) for $\left(8\left(\frac{b}{a}\right)^3 - \frac{32cb}{a^2} + \frac{64d}{a}\right) < 0$, the solutions for equation (eq.31) are as shown in (eq. 53), (eq. 54), (eq. 55) and (eq. 56).

$$\text{Solution 1: } S_{1,1} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \qquad (53)$$

$$\text{Solution 2: } S_{1,2} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (54)$$

$$\text{Solution 3: } S_{1,3} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (55)$$

$$\text{Solution 4: } S_{1,4} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (56)$$

By using the expression $x = \frac{-\frac{b}{a}+y}{4}$ while relying on the expressions (eq.36) and (eq.38) for $\left(8\left(\frac{b}{a}\right)^3 - \frac{32cb}{a^2} + \frac{64d}{a}\right) > 0$, the proposed solutions for equation (eq.31) are as shown in (eq.57), (eq.58), (eq.59) and (eq.60).

$$\text{Solution 1: } S_{2,1} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (57)$$

$$\text{Solution 2: } S_{2,2} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (58)$$

$$\text{Solution 3: } S_{2,3} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (59)$$

$$\text{Solution 4: } S_{2,4} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (60)$$

By using the expression $x = \frac{-\frac{b}{a}+y}{4}$ while relying on the expressions (eq.36) and (eq.38) for $\left(8\left(\frac{b}{a}\right)^3 - \frac{32cb}{a^2} + \frac{64d}{a}\right) = 0$, the proposed solutions for equation (eq.31) are as shown in (eq.61), (eq.62), (eq.63) and (eq.64).

$$\text{Solution 1: } S_{3,1} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\left(\frac{P}{2}+y_{0,1}\right)} \quad (61)$$

$$\text{Solution 2: } S_{3,2} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\left(\frac{P}{2}+y_{0,1}\right)} \quad (62)$$

$$\text{Solution 3: } S_{3,3} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\left(\frac{P}{2}+y_{0,1}\right)} \quad (63)$$

$$\text{Solution 4: } S_{3,4} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\left(\frac{P}{2}+y_{0,1}\right)} \quad (64)$$

3.3. Fourth Proposed Theorem

In this subsection, we propose a fourth theorem based on the value of $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right)$ along with proof. This theorem is an extending of second proposed theorem after reducing the form of fourth degree polynomial from expression (eq.31) to expression (eq.34) by replacing x with $x = \frac{-\frac{b}{a}+y}{4}$.

Theorem 4

Considering the polynomial equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ where all coefficients belong to the group of numbers $\mathbb{R}$, if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right) > 0$; then, this fourth degree polynomial equation accepts at least two complex solutions, where the imaginary parts are different from zero and dependent on the group of coefficients $\{a, b, c\}$.

3.4. Proof of Theorem 4

Considering the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ where we supposed $x = \frac{-\frac{b}{a}+y}{4}$ to have the expression (eq.34) and all coefficients belong to the group of numbers $\mathbb{R}$. By projecting the proposed expression in (eq.6) on the value of P in (eq.35) and (eq.36), we have the following

expression: $-6\left(\frac{b}{a}\right)^2 + \frac{16c}{a} = -2[y_0{}^2 + y_1{}^2 + y_2{}^2]$.

In this proof, we rely on the shown expressions of P, Q and R in (eq.36). We refer to real part of numbers by Re().

If $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right) > 0$; then, the fourth degree polynomial equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ accepts at least two complex solutions where the imaginary parts are different from zero and dependent on the group of coefficients $\{a, b, c\}$, because:

$$\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right) > 0 \Rightarrow -2[y_0{}^2 + y_1{}^2 + y_2{}^2] > 0$$

$$\Rightarrow y_0 + y_1 + y_2 < 0$$

$$\Rightarrow Re(y_0) < 0 \ or \ Re(y_1) < 0 \ or \ Re(y_2) < 0$$

$$\Rightarrow \left(\sqrt{y_0} \in (\mathbb{C}\backslash\mathbb{R})\right) or \left(\sqrt{y_1} \in (\mathbb{C}\backslash\mathbb{R})\right) or \left(\sqrt{y_2} \in (\mathbb{C}\backslash\mathbb{R})\right)$$

$$\Rightarrow \exists \, y \in (\mathbb{C}\backslash\mathbb{R}) \mid y^4 + Py^2 + Qy + R = 0 \ (Considering \ the \ expressions \ of \ proposed \ solutions \ in \ Theorem \ 3)$$

$$\Rightarrow \exists \, \{y, \overline{y}\} \in (\mathbb{C}\backslash\mathbb{R}) * (\mathbb{C}\backslash\mathbb{R}) \mid y^4 + Py^2 + Qy + R = 0 \ and \ \overline{y}^4 + P\overline{y}^2 + Q\overline{y} + R = 0$$

$$\Rightarrow \exists \, \{x, \overline{x}\} \in (\mathbb{C}\backslash\mathbb{R}) * (\mathbb{C}\backslash\mathbb{R}) \mid ax^4 + bx^3 + cx^2 + dx + e = 0 \ and \ a\overline{x}^4 + b\overline{x}^3 + c\overline{x}^2 + d\overline{x} + e = 0$$

3.5. Fifth Proposed Theorem

In this subsection, we propose a fifth theorem based on the value of $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right)$ along with proof. This theorem is an extending of fourth proposed theorem after passing from expression (eq.31) to expression (eq.65) by supposing $= \frac{1}{x}$.

Theorem 5

Considering the polynomial equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ where all coefficients belong to the group of numbers $\mathbb{R}$, if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right) > 0$; then, this fourth degree polynomial equation accepts at least two complex solutions, where the imaginary parts are different from zero and dependent on the group of coefficients $\{c, d, e\}$.

3.6. Proof of Theorem 5

We use the expression $z = \frac{1}{x}$ in the polynomial equation $(eq. 31)$, in order to extend the used logic in Theorem 4 by projection on expression (eq.65). We consider all coefficients of quartic equation belong to the group of numbers $\mathbb{R}$.

$$ax^4 + bx^3 + cx^2 + dx + e = 0 \ where \ e \neq 0 \Rightarrow z^4 + \frac{d}{e}z^3 + \frac{c}{e}z^2 + \frac{b}{e}z + \frac{a}{e} = 0 \qquad (65)$$

We use the expression $z = \frac{-\frac{d}{e}+w}{4}$ to replace z in $z^4 + \frac{d}{e}z^3 + \frac{c}{e}z^2 + \frac{b}{e}z + \frac{a}{e} = 0$, in order to reduce the form of

polynomial. Thereby, we have the presented polynomial equation in (eq.66).

$$w^4 + M w^2 + Nw + O = 0 \qquad (66)$$

The coefficients of shown polynomial in (eq.66) are as follow:

$$M = \left[-6\left(\frac{d}{e}\right)^2 + \frac{16c}{e}\right]; N = \left[8\left(\frac{d}{e}\right)^3 - \frac{32cd}{e^2} + \frac{64b}{e}\right]; O = -3\left(\frac{d}{e}\right)^4 + \frac{16cd^2}{e^3} - \frac{64bd}{e^2} + \frac{256a}{e}$$

By projecting the proposed expression in (eq.6), we have the next expression : $\left(-6\left(\frac{d}{e}\right)^2 + \frac{16c}{e}\right) = -2[w_0{}^2 + w_1^2 + w_2^2]$, where w is the solution for polynomial equation shown in (eq.66). The solution w is expressed as follow:

$$w = \sqrt{w_0} + \sqrt{w_1} + \sqrt{w_2} \ if \ \left[8\left(\frac{d}{e}\right)^3 - \frac{32cd}{e^2} + \frac{64b}{e}\right] \leq 0, \ and$$

$$w = -\sqrt{w_0} - \sqrt{w_1} - \sqrt{w_2} \ if \ \left[8\left(\frac{d}{e}\right)^3 - \frac{32cd}{e^2} + \frac{64b}{e}\right] \geq 0.$$

If $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right) > 0$; then, the fourth degree polynomial equation in (eq.31) accepts at least two complex solutions where the imaginary parts are different from zero and dependent on the group of coefficients $\{c, d, e\}$, because:

$$\left(-6\left(\frac{d}{e}\right)^2 + \frac{16c}{e}\right) > 0 \Rightarrow -2[w_0{}^2 + w_1{}^2 + w_2{}^2] > 0$$

$$\Rightarrow w_0 + w_1 + w_2 < 0$$

$$\Rightarrow Re(w_0) < 0 \ or \ Re(w_1) < 0 \ or \ Re(w_2) < 0$$

$$\Rightarrow \left(\sqrt{w_0} \in (\mathbb{C}\backslash\mathbb{R})\right) or \left(\sqrt{w_1} \in (\mathbb{C}\backslash\mathbb{R})\right) or \left(\sqrt{w_2} \in (\mathbb{C}\backslash\mathbb{R})\right)$$

$$\Rightarrow \exists \, w \in (\mathbb{C}\backslash\mathbb{R}) \mid w^4 + M w^2 + Nw + O = 0 \ (Considering \ the \ expressions \ of \ proposed \ solutions \ in \ Theorem \ 3.)$$

$$\Rightarrow \exists \, \{w, \overline{w}\} \in (\mathbb{C}\backslash\mathbb{R}) * (\mathbb{C}\backslash\mathbb{R}) \mid w^4 + M w^2 + Nw + O = 0 \ and \ \overline{w}^4 + M\overline{w}^2 + N\overline{w} + O = 0$$

$$\Rightarrow \exists \, \{x, \overline{x}\} \in (\mathbb{C}\backslash\mathbb{R}) * (\mathbb{C}\backslash\mathbb{R}) \mid ax^4 + bx^3 + cx^2 + dx + e = 0 \ and \ a\overline{x}^4 + b\overline{x}^3 + c\overline{x}^2 + d\overline{x} + e = 0$$

3.7. Sixth Proposed Theorem

In this subsection, we propose a sixth theorem as an extending to Theorem 4 and Theorem 5. This sixth theorem is based on the values of $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right)$ and $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right)$, and it aims to determine whether a fourth degree polynomial equation accepts four complex solutions with imaginary parts different from zero or not.

Theorem 6

Considering the polynomial equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ where all coefficients belong to the group of

numbers $\mathbb{R}$, if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right) > 0$ and $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right) > 0$; then, this fourth degree polynomial equation accepts four complex solutions with imaginary parts different from zero.

3.8. Proof of Theorem 6

Basing on fourth proposed theorem (Theorem 4) where all coefficients of quartic equation belong to the group of numbers $\mathbb{R}$, if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right) > 0$; then, the concerned polynomial equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ accepts at least two complex solutions $\{X_1, \overline{X_1}\} \in (\mathbb{C}\backslash\mathbb{R}) * (\mathbb{C}\backslash\mathbb{R})$. The imaginary parts of solutions $\{X_1, \overline{X_1}\}$ are different from zero and dependent on the value of $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right)$.

Basing on fifth proposed theorem (Theorem 5) where all coefficients of quartic equation belong to the group of numbers $\mathbb{R}$, if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right) > 0$; then, the concerned polynomial equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ accepts at least two complex solutions $\{X_2, \overline{X_2}\} \in (\mathbb{C}\backslash\mathbb{R}) * (\mathbb{C}\backslash\mathbb{R})$. The imaginary parts of solutions $\{X_2, \overline{X_2}\}$ are different from zero and dependent on the value of $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right)$.

The value of $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right)$ is dependent on the group of coefficients $\{b, a, c\}$, whereas the value of $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right)$ is dependent on the group of coefficients $\{d, e, c\}$. Thereby, the imaginary parts of $\{X_1, \overline{X_1}\}$ and $\{X_2, \overline{X_2}\}$ are dependent on different groups of coefficients, which are used to express the values of these imaginary parts.

As a result, we deduce that if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right) > 0$ and $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right) > 0$; then, the fourth degree polynomial equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ accepts four complex solutions $\{X_1, \overline{X_1}, X_2, \overline{X_2}\}$ with imaginary parts different from zero, where the imaginary parts of $\{X_1, \overline{X_1}\}$ are dependent on the value of $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right)$ and the imaginary parts of $\{X_2, \overline{X_2}\}$ are dependent on the value of $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right)$.

4. Conclusion

The proposed theorems in this paper are developed basing on extended logic where Theorem 3 and Theorem 4 are based on Theorem 1 and Theorem 2 successively. The fifth proposed theorem (Theorem 5) is an extending of Theorem 4, whereas Theorem 6 is an extending by conjunction between Theorem 4 and Theorem 5. All proposed theorems are based on the mathematical expressions and calculations in the proofs of Theorem 1 and Theorem 3.

The proposed expressions as solutions for fourth degree polynomial equations under the form $ax^4 + bx^3 + cx^2 +$

$dx + e = 0$ $with$ $a \neq 0$ are developed by replacing x with $\frac{-b}{a}+y}{4}$, and proposing $y = \sqrt{y_0} + \sqrt{y_1} + \sqrt{y_2}$ for $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) \leq 0$ and $y = -\sqrt{y_0} - \sqrt{y_1} - \sqrt{y_2}$ for $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) \geq 0$.

The proposed expressions $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right) = -2[y_0^2 + y_1^2 + y_2^2]$, $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) = -8\sqrt{x_0}\sqrt{x_1}\sqrt{x_2}$ for $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) \leq 0$ and $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) = 8\sqrt{x_0}\sqrt{x_1}\sqrt{x_2}$ for $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) \geq 0$ are helping to pass from a fourth degree polynomial expression in complete form (eq.31) to a third degree polynomial equation (eq.37).

The proposed solutions for fourth degree polynomial equations in simple form and complete form are including quadratic roots and cubic roots as subparts. The first and third proposed theorems enable to calculate the roots of quartic equations nearly simultaneously, whereas the other proposed theorems enable to define how many complex roots from the group of numbers $(\mathbb{C}\backslash\mathbb{R})$ a quartic equation may accept.

All the expressions of proposed roots for quartic equations include cubic roots and quadratic roots as subparts, and the only difference from one proposed quartic root to other three proposed roots for a fourth degree polynomial equation is the signs of included subparts. Thereby, determining the values of these subparts opens the way to calculate the four solutions of concerned quartic polynomial equation nearly simultaneously.

Conflicts of Interest

The author declares no conflicts of interest.

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