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Application of GRA Method in Multi-Attribute Decision Making Problem Base on Picture Fuzzy Choquet Integral Information

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Abstract: Picture fuzzy set (PFS) is the generalize structure over existing structures of fuzzy sets to arrange uncertainty and imprecise information in the decision making problem. Picture fuzzy set is a direct extension of the intuitionistic fuzzy set (IFS) that can model uncertainty in situations with multiple types of answers, such as yes, no, abstain, and refuse. In this paper, we propose Choquet integral (CI) for the picture fuzzy set (PFS). Then, we defined some basic operational laws for picture fuzzy set. Also, based on the defined operational laws, we proposed picture fuzzy averaging (PFA) and picture fuzzy Choquet integral weighted averaging operator (PFCIWAO) along with their basic properties. Also, we propose the normalized Hamming distance and normalized Euclidean distance for the picture fuzzy numbers. Viewing the electiveness of the picture fuzzy set, we proposed a decision-making approach for the multi-criteria decision-making problems. We also propose the GRA method using Choquet integral for dealing uncertainty in decision making problems under picture fuzzy information. Lastly, we illustrate an example show the effectiveness and reliability of the proposed method.

Keywords: IFS, CI, PFS, GRA Method, PFA and PFCIWAO

1. Introduction

Cuong [2, 3] introduced a novel concept of photograph fuzzy set, which dignified in 3 different functions providing the wonderful, neutral, and poor membership degrees. Cuong [4], discussed some feature of PF sets and additionally permitted their distance measures. Cuong and Hai [5] de need rest fuzzy logic operators and implications on PFSs and added precept operations for fuzzy derivation bureaucracy inside the image fuzzy common sense. Cuong, Kreinovich and Ngan [6] examined the feature of photo fuzzy t-norm and t-conorm. Phong et al. [7] explored positive con guration of photo fuzzy family members. Wei et al. [12-14] de need many techniques to compute the closeness among PFSs. Presently, many researchers have advanced more fashions in the PFSs condition: Correlation coefficient of PFS are proposed by way of Sing [8] and use it on clustering analysis. Son et al. Son [9, 10] de need PF separation measures, generalized photo fuzzy distance measures and photo fuzzy affiliation measures, and blended it to address grouping examination below PF unit's condition. Using the picture fuzzy weighted go-entropy idea, Wei [11] studied fundamental management technique and used this method to rank the choices. He additionally de need the notion of advantageous-internal, impartial-inner, terrible-inner, and fantastic-external, neutral-outside, and negative-outside cubic photo fuzzy units. Basic de nations and result about Coquet integral, fuzzy set, Intuitionistic fuzzy set (IFS) and picture fuzzy sets are present in Sec. 2. In Sec. 3, we developed the GRA technique for picture fuzzy MAGDM problems with incomplete weight information. In Sec. 4, we strengthen our developed algorithmic method with a descriptive example.

2. Preliminaries

Definition 1. [15] An FS $\mathfrak{I}_{\breve{u}}$ on a universal set $\mathbb{Z} \neq \phi$ is define as

$$\mathfrak{I} = \{\langle \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathscr{k}) | \mathscr{k} \in \mathbb{Z} \rangle\}.$$

An FS in a set C is indicated by $P_{\mathfrak{I}}(\dot{c}): C \longrightarrow [0, 1]$ are the membership grades of each $c \in C$, and $P_{\mathfrak{I}}(c)$ satisfy the condition that $0 \leq P_{\mathfrak{I}}(c) \dot{\leq} 1$, $\forall\, c \dot{\in} C$.

Definition 2. [1] An IFS $\mathfrak{I}_{\breve{u}}$ on the universe of discourse $\mathbb{Z} \neq \phi$ is defined as,

$$\mathfrak{I}_{\breve{u}} = \{\langle \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathscr{k}), \mathcal{M}_{\breve{e}_{\breve{u}}}(\mathscr{k}) | \mathscr{k} \in \mathbb{Z} \rangle\}.$$

An IFS in a set $\mathbb{Z}$ is defined by $\mathcal{L}_{\breve{e}_{\breve{u}}}(\mathscr{k}): \mathbb{Z} \longrightarrow \Theta$ and $\mathcal{M}_{\breve{e}_{\breve{u}}}(\mathscr{k}): \mathbb{Z} \longrightarrow \Theta$ are the membership grade and membership grade of each $\mathscr{k} \in \mathbb{Z}$, respectively. Furthermore $\mathcal{L}_{\breve{e}_{\breve{u}}}(\mathscr{k})$ and $\mathcal{M}_{\breve{e}_{\breve{u}}}(\mathscr{k})$ satisfy $0 \leq \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathscr{k}) + \mathcal{M}_{\breve{e}_{\breve{u}}}(\mathscr{k}) \leq 1, \forall \mathscr{k} \in \mathbb{Z}$.

Definition 3. [7] A PFS $\mathfrak{I}_{\breve{u}}$ on the universe of discourse $\mathbb{Z} \neq \phi$ is define as,

$$\mathfrak{I}_{\breve{u}} = \{\langle \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathscr{k}), \mathcal{M}_{\breve{e}_{\breve{u}}}(\mathscr{k}), \mathcal{O}_{\breve{e}_{\breve{u}}}(\mathscr{k}) \;| \mathscr{k} \in \mathbb{Z} \rangle\}.$$

A PFS in a set $\mathbb{Z}$ is defined by $\mathcal{L}_{\breve{e}_{\breve{u}}}(\mathscr{k}): \mathbb{Z} \longrightarrow \Theta$, $\mathcal{M}_{\breve{e}_{\breve{u}}}(\mathscr{k}): \mathbb{Z} \longrightarrow \Theta$ and $\mathcal{O}_{\breve{e}_{\breve{u}}}(\mathscr{k}): \mathbb{Z} \longrightarrow \Theta$ are the positive grade, neutral grade, and negative grade of each $\mathscr{k} \in \mathbb{Z}$, respectively. Furthermore $\mathcal{L}_{\breve{e}_{\breve{u}}}(\mathscr{k})$, $\mathcal{M}_{\breve{e}_{\breve{u}}}(\mathscr{k})$ and $\mathcal{O}_{\breve{e}_{\breve{u}}}(\mathscr{k})$ satisfy $0 \leq \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathscr{k}) + \mathcal{M}_{\breve{e}_{\breve{u}}}(\mathscr{k}) + \mathcal{O}_{\breve{e}_{\breve{u}}}(\mathscr{k}) \leq 1, \forall \mathscr{k} \in \mathbb{Z}$.

3. Fuzzy Measure and Choquet Integral

Definition 1. [14] Let $\mathbb{Z} = \{\mathscr{k}_1, \mathscr{k}_2, \ldots\ldots, \mathscr{k}_n\} \neq \phi$ be a universal of non-empty set and $p(\mathbb{Z})$ denote the power set of $\mathbb{Z}$. Then, a function $\mathcal{L}_{\breve{e}_{\breve{u}}}(\mathscr{k}): p(\mathbb{Z}) \longrightarrow \Theta$ is said to be a fuzzy measure $\mathcal{L}_{\breve{e}_{\breve{u}}}$ on $\mathbb{Z}$, if the following conditions are satisfy,

1) $\mathcal{L}_{\breve{e}_{\breve{u}}}(\phi) = 0$, $\mathcal{L}_{\breve{e}_{\breve{u}}}(\mathbb{Z}) = 1$.
2) If $\mathfrak{I}_{\breve{u}_1}$, $\mathfrak{I}_{\breve{u}_2} \in \wp(\mathbb{Z})$ and $\mathfrak{I}_{\breve{u}_1} \subseteq \mathfrak{I}_{\breve{u}_2}$ then $\mathcal{L}_{\breve{e}_{\breve{u}}}(\mathfrak{I}_{\breve{u}_1}) \leq \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathfrak{I}_{\breve{u}_2})$.

It is mandatory to consider the adage of continuity when $\mathbb{Z}$ is in nite, it is enough to assume a nite universe of discourse in exercise. For decision attribute set $\{\mathscr{k}_1, \mathscr{k}_2, \ldots\ldots, \mathscr{k}_n\}$, $\mathcal{L}_{\breve{e}_{\breve{u}}}(\{\mathscr{k}_1, \mathscr{k}_2, \ldots\ldots, \mathscr{k}_n\})$ can be deem as the degree of subjective importance. Thus, weights of any set of attributes can also be obtained with the separate weights of attributes. Instinctively, we say that the following about any pair of criteria sets $\mathfrak{I}_{\breve{u}_1}$, $\mathfrak{I}_{\breve{u}_2} \in \wp(\mathbb{Z})$, $\mathfrak{I}_{\breve{u}_1} \cap \mathfrak{I}_{\breve{u}_2} = \phi$; $\mathfrak{I}_{\breve{u}_1}$ and $\mathfrak{I}_{\breve{u}_2}$ are assumed to be without interaction (or to be independent) and called it additive measure if

$$\mathcal{L}_{\breve{u}_1}(\mathfrak{I}_{\breve{u}_1} \cup \mathfrak{I}_{\breve{u}_2}) = \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathfrak{I}_{\breve{u}_1}) + \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathfrak{I}_{\breve{u}_2}). \quad (1)$$

$\mathfrak{I}_{\breve{u}_1}$ and $\mathfrak{I}_{\breve{u}_2}$ reveals a positive synergetic interaction among them (or are complementary) and called a super additive measure if,

$$\mathcal{L}_{\breve{e}_{\breve{u}}}(\mathfrak{I}_{\breve{u}_1} \cup \mathfrak{I}_{\breve{u}_2}) > \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathfrak{I}_{\breve{u}_1}) + \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathfrak{I}_{\breve{u}_2}) \quad (2)$$

$\mathfrak{I}_{\breve{u}_1}$ and $\mathfrak{I}_{\breve{u}_2}$ called to be a sub-additive measure if

$$\mathcal{L}_{\breve{e}_{\breve{u}}}(\mathfrak{I}_{\breve{u}_1} \cup \mathfrak{I}_{\breve{u}_2}) > \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathfrak{I}_{\breve{u}_1}) + \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathfrak{I}_{\breve{u}_2}) \quad (3)$$

From the dentition 4, therefore, Sagano defined the following measure to confirm a fuzzy measure in MAGDM problems:

$$\mathcal{L}_{\breve{u}_1}(\mathfrak{I}_{\breve{u}_1} \cup \mathfrak{I}_{\breve{u}_2}) = \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathfrak{I}_{\breve{u}_1}) + \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathfrak{I}_{\breve{u}_2}) + \lambda \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathfrak{I}_{\breve{u}_1}) \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathfrak{I}_{\breve{u}_2}) \quad (4)$$

$\lambda \in [-1, \infty)$, $\mathfrak{I}_{\breve{u}_1} \cap \mathfrak{I}_{\breve{u}_2 = \emptyset}$. The interaction between the attributes is determine by the parameter λ. Simply an additive measure is obtained when $\lambda = 0$ in Equation 4. Meantime, if all the elements in $\mathbb{Z}$ are independent, and we have

$$\mathcal{L}_{\breve{e}_{\breve{u}}}(\mathfrak{I}_{\breve{u}}) = \sum_{p=1}^{n} \mathcal{L}_{\breve{e}_{\breve{u}}}(\{\mathscr{k}_p\}) \quad (5)$$

If $\mathbb{Z}$ is a finite set, then $\bigcup_{p=1}^{n} \mathscr{k}_p = \mathbb{Z}$. The λ-fuzzy measure $\mathcal{L}_{\breve{e}_{\breve{u}}}$ satisfies the following,

$$\mathcal{L}_{\breve{e}_{\breve{u}}}(\mathbb{Z}) = \mathcal{L}_{\breve{e}_{\breve{u}}}(\bigcup_{p=1}^{n} \mathscr{k}_i) = \begin{cases} \frac{1}{\lambda}\left(\prod_{p=1}^{n}[1 + \lambda \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathscr{k}_p)] - 1\right) if\ \lambda \neq 0 \\ \sum_{p=1}^{n} \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathscr{k}_p)\ if\ \lambda = 0 \end{cases} \quad (6)$$

Where $\mathscr{k}_p \cap r_d = \phi$ for all $p, d = 1, \ldots\ldots, n$ and $p \neq d$. Noted that $\mathcal{L}_{\breve{e}_{\breve{u}}}(\mathscr{k}_p)$ a single member $\mathscr{k}_p$ is said to be a fuzzy density, and can be signified as $\mathcal{L}_{\breve{e}_{\breve{u}}} = \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathscr{k}_p)$.

Particularly for every subset $\mathfrak{I}_{\breve{u}_1} \in p(\mathbb{Z})$, we have

$$\mathcal{L}_{\breve{e}_{\breve{u}}}(\mathfrak{I}_{\breve{u}_1}) = \begin{cases} \frac{1}{\lambda}\left(\prod_{p=1}^{n}[1 + \lambda \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathscr{k}_p)] - 1\right) if\ \lambda \neq 0 \\ \sum_{p=1}^{n} \mathcal{L}_{\breve{e}_{\breve{u}}}(\mathscr{k}_p)\ if\ \lambda = 0 \end{cases} \quad (7)$$

A uniquely value of λ is determined from $\mathcal{L}_{\check{e}_{\check{u}}}(\mathbb{Z}) = 1$, based on Equation 2 as,

$$\lambda + 1 = \prod_{p=1}^{n}[1 + \lambda\mathcal{L}_{\check{e}_{\check{u}}}] \tag{8}$$

It can be seen that λ are uniquely obtained by $\mathcal{L}_{\check{e}_{\check{u}}}(\mathbb{Z}) = 1$.

Definition 5. [14] Let g and $\mathcal{L}_{\check{e}_{\check{u}}}$ be a positive real-valued function on the fuzzy measure $\mathbb{Z}$, respectively. Then, the discrete Choquet integral of g with respect to $\mathcal{L}_{\check{e}_{\check{u}}}$ is defined by

$$C_{\mu}(g) = \sum_{p=1}^{n} q_{\sigma(p)}\left[\mathcal{L}_{\check{e}_{\check{u}}}\left(A_{\sigma(p)}\right) - \mathcal{L}_{\check{e}_{\check{u}}}\left(A_{\sigma(p-1)}\right)\right] \tag{9}$$

Where $\sigma(p)$ shows a permutation on $\mathbb{Z}$ such that $g_{\sigma(1)} \geq g_{\sigma(2)} \geq \ldots \geq g_{\sigma(n)}$, $A_{\sigma(p)} = \{1, 2, \ldots\ldots, p\}$, $A_{\sigma(0)} = \emptyset$.

Definition 6. Let $\mathfrak{J}_{\check{u}_1} = \langle\mathcal{L}_{\check{e}_{\check{u}}}, \mathcal{M}_{\check{e}_{\check{u}}}, \mathcal{O}_{\check{e}_{\check{u}}}\rangle$ and $\mathfrak{J}_{\check{u}_2} = \langle\mathcal{L}_{\check{e}_{\check{u}}}, \mathcal{M}_{\check{e}_{\check{u}}}, \mathcal{O}_{\check{e}_{\check{u}}}\rangle$ are two PFNs define on the universe of discovers $\mathbb{Z} \neq \phi$, some operations on SFNs are defined as follows:

$$\mathfrak{J}_{\check{u}_1} \subseteq \mathfrak{J}_{\check{u}_2}\ iff\ \forall\, r \in \mathcal{R}, \mathcal{L}_{\check{e}_{\check{u}}} \leq \mathfrak{J}_{\check{u}_2}, \mathcal{M}_{\check{e}_{\check{u}}} \leq \mathcal{M}_{\check{e}_{\check{u}}} \text{ and } \mathcal{O}_{\check{e}_{\check{u}}} \geq \mathcal{O}_{\check{e}_{\check{u}}} \tag{10}$$

$$\mathfrak{J}_{\check{u}_1} \subseteq \mathfrak{J}_{\check{u}_2}\ iff\ \mathfrak{J}_{\check{u}_1} \subseteq \mathfrak{J}_{\check{u}_2} \text{ and } \mathfrak{J}_{\check{u}_2} \subseteq \mathfrak{J}_{\check{u}_1} \tag{11}$$

Union

$$\mathfrak{J}_{\check{u}_1} \cup \mathfrak{J}_{\check{u}_2} = \langle\max\left(\mathcal{L}_{\check{e}_{\check{u}}}, \mathcal{L}_{\check{e}_{\check{u}}}\right), min\left(\mathcal{M}_{\check{e}_{\check{u}}}, \mathcal{M}_{\check{e}_{\check{u}}}\right), min\left(\mathcal{O}_{\check{e}_{\check{u}}}, \mathcal{O}_{\check{e}_{\check{u}}}\right)\rangle; \tag{12}$$

Intersection

$$\mathfrak{J}_{\check{u}_1} \cap \mathfrak{J}_{\check{u}_2} = \langle\min\left(\mathcal{L}_{\check{e}_{\check{u}}}, \mathcal{L}_{\check{e}_{\check{u}}}\right), min\left(\mathcal{M}_{\check{e}_{\check{u}}}, \mathcal{M}_{\check{e}_{\check{u}}}\right), max\left(\mathcal{O}_{\check{e}_{\check{u}}}, \mathcal{O}_{\check{e}_{\check{u}}}\right)\rangle \tag{13}$$

Compliment

$$\mathfrak{J}_{\check{u}}^{c} = \langle\mathcal{O}_{\check{e}_{\check{u}}}, \mathcal{M}_{\check{e}_{\check{u}}}, \mathcal{L}_{\check{e}_{\check{u}}}\rangle. \tag{14}$$

Definition 7. Let $\mathfrak{J}_1 = \langle P_{\mathfrak{J}_1}, \mathcal{I}_{\mathfrak{J}_1}, \mathcal{N}_{\mathfrak{J}_1}\rangle$ and $\mathfrak{J}_2 = \langle P_{\mathfrak{J}_2}, \mathcal{I}_{\mathfrak{J}_2}, \mathcal{N}_{\mathfrak{J}_2}\rangle$ are two PFNs define $\mathcal{C} \neq \phi$, few operations on PFNs are defined as follows with $\tau \geq 0$.

(1) $\mathfrak{J}_1 \oplus \mathfrak{J}_2 = \left\{P_{\mathfrak{J}_1} + P_{\mathfrak{J}_2} - P_{\mathfrak{J}_1} \cdot P_{\mathfrak{J}_2}, \mathcal{I}_{\mathfrak{J}_1}\mathcal{I}_{\mathfrak{J}_2}, \mathcal{N}_{\mathfrak{J}_1} \cdot \mathcal{N}_{\mathfrak{J}_2}\right\}$

(2) $\tau \cdot \mathfrak{J}_1 = \left\{1 - (1 - P)^{\tau}, \left(\mathcal{I}_{\mathfrak{J}_1}\right)^{\tau}, \left(\mathcal{N}_{\mathfrak{J}_1}\right)^{\tau}\right\}$;

(3) $\mathfrak{J}_1 \otimes \mathfrak{J}_2 = \left\{P_{\mathfrak{J}_1} \cdot P_{\mathfrak{J}_2}, \mathcal{I}_{\mathfrak{J}_1} \cdot \mathcal{I}_{\mathfrak{J}_1}, \mathcal{N}_{\mathfrak{J}_1} + \mathcal{N}_{\check{e}_{\check{u}_2}} - \mathcal{N}_{\mathfrak{J}_1} \cdot \mathcal{N}_{\mathfrak{J}_1}\right\}$;

(4) $(\mathfrak{J}_1)^{\tau} = \left\{\left(p_{\mathfrak{J}_1}\right)^{\tau}, \left(\mathcal{I}_{\mathfrak{J}_1}\right)^{\tau}, 1 - \left(1 - \mathcal{N}_{\mathfrak{J}_1}\right)^{\tau}\right\}$.

Comparison Rules for PFNs. For ranking the PFNs, different function is introduced in this section described as.

Definition 8. Let $\mathfrak{J}_{\check{u}} = \langle\mathcal{O}_{\check{e}_{\check{u}}}, \mathcal{M}_{\check{e}_{\check{u}}}, \mathcal{L}_{\check{e}_{\check{u}}}\rangle$ be any PFNs. Then

(1) Score function is defined as $sc(\mathfrak{J}_{\check{u}}) = \frac{\left(\mathcal{L}_{\check{e}_{\check{u}}} + 1 - \mathcal{M}_{\check{e}_{\check{u}}} + 1 - \mathcal{O}_{\check{e}_{\check{u}}}\right)}{3} = \frac{1}{3}\left(2 + \mathcal{L}_{\check{e}_{\check{u}}} - \mathcal{M}_{\check{e}_{\check{u}}} - \mathcal{O}_{\check{e}_{\check{u}}}\right)$.

(2) Accuracy function is defined as $acu(\mathfrak{J}_{\check{u}}) = \mathcal{L}_{\check{e}_{\check{u}}} - \mathcal{O}_{\check{e}_{\check{u}}}$.

(3) Certainty function is defined as $cr(\mathfrak{J}_{\check{u}}) = \mathcal{L}_{\check{e}_{\check{u}}}$.

Ranking of PFNs described from definition 8.

Definition 9. Let $\mathfrak{J}_{\check{u}_1} = \langle\mathcal{O}_{\check{e}_{\check{u}}}, \mathcal{M}_{\check{e}_{\check{u}}}, \mathcal{L}_{\check{e}_{\check{u}}}\rangle$ and $\mathfrak{J}_{\check{u}_2} = \langle\mathcal{O}_{\check{e}_{\check{u}}}, \mathcal{M}_{\check{e}_{\check{u}}}, \mathcal{L}_{\check{e}_{\check{u}}}\rangle$ are two PFNs define on the universe of discourse $\mathbb{Z} \neq \phi$. Then Ranking of PFNs described from definition 8,

(1) If $sc\left(\mathfrak{J}_{\check{u}_1}\right) \succ sc\left(\mathfrak{J}_{\check{u}_2}\right)$, then $\mathfrak{J}_{\check{u}_1} \succ \mathfrak{J}_{\check{u}_2}$.

(2) If $sc\left(\mathfrak{J}_{\check{u}_1}\right) \approx sc\left(\mathfrak{J}_{\check{u}_2}\right)$, and $acu\left(\mathfrak{J}_{\check{u}_1}\right) \succ acu\left(\mathfrak{J}_{\check{u}_2}\right)$, then $\mathfrak{J}_{\check{u}_1} \succ \mathfrak{J}_{\check{u}_2}$.

(3) If $sc\left(\mathfrak{J}_{\check{u}_1}\right) \approx sc\left(\mathfrak{J}_{\check{u}_2}\right)$, $acu\left(\mathfrak{J}_{\check{u}_1}\right) \approx acu\left(\mathfrak{J}_{\check{u}_2}\right)$ and $cr\left(\mathfrak{J}_{\check{u}_1}\right) \succ cr\left(\mathfrak{J}_{\check{u}_2}\right)$, then $\mathfrak{J}_{\check{u}_1} \succ \mathfrak{J}_{\check{u}_2}$.

(4) If $sc\left(\mathfrak{J}_{\check{u}_1}\right) \approx sc\left(\mathfrak{J}_{\check{u}_2}\right)$, $acu\left(\mathfrak{J}_{\check{u}_1}\right) \approx acu\left(\mathfrak{J}_{\check{u}_2}\right)$ and $cr\left(\mathfrak{J}_{\check{u}_1}\right) \approx cr\left(\mathfrak{J}_{\check{u}_2}\right)$, then $\mathfrak{J}_{\check{u}_1} \approx \mathfrak{J}_{\check{u}_2}$.

Definition 10. Let any collections $\mathfrak{J}_{\check{u}_p} = \langle\mathcal{L}_{\check{e}_{\check{u}}}, \mathcal{M}_{\check{e}_{\check{u}}}, \mathcal{O}_{\check{e}_{\check{u}}}\rangle$, $p \in \mathcal{N}$ be the PFNs and $PFWA: PFN^{n} \times PFN^{n} \longrightarrow PFN$, then PFWA describe as,

$$PFWA\left(\mathfrak{J}_{\check{u}_1}, \mathfrak{J}_{\check{u}_2}, \ldots\ldots, \mathfrak{J}_{\check{u}_n}\right) = \sum_{p=1}^{n} \tau_p\, \mathfrak{J}_{\check{u}_p}, \tag{15}$$

In which $\tau = (\tau_1, \ldots\ldots, \tau_n)^T$ be the weight vector of $\mathfrak{J}_{\check{u}_p} = \langle\mathcal{L}_{\check{e}_{\check{u}}}, \mathcal{M}_{\check{e}_{\check{u}}}, \mathcal{O}_{\check{e}_{\check{u}}}\rangle$, $p \in \mathcal{N}$, with $\tau_p \geq 0$ and $\sum_{p=1}^{n} \tau_p = 1$.

Theorem 1. Let any collections $\mathfrak{J}_{\check{u}_p} = \langle\mathcal{L}_{\check{e}_{\check{u}}}, \mathcal{M}_{\check{e}_{\check{u}}}, \mathcal{O}_{\check{e}_{\check{u}}}\rangle$, $p \in \mathcal{N}$ be the PFNs. Then operational properties of PFNs can obtained by utilizing the definition as,

$$PFWA(\Im_{\breve{u}_1}, \Im_{\breve{u}_2}, \dots\dots, \Im_{\breve{u}_n}) = \begin{Bmatrix} 1 - \prod_{p=1}^{n}(1 - \mathcal{L}_{\breve{e}_{\breve{u}}})^{\tau p}, \\ \prod_{p=1}^{n}(\mathcal{M}_{\breve{e}_{\breve{u}}})^{\tau p}, \\ \prod_{p=1}^{n}(\mathcal{O}_{\breve{e}_{\breve{u}}})^{\tau p} \end{Bmatrix} \tag{16}$$

Definition 11. Let any collections $\Im_{\breve{u}_p} = \langle \mathcal{L}_{\breve{e}_{\breve{u}}}, \mathcal{M}_{\breve{e}_{\breve{u}}}, \mathcal{O}_{\breve{e}_{\breve{u}}} \rangle, p \in \mathcal{N}$ be the PFNs and $PFOWA: PFN^n \times PFN^n \longrightarrow PFN$, then PFOWA describe as,

$$PFWA(\Im_{\breve{u}_1}, \Im_{\breve{u}_2}, \dots\dots, \Im_{\breve{u}_n}) = \sum_{p=1}^{n} \tau_p \Im_{\breve{u}_{\sigma(p)}} \tag{17}$$

In which $\tau = \{\tau_1, \dots\dots, \tau_n\}$ the weight vector of $\Im_{\breve{u}_p} = \langle \mathcal{L}_{\breve{e}_{\breve{u}}}, \mathcal{M}_{\breve{e}_{\breve{u}}}, \mathcal{O}_{\breve{e}_{\breve{u}}} \rangle$, $p \in \mathcal{N}$, with $\tau_p \geq 0$ and $\sum_{p=1}^{n} \tau_p = 1$, $\sigma(p)$ present a permutation on $\mathbb{Z}$.

Theorem 2. Let any collections $\Im_{\breve{u}_p} = \langle \mathcal{L}_{\breve{e}_{\breve{u}}}, \mathcal{M}_{\breve{e}_{\breve{u}}}, \mathcal{O}_{\breve{e}_{\breve{u}}} \rangle$, $p \in \mathcal{N}$ be the PFNs. Then operational properties of PFNs can obtained by utilizing the definition as,

$$PFOWA(\Im_{\breve{u}_1}, \Im_{\breve{u}_2}, \dots\dots, \Im_{\breve{u}_n}) = \begin{Bmatrix} 1 - \prod_{p=1}^{n}(1 - \mathcal{L}_{\breve{e}_{\breve{u}}})^{\tau p}, \\ \prod_{p=1}^{n}(\mathcal{M}_{\breve{e}_{\breve{u}}})^{\tau p}, \\ \prod_{p=1}^{n}(\mathcal{O}_{\breve{e}_{\breve{u}}})^{\tau p} \end{Bmatrix} \tag{18}$$

Theorem 3. Let any collections $\Im_{\breve{u}_p} = \langle \mathcal{L}_{\breve{e}_{\breve{u}}}, \mathcal{M}_{\breve{e}_{\breve{u}}}, \mathcal{O}_{\breve{e}_{\breve{u}}} \rangle$, $p \in \mathcal{N}$ be the PFNs and λ be a fuzzy measure on $\mathbb{Z}$. A picture fuzzy Choquet integral weighted averaging (PFCIWA) operator as,

$$PFCIWA: PFN^n \times PFN^n \longrightarrow PFN \text{ such that, } PFCIWA(\Im_{\breve{u}_1}, \Im_{\breve{u}_2}, \dots\dots, \Im_{\breve{u}_n}) = \begin{Bmatrix} 1 - \prod_{p=1}^{n}(1 - \mathcal{L}_{\breve{e}_{\breve{u}}})^{\lambda(A_{\sigma(p)}) - \lambda(A_{\sigma(p-1)})}, \\ \prod_{p=1}^{n}(\mathcal{M}_{\breve{e}_{\breve{u}}})^{\lambda(A_{\sigma(p)}) - \lambda(A_{\sigma(p-1)})}, \\ \prod_{P=1}^{n}\left(\mathcal{O}_{\breve{e}_{\breve{u}_{\sigma(p)}}}\right)^{\lambda(A_{\sigma(p)}) - \lambda(A_{\sigma(P-1)})} \end{Bmatrix} \tag{19}$$

$\sigma(p)$ show a permutation on $\mathbb{Z}$ and $A_{\sigma(n)} = \{1, \dots\dots, p\}$, $A_{\sigma(0)} = \phi$.

Definition 12. Let $\mathbb{Z} \neq \phi$ be the universal set and any two picture fuzzy sets $\Im_j, \Im_\ell$. Then normalized Hamming distance $f_{NHD}(\Im_j, \Im_\ell)$ is given as for all $k \in \mathbb{Z}$,

$$f_{NHD}(\Im_j, \Im_\ell) = \frac{1}{n}\sum_{p=1}^{n}\left(\left|\mathcal{L}_{\Im_j}(k_P) - \mathcal{L}_{\Im_\ell}(k_p)\right| + \left|\mathcal{M}_{\Im_j}(k_p) - \mathcal{M}_{\Im_\ell}(k_p)\right| + \left|\mathcal{O}_{\Im_j}(k_p) - \mathcal{O}_{\Im_\ell}(k_p)\right|\right) \tag{20}$$

Definition 13. Let $\mathbb{Z} \neq \phi$ be the universal set and any two picture fuzzy sets $\Im_j$, $\Im_\ell$. Then normalized Euclidean distance $f_{NHD}(\Im_j, \Im_\ell)$ is given as for all $k \in \mathbb{Z}$,

$$f_{NHD}(\Im_j, \Im_\ell) = \sqrt{\frac{1}{n}\sum_{p=1}^{n}\begin{pmatrix} \left(\mathcal{L}_{\Im_j}(k_p) - \mathcal{L}_{\Im_\ell}(k_p)\right)^2 + \\ \left(\mathcal{M}_{\Im_j}(k_p) - \mathcal{M}_{\Im_\ell}(k_p)\right)^2 + \\ \left(\mathcal{O}_{\Im_j}(k_p) - \mathcal{O}_{\Im_\ell}(k_P)\right)^2 \end{pmatrix}}.$$

GRA Approach for Multi-Attribute Decision Making with incomplete weight information in picture fuzzy setting

Suppose that $A = \{b_1, \dots\dots, b_n\}$, n alternatives and $C = \{c_1, \dots\dots, c_m\}$, m alternatives, weight vector for parameter is $V = (v_1, \dots\dots, v_m)$, where $v_k \geq 0$ $(k = 1, \dots\dots, m)$, $\sum_{k=1}^{n} v_k = 1$.

Assume that the decision maker gives their information approximately the weights of criteria are denoted as, for $j \neq k$,

(1) If $\{v_j \geq v_k\}$ (weak ranking)

(2) If $\{v_j - v_k \geq \lambda_j (> 0)\}$, (strict ranking).

(3) If $\{v_j \geq \lambda_j v_k\}$, $0 \leq \lambda_j \leq 1$, (multiple ranking).

(4) If $\{\lambda_j \leq v_j \leq \lambda_j + \delta_j\}$, $0 \leq \lambda_j \leq \lambda_j + \delta_j \leq 1$, (interval ranking)

The decision maker f_k $(k = 1, \dots, l)$ give the following decision matrix.

$$R^k = \left[\Im_{\breve{u}_{pq}}^{(k)}\right]_{m \times n} = \begin{array}{c|cccc} & d_1 & d_2 & \cdots & d_n \\ \hline b_1 & \Im_{u_{11}}^{(k)} & \cdot & \cdots & \Im_{u_{1n}}^{(k)} \\ b_2 & \Im_{u_{21}}^{(k)} & \cdot & \cdots & \Im_{u_{21}}^{(k)} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ b_m & \Im_{u_{m1}}^{(k)} & \cdot & \cdots & \Im_{u_{mn}}^{(k)} \end{array}$$

Where $\Im_{\breve{u}_{pq}}^{(k)} = \left(\mathcal{L}_{\breve{u}_{pq}}^{(k)}, \mathcal{M}_{\breve{u}_{pq}}^{(k)}, \mathcal{O}_{\breve{u}_{pq}}^{(k)}\right)$ is an PFN representing the perform rating of the alternative $a_p \in A$ with the attribute $c_p \in C$ provided by the decision makers d_k.

To apply GRA approach at the organization selection making hassle, first we used PFCIW operator to discover individual decision matrices into a collective selection matrix.

Step 1: Suppose that we have m alternative, $A = \{b_1, \dots\dots, b_n\}$, and n attributes $C_q (q = 1, \dots, n)$, now we invited each expert $d_k (k = 1, \dots, r)$ to express their individual preference according to each by an picture fuzzy numbers $\Im_{\breve{u}_{pq}}^{(k)} = \left(\mathcal{L}_{\breve{u}_{pq}}^{(k)}, \mathcal{M}_{\breve{u}_{pq}}^{(k)}, \mathcal{O}_{\breve{u}_{pq}}^{(k)}\right)$ $(p = 1, \ldots, m;\ q = 1, \ldots, n, r = 1, \ldots, k)$ expressed by f_r. Then, we get a decision-making matrix, $D^S = \left[E_{iP}^{(S)}\right]_{m\times n} (s = 1, \ldots, r)$ for decision. But there are 2 kinds of criteria, as benefit and cost criteria, such as benefit and cost criteria, then we convert the decision matrices, $D^S = \left[E_{iP}^{(S)}\right]_{m\times n}$ into the normalized picture fuzzy decision matrices, $R^r = \left[\Im_{\breve{u}_{pq}}^{(r)}\right]_{m\times n}$, by the following rules; $\Im_{\breve{u}_{pq}}^{(r)} = \left\{\begin{matrix} \Im_{\breve{u}_{pq}}^{r}, for\ benefit\ criteria\ A_p \\ \overline{\Im_{\breve{u}_{pq}}^{(r)}},\ for\ cost\ crieria\ A_p \end{matrix}\right\}$, $j = 1, \ldots, n$, and $\overline{\Im_{\breve{u}_{pq}}^{(r)}}$ is the complement of $\Im_{\breve{u}_{pq}}^{(r)}$. If the criteria are the same type, then normalization are not required. Then, the following decision matrix is obtained:

$$\begin{array}{c|cccc} & d_1 & d_2 & \cdots & d_n \\ \hline b_1 & \Im_{u_{11}}^{(k)} & \cdot & \cdots & \Im_{u_{1n}}^{(k)} \\ b_2 & \Im_{u_{21}}^{(k)} & \cdot & \cdots & \Im_{u_{21}}^{(k)} \\ \vdots & \vdots & & \ddots & \vdots \\ b_m & \Im_{u_{m1}}^{(k)} & \cdot & \cdots & \Im_{u_{mn}}^{(k)} \end{array}$$

Step 2: Confirm the fuzzy density $\mathcal{L}_{\breve{e}_{\breve{u}}} = \mathcal{L}_{\breve{e}_{\breve{u}}}(a_p)$. From Eq. (8), parameter λ_1 of expert can be determined.

Step 3: $\Im_{\breve{u}_{pq}}^{(r)}$ is reordered such that $\Im_{\breve{u}_{pq}}^{(r)} \geq \Im_{\breve{u}_{pq}}^{(r-1)}$. Using the PF Choquet integral average operator.

$$PFCIWA\left(\Im_{\breve{u}_{pq}}^{(1)}, \Im_{\breve{u}_{pq}}^{(2)}, \ldots, \Im_{\breve{u}_{pq}}^{(r)}\right) = \left\{\begin{matrix} 1 - \prod_{p=1}^{r}\left(1 - \mathcal{L}_{\breve{e}_{\breve{u}}}\right)^{\lambda(A_{\sigma(p)}) - \lambda(A_{\sigma(p-1)})}, \\ \prod_{p=1}^{r}\left(\mathcal{M}_{\breve{e}_{\breve{u}}}\right)^{\lambda(A_{\sigma(p)}) - \lambda(A_{\sigma(P-1)})}, \\ \prod_{p=1}^{r}\left(\mathcal{O}_{\breve{e}_{\breve{u}\sigma(p)}}\right)^{\lambda(A_{\sigma(p)}) - \lambda(A_{\sigma(p-1)})} \end{matrix}\right\} \quad (21)$$

to aggregate all the picture fuzzy decision matrices $R^r = \left[\Im_{\breve{u}_{pq}}^{(r)}\right]_{m\times n}$ $(r = 1, \ldots, k)$ into a collective picture fuzzy decision matrix $R = \left[\Im_{\breve{u}_{pq}}^{(r)}\right]_{m\times n}$ where $\Im_{\breve{u}_{pq}}^{(r)} = \left(\mathcal{L}_{\breve{u}_{pq}}^{(r)}, \mathcal{M}_{\breve{u}_{pq}}^{(r)}, \mathcal{O}_{\breve{u}_{pq}}^{(r)}\right)$ $(p = 1, \ldots, m;\ q = 1, \ldots, n, r = 1, \ldots, k)$, $\sigma(p)$ show permutation on $\mathbb{Z}$ and $A_{\sigma(n)} = \{1, \ldots, p\}$, $A_{\sigma(0)} = \phi$ and $\mathcal{L}_{\breve{e}_{\breve{u}}}(a_p)$ can be calculated by Eq. (2.9).

Step 4: $L^+ = \{L_1^+, L_2^+, \ldots, L_m^+\}$ and $P^- = \{P_1^-, P_2^-, \ldots, P_m^-\}$ are the PFPIS and PFNIS, respectively.

$$L_p^+ = \max_q sc_{pq} \quad (22)$$

And

$$P_p^- = \min_q sc_{pq}, \quad (23)$$

Where $L^+ = \left(L_{\breve{u}_p}^+, I_{\breve{u}_p}^+, N_{\breve{u}_p}^+\right)$ and $P^- = \left(P_{\breve{u}_p}^-, I_{\breve{u}_p}^-, N_{\breve{u}_p}^-\right)$ $p = 1, \ldots, m$

Step 5: Calculate the distance between the alternative a_P and the PEPIS L^+, and PFNIS P^-.

$$f(e_j, e_k) = \frac{1}{n}\sum_{p=1}^{n}\left(\left|P_{ej}(a_P) - P_{ek}(a_p)\right| + \left|I_{ej}(a_p) - I_{ek}(a_p)\right| + \left|N_{ej}(a_P) - N_{ek}(a_P)\right|\right). \quad (24)$$

This is known to be Normalized Hamming distance $d(e_j, e_k)$, and construct matrix D^+ and matrix D^- as,

$$\begin{array}{cccc} f\left(\Im_{u_{11}}, L_1^+\right) & f\left(\Im_{u_{12}}, L_2^+\right) & \cdots & f\left(\Im_{u_{1n}}, L_n^+\right) \\ \hline f\left(\Im_{u_{21}}, L_1^+\right) & f\left(\Im_{u_{22}}, L_2^+\right) & \cdots & f\left(\Im_{u_{1n}}, L_n^+\right) \\ \vdots & \vdots & \ddots & \vdots \\ f\left(\Im_{u_{m1}}, L_1^+\right) & f\left(\Im_{u_{m2}}, L_2^+\right) & \cdots & f\left(\Im_{u_{mn}}, L_n^+\right) \end{array} \quad (25)$$

and

$$\begin{array}{cccc} f\left(\Im_{u_{11}}, P_1^-\right) & f\left(\Im_{u_{12}}, P_2^-\right) & \cdots & f\left(\Im_{u_{1n}}, P_n^-\right) \\ \hline f\left(\Im_{u_{21}}, L_1^+\right) & f\left(\Im_{u_{22}}, P_2\right) & \cdots & f\left(\Im_{u_{1n}}, P_n^-\right) \\ \vdots & \vdots & \ddots & \vdots \\ f\left(\Im_{u_{m1}}, P_1\right) & f\left(\Im_{u_{m2}}, P_2\right) & \cdots & f\left(\Im_{u_{mn}}, P_n^-\right) \end{array} \quad (26)$$

Step 6: From PIS and NIS by means of using the following equation.

$$\xi_{pq}^{+} = \frac{min_{1\leq p\leq m} min_{1\leq q\leq n} d\left(\Im_{\breve{u}_{pq}}, L_p^{+}\right) + \sigma max_{1\leq p\leq m} max_{1\leq q\leq n} d\left(\Im_{\breve{u}_{pq}}, L_p^{+}\right)}{d\left(\Im_{\breve{u}_{pq}}, L_p^{+}\right) + \sigma max_{1\leq p\leq m} max_{1\leq q\leq n} d\left(\Im_{\breve{u}_{Pq}}, L_p^{+}\right)}. \tag{27}$$

Where $p = 1, \ldots., m;\ q = 1, \ldots, n$. Same, the grey coefficient of each alternative calculated from NIS is provided as

$$\xi_{pq}^{-} = \frac{min_{1\leq p\leq m} min_{1\leq q\leq n} d\left(\Im_{\breve{u}_{pq}}, p_p^{-}\right) + \sigma max_{1\leq p\leq m} max_{1\leq q\leq n} d\left(\Im_{\breve{u}_{pq}}, P_p^{-}\right)}{d\left(\Im_{\breve{u}_{pq}}, p_p^{-}\right) + \sigma max_{1\leq p\leq m} max_{1\leq q\leq n} d\left(\Im_{\breve{u}_{pq}}, p_P^{-}\right)} \tag{28}$$

Where $p = 1, \ldots., m;\ q = 1, \ldots, n$ and the identification coefficient $\sigma = 0.5$.

Step 7: For each opportunity from PIS and NIS through using respectively,

$$\xi_p^{+} = \sum_{q=1}^{n} v_q \xi_{pq}^{+} \tag{29}$$

$$\xi_p^{-} = \sum_{q=1}^{n} v_q \xi_{pq}^{-}$$

Obviously, for the weights are acknowledged, the smaller $\xi_{\wp}^{-}$ and the larger $\xi_{\wp}^{+}$, the finest alternative $a_{\wp}$. But incomplete records about weights of alternatives is thought. So, on this condition the ξ_p^{-} and ξ_p^{+}, information about weight calculated initially.

$$(OM1)\begin{Bmatrix} min\xi_p^{-} = \sum_{q=1}^{n} v_q \xi_{pq}^{-}, P = 1, \ldots., m \\ max\xi_p^{+} = \sum_{q=1}^{n} v_q \xi_{pq}^{+}, P = 1, \ldots., m \end{Bmatrix} \tag{30}$$

$$(OM2)\left\{ min\xi_p = \sum_{p=1}^{m}\sum_{q=1}^{n}\left(\xi_{pq}^{-} - \xi_{pq}^{+}\right) v_q \right\} \tag{31}$$

To finding the solution of OM2, we obtain optimal solution $v = (v_1, \ldots\ldots, v_m)$, using the weight information's of provided alternatives. Then, we obtain $\xi_{\wp}^{+}(P = 1, \ldots\ldots, m)$ and $\xi_{\wp}^{-}(P = 1, \ldots\ldots, m)$ as utilizing above formula, respectively.

Step 8: From the PIS and NIS, find the relative degree, using the following equation.

$$\xi_p = \frac{\xi_p^{+}}{\xi_p^{-} + \xi_p^{+}} (P = 1, \ldots\ldots, m) \tag{32}$$

Step 9: According to the $\xi_p(P = 1, \ldots\ldots, m)$ value, select the best one.

4. Descriptive Example

Example 1. Suppose there is a panel with four possible emerging technology enterprises $\check{Z}_i (i = 1, 2, 3, 4)$ to select. There are three experts, selects four attributes to evaluate the four possible emerging technology enterprises,

(1) Technical advancement is denoted by $\overline{A}_1$;
(2) Potential market risk is denoted by $\overline{A}_2$;
(3) Industrialization infrastructure, human resources and economic circumstance is denoted by means of $\overline{A}_3$;
(4) Employment creation and the improvement of technology and technology is denoted by $\overline{A}_4$.

Step 1: From the results received with every rising era business enterprise, the three experts imparting their personal evaluations which are shown in tables 1-3.

Table 1. *Picture fuzzy information D^1.*

	$\overline{A}_1$	$\overline{A}_2$	$\overline{A}_3$	$\overline{A}_4$
Z_1	(0.2, 0.1, 0.6)	(0.5, 0.3, 0.1)	(0.3, 0.1, 0.5)	(0.4, 0.3, 0.2)
Z_2	(0.1, 0.4, 0.4)	(0.6, 0.2, 0.1)	(0.2, 0.2, 0.5)	(0.2, 0.1, 0.6)
Z_3	(0.2, 0.3, 0.3)	(0.4, 0.3, 0.2)	(0.3, 0.1, 0.4)	(0.3, 0.2, 0.4)
Z_4	(0.3, 0.1, 0.6)	(0.3, 0.2, 0.4)	(0.1, 0.3, 0.5)	(0.2, 0.3, 0.3)

Table 2. *Picture fuzzy information D^2.*

	$\overline{A}_1$	$\overline{A}_2$	$\overline{A}_3$	$\overline{A}_4$
Z_1	(0.1, 0.3, 0.5)	(0.4, 0.3, 0.2)	(0.1, 0.1, 0.6)	(0.2, 0.3, 0.4)
Z_2	(0.2, 0.2, 0.4)	(0.4, 0.3, 0.2)	(0.3, 0.2, 0.4)	(0.4, 0.1, 0.4)
Z_3	(0.1, 0.2, 0.6)	(0.6, 0.1, 0.1)	(0.2, 0.2, 0.4)	(0.5, 0.2, 0.2)
Z_4	(0.4, 0.1, 0.5)	(0.5, 0.1, 0.3)	(0.3, 0.3, 0.3)	(0.6, 0.2, 0.1)

Table 3. *Picture fuzzy information D^3.*

	$\overline{A}_1$	$\overline{A}_2$	$\overline{A}_3$	$\overline{A}_4$
Z_1	(0.3, 0.1, 0.3)	(0.4, 0.2, 0.1)	(0.2, 0.3, 0.4)	(0.5, 0.2, 0.1)
Z_2	(0.1, 0.5, 0.3)	(0.6, 0.1, 0.2)	(0.1, 0.1, 0.7)	(0.3, 0.1, 0.3)
Z_3	(0.4, 0.2, 0.3)	(0.4, 0.2, 0.2)	(0.2, 0.2, 0.5)	(0.6, 0.2, 0.1)
Z_4	(0.1, 0.2, 0.6)	(0.6, 0.2, 0.1)	(0.3, 0.1, 0.4)	(0.7, 0.1, 0.1)

Since C_1, C_3 are cost-type criteria and C_2, C_4 are bene t-type criteria. So, we have need to normalize the photo fuzzy statistics. Normalized photograph fuzzy information is shown in tables 4-6.

Table 4. *Normalized picture fuzzy information R^1.*

	$\overline{A}_1$	$\overline{A}_2$	$\overline{A}_3$	$\overline{A}_4$
Z_1	(0.6, 0.1, 0.2)	(0.5, 0.3, 0.1)	(0.5, 0.1, 0.3)	(0.4, 0.3, 0.2)
Z_2	(0.4, 0.4, 0.1)	(0.6, 0.2, 0.1)	(0.5, 0.2, 0.2)	(0.2, 0.1, 0.6)
Z_3	(0.3, 0.3, 0.2)	(0.4, 0.3, 0.2)	(0.4, 0.1, 0.3)	(0.3, 0.2, 0.4)
Z_4	(0.6, 0.1, 0.3)	(0.3, 0.2, 0.4)	(0.5, 0.3, 0.1)	(0.2, 0.3, 0.3)

Table 5. *Normalized picture fuzzy information R^2.*

	$\overline{A}_1$	$\overline{A}_2$	$\overline{A}_3$	$\overline{A}_4$
Z_1	(0.5, 0.3, 0.1)	(0.4, 0.3, 0.2)	(0.6, 0.1, 0.1)	(0.2, 0.3, 0.4)
Z_2	(0.4, 0.2, 0.2)	(0.4, 0.3, 0.2)	(0.4, 0.2, 0.3)	(0.4, 0.1, 0.4)
Z_3	(0.6, 0.2, 0.1)	(0.6, 0.1, 0.1)	(0.4, 0.2, 0.2)	(0.5, 0.2, 0.2)
Z_4	(0.5, 0.1, 0.4)	(0.5, 0.1, 0.3)	(0.3, 0.3, 0.3)	(0.6, 0.2, 0.1)

Table 6. *Normalized picture fuzzy information R^3.*

	$\overline{A}_1$	$\overline{A}_2$	$\overline{A}_3$	$\overline{A}_4$
Z_1	(0.3, 0.1, 0.3)	(0.4, 0.2, 0.1)	(0.4, 0.3, 0.2)	(0.5, 0.2, 0.1)
Z_2	(0.3, 0.5, 0.1)	(0.6, 0.1, 0.2)	(0.7, 0.1, 0.1)	(0.3, 0.1, 0.3)
Z_3	(0.3, 0.2, 0.4)	(0.4, 0.2, 0.2)	(0.5, 0.2, 0.2)	(0.6, 0.2, 0.1)
Z_4	(0.6, 0.2, 0.1)	(0.6, 0.2, 0.1)	(0.4, 0.1, 0.3)	(0.7, 0.1, 0.1)

As the records approximately characteristic weights, given by using specialists, is in part recognized,

$$\Delta=\begin{Bmatrix} 0.2 \le \omega_1 \le 0.25, \\ 0.15 \le \omega_2 \le 0.2, \\ 0.28 \le \omega_3 \le 0.32, \\ 0.35 \le \omega_4 \le 0.4 \end{Bmatrix}, \omega_p \ge 0, p = 1,2,3,4, \sum_{p=1}^{4} \omega_p = 1$$

Step 2: Firstly, define fuzzy density of every decision maker, and its λ parameter. Let $P(A_1) = 0.30, P(A_2) = 0.40, P(A_3) = 0.50$. Then λ of expert can be determined: $\lambda = -0.45$. By Eq. (6), we have $P(A_1, A_2) = 0.65$, $P(A_1, A_3) = 0.73, P_{\breve{\varepsilon}_{\breve{u}}}(A_2, A_3) = 0.81, P(A_1, A_2, A_3) = 1$.

Step 3: From definition 9, $\mathfrak{I}_{\breve{u}_{\text{pq}}}^{(\mathscr{k})}$ is reordered such that $\mathfrak{I}_{\breve{u}_{\text{pq}}}^{(\mathscr{k})} \ge \mathfrak{I}_{\breve{u}_{\text{pq}}}^{(\mathscr{k}-1)}$. Then utilize the picture fuzzy Choquet integral weighted operator

$$PFCIWA(\mathfrak{I}_1, \dots, \mathfrak{I}_n) = \begin{Bmatrix} 1-\prod_{p=1}^{r}\left(1-P_{\mathfrak{I}_{\sigma(p)}}\right)^{\lambda(A_{\sigma(p)})-\lambda(A_{\sigma(p-1)})}, \\ \prod_{p=1}^{r}\left(I_{\mathfrak{I}_{\sigma(P)}}\right)^{\lambda(A_{\sigma(p)})-\lambda(A_{\sigma(p-1)})}, \\ \prod_{p=1}^{r}\left(N_{\mathfrak{I}_{\sigma(p)}}\right)^{\lambda(A_{\sigma(p)})-\lambda(A_{\sigma(p-1)})} \end{Bmatrix}$$

To mixture all the image fuzzy choice matrices $R^{\mathscr{k}} = \left[\mathfrak{I}_{\text{pq}}^{(\mathscr{k})}\right]_{m\times n}$ into a collective photo fuzzy selection matrix as follows,

Table 7. *Collective picture fuzzy information.*

	$\bar{A}_1$	$\bar{A}_2$	$\bar{A}_3$	$\bar{A}_4$
Z_1	(0.473, 0.146, 0.180)	(0.431, 0.260, 0.127)	(0.507, 0.146, 0.177)	(0.377, 0.260, 0.200)
Z_2	(0.366, 0.339, 0.127)	(0.539, 0.180, 0.162)	(0.554, 0.156, 0.180)	(0.309, 0.100, 0.408)
Z_3	(0.424, 0.225, 0.200)	(0.465, 0.177, 0.156)	(0.437, 0.162, 0.225)	(0.488, 0.200, 0.193)
Z_4	(0.567, 0.127, 0.225)	(0.488, 0.156, 0.222)	(0.400, 0.204, 0.215)	(0.554, 0.177, 0.139)

Step 4: Utilize eq. (22) and eq. (23) we get,

$$P^+ = \{\langle 0.567.0.127, 0.225\rangle, \langle 0.539, 0.180, 0.162\rangle, \langle 0.554, 0.156, 0.180\rangle, \langle 0.554, 0.177, 0.139\rangle\}$$

$$P^- = \{\langle 0.366, .0.339, 0.127\rangle, \langle 0.431, 0.260, 0.127\rangle, \langle 0.400, 0.204, 0.215\rangle, \langle 0.309, 0.100, 0.408\rangle\}$$

Step 5: Equation (25) and (26) as follow,

Table 8. *Positive-ideal separation matrix.*

	A_1	A_2	A_3	A_4
Z_1	0.0395	0.0553	0.0152	0.0803
Z_2	0.1277	0.0000	0.0000	0.1479
Z_3	0.0668	0.0205	0.0419	0.0357
Z_4	0.0000	0.0336	0.0590	0.0000

Table 9. *Negative-ideal separation.*

	A_1	A_2	A_3	A_4
Z_1	0.0882	0.0000	0.0506	0.1091
Z_2	0.0000	0.0553	0.0590	0.0000
Z_3	0.0609	0.0366	0.0221	0.1235
Z_4	0.1277	0.0637	0.0687	0.1479

Step 6: Utilize equations (27) and (28) we get PIS and NIS as follow,

$$\left[\xi_{ij}^{+}\right] =$$

0.6518	0.5721	0.8295	0.4711
0.3667	1.0000	1.0000	0.3333
0.5254	0.7829	0.6383	0.6744
1.0000	0.6875	0.5562	1.0000

$$\left[\xi_{ij}^{-}\right] =$$

0.0395	0.0553	0.0152	0.0803
0.1277	0.0000	0.0000	0.1479
0.0668	0.0205	0.0419	0.0357
0.0000	0.0336	0.0590	0.0000

Step 7: Model (M2) as:

$$min\xi(\omega) = -0.0708\omega_1 + 0.4283\omega_2 - 0.2594\omega_3 - 1.5440\omega_4$$

We get that,

$$\omega = (0.330, 0.144, 0.366, 0.157)$$

Then, we can get the degree of gray relational coefficient of every opportunity from PIS and NIS:

$$\xi_1^+ = 0.6001, \xi_2^+ = 0.5439, \xi_3^+ = 0.6331, \xi_4^+ = 0.8823$$

$$\xi_1^- = 0.5355, \xi_2^- = 0.8657, \xi_3^- = 0.5352, \xi_4^- = 0.4016$$

Step 8: Utilize equation 32, we get PIS and NIS:

$$\xi_1 = \frac{\xi_1^+}{\xi_1^- + \xi_1^+} = \frac{0.6001}{0.5355+0.6001} = 0.5284$$

$$\xi_2 = \frac{\xi_2^+}{\xi_2^- + \xi_2^+} = \frac{0.5439}{0.8657+0.5439} = 0.3858$$

$$\xi_3 = \frac{\xi_3^+}{\xi_3^- + \xi_3^+} = \frac{0.6331}{0.5352+0.6331} = 0.5418$$

$$\xi_4 = \frac{\xi_4^+}{\xi_4^- + \xi_4^+} = \frac{0.8823}{0.4016+0.8823} = 0.6872$$

Step 9: Ranking order of the 4 options, in step with the relative relational degree are:

$$\check{Z}_4 > \check{Z}_3 > \check{Z}_1 > \check{Z}_2 >$$

and thus, the most desirable alternative is $\check{Z}_4$.

5. Conclusion

In this paper, we proposed decision making approach to deal with picture fuzzy information. As picture fuzzy set is the generalization of all the existing structure of fuzzy sets, so an algorithm based on GRA approach to deal with uncertainty and inaccurate information in decision making problems using picture fuzzy environments. Final, a numerical application is illustrated to shows the how our proposed technique is effective and reliable to deal with uncertainty. In future, we use TOPSIS approach to deal with uncertainty using picture fuzzy information.

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