American Journal of Mathematical and Computer Modelling
2022; 7(2): 20-30
http://www.sciencepublishinggroup.com/j/ajmcm
doi: 10.11648/j.ajmcm.20220702.11
ISSN: 2578-8272 (Print); ISSN: 2578-8280 (Online)

A Reliable Method for Obtaining Analytical Solutions to Some Ordinary and Partial Differential Equations with Initial Values

Galal Mahrous Moatimid[1], Mohamed Abdel-Latif Ramadan[2], Mahmoud Hamed Taha[1, *], Elsayed Eladdad[3]

[1]Department of Mathematics, Ain Shams University, Cairo, Egypt

[2]Department of Mathematics and Computer Science, Menoufia University, Menoufia, Egypt

[3]Department of Mathematics, Tanta University, Tanta, Egypt

Email address:

gal_moa@hotmail.com (G. M. Moatimid), ramadanmohamed13@yahoo.com (M. Abdel-Latif R.),
mahmoud.h.taha@edu.asu.edu.eg (M. H. Taha), elsayedeladdad@yahoo.com (E. Eladdad)

[*]Corresponding author

To cite this article:

Galal Mahrous Moatimid, Mohamed Abdel-Latif Ramadan, Mahmoud Hamed Taha, Elsayed Eladdad. A Reliable Method for Obtaining Analytical Solutions to Some Ordinary and Partial Differential Equations with Initial Values. *American Journal of Mathematical and Computer Modelling.* Vol. 7, No. 2, 2022, pp. 20-30. doi: 10.11648/j.ajmcm.20220702.11

Received: February 19, 2022; **Accepted**: March 28, 2022; **Published**: April 20, 2022

Abstract: In this communication, the homotopy perturbation method is modified and extended to obtain the analytical solutions of some nonlinear differential equations. Differential equations are used to mathematically formulate, and thus aid the solution of, physical and other problems involving functions of several variables, such as the propagation of heat or sound, fluid flow, elasticity, electrostatics, electrodynamics, etc. Fluid mechanics, heat and mass transfer, and electromagnetic theory are all modeled by partial differential equations (PDE) and all have plenty of real life applications. dynamic meteorology and numerical weather forecasting: the weather report you see every night on TV has been obtained from the numerical solution of a complex set of nonlinear PDEs. The numerical solution of nonlinear differential equations is extremely difficult. Here, the proposed technique is implemented to obtain the analytical solutions of the initial-value ordinary and partial differential equations. In the current study, some problems are solved using a newly modified method that outperforms all other known methods, with approximate results in the form of power series. The method's algorithm is described and illustrated using some well-known problems. The obtained results demonstrate the method's efficiency. Furthermore, those results implies that this new method is simpler to implement. The approach is powerful, effective, and promising in analyzing some classes of differential equations for heat conduction problems and other dynamical systems. To crystallize the new approach, some illustrated examples are introduced.

Keywords: Homotopy, Perturbation, Klein-Gordon, Duffing Equation, Schrödinger Equation, Picard Method

1. Introduction

Linear and nonlinear differential equations are very important classes of evolution equations that have been developed in recent years. These classes have wide applications in chemistry, plasma physics, thermo-elasticity, and engineering mechanics. Several recent numerical techniques have been used to treat these problems. In addition, many powerful methods have been developed and modified to get the exact solution of such equations or to get a better approximate solution than the already existing one. The methods for solving linear differential equations are quite straightforward and well-established. Nonlinear differential equations, on the other hand, have fewer approaches for solving them, and linear approximations are typically required. Some scientists are currently attempting to devise a way for approaching the exact solutions of such nonlinear differential equations. Some of the researchers managed to get methods that solve some classes of nonlinear problems. Others modified the existing methods to get a better solution or, in some cases, the exact solution. The homotopy perturbation method is one of these methods (HPM). This work is considered a continuation of El-Dib and Moatimid's work [1]. He [2] proposed the HPM for solving linear, nonlinear differential, and integral equations for the first time. That isn't to suggest that it always gets the exact answer. If a precise solution cannot be found, an accurate approximation of the answer is calculated. He [3-7] and He et al. [8] have carried out more changes to the HPM. HPM has undergone a number of changes. See, for example, Siddiqui et al. [9]. By providing a new operator to solve Lane-Emden equations, Wazwaz [10] developed a straightforward adaptation of the Adomian decomposition approach.

Those methods can be used effectively to solve the equations with many types of linearities such as the Klein-Gordon equations. The Klein-Gordon equation is used in a variety of scientific domains, including solid-state physics. The HPM was employed to attain approximate analytical solutions for the Klein–Gordon and sine-Gordon equations by Chowdhury and Hashim [11]. The potential of HPM in solving nonlinear partial differential equations was demonstrated by comparisons between exact solutions, solutions produced by the Adomian decomposition technique (ADM), and the variational iteration method (VIM). In addition, the motion of a rigid pendulum is attached to a stretched wire, nonlinear optics, and optical solitons [12–15]. Condensed matter physics [16], the interaction of solitons in a collisionless plasma and the recurrence of initial states, and quantum field theory [17] are considered as Klein-Gordon direct applications.

The rest of the paper is divided into the following sections: The analysis of the approach is presented in Section 2. The method's uniqueness and exactness are discussed in this section. In Section 3, the solution's algorithm is shown. Some examples are provided throughout Section 4 to demonstrate our methodology. The current work's conclusions are reported in Section 5.

2. Analysis of the Method

This section proves the uniqueness of the solution of

$$\frac{\partial^2 u(x,t)}{\partial t^2} = L\big(u(x,t)\big) + N\big(u(x,t)\big) + f(x,t), \tag{1}$$

with conditions $u(x,0) = \alpha\ and\ u_t(x,0) = \beta$,
where $L\big(u(x,t)\big)$ is the linear part of the differential equation, $N\big(u(x,t)\big)$ is the nonlinear part of the partial differential equation, and $f(x,t)$ is a given function.

Definition.

Let $H = C[a,b]$ be the set of all continuous functions defined on the closed interval $[a,b]$. The distance between any arbitrary functions $\propto(t), \beta(t) \in H$ is defined in the form $d\big(\propto(t), \beta(t)\big) = max_{a\le t\le b}|\propto(t) - \beta(t)|$. It is known that (H,d) is a complete metric space and the following properties are satisfied:

1) $d(\propto,\beta) = 0 \leftrightarrow \propto = \beta\ \forall \propto,\beta \in H$
2) $d(\propto+\gamma, \beta+\gamma) = d(\alpha,\beta)\ \forall \propto,\beta,\gamma \in H$
3) $d(\propto+\gamma, \beta+e) \le d(\alpha,\beta) + d(\gamma,e)\ \forall \propto,\beta,\gamma,e \in H$

Now, consider that $f(x,t)$ is a bounded function for all $(x,t) \in R\times R$. Also, it is supposed that the linear and nonlinear operators $L\ and\ K$ satisfy Lipchitz conditions with

$$d\Big(L\big(t,u(x)\big), L\big(t,u(y)\big)\Big) \le M_1\, d\big(u(x),u(y)\big), M_1 \ge 0, \tag{2}$$

$$d\Big(K\big(t,u(x)\big), K\big(t,u(y)\big)\Big) \le M_2\, d\big(u(x),u(y)\big), M_2 \ge 0, \tag{3}$$

Let $m = \frac{(M_1+M_2)\,t^2}{2}$.

Theorem.

Assume that equations (2), (3) hold in such a way that $0 < m < 1$, then there exists a unique solution to the problem given by the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = L\big(u(x,t)\big) + K\big(u(x,t)\big) + f(x,y), \tag{4}$$

with the initial conditions

$$u(x,0) = A, \frac{\partial u(x,0)}{\partial t} = B. \tag{5}$$

Proof.

Let $u\ and\ u^*$ be two different solutions for equations (4) and (5), then we can write

$$u(x,t) = A + B\,t + \int_0^t\int_0^t \Big(L\big(u(x,t)\big) + K\big(u(x,t)\big) + f(x,t)\Big)\, dt\, dt\,,$$

$$u^*(x,t) = A + B\,t + \int_0^t\int_0^t \Big(L\big(u^*(x,t)\big) + K\big(u^*(x,t)\big) + f(x,t)\Big)\, dt\, dt\,,$$

and

$$d(u,u^*) = d\Big(A + B\,t + \int_0^t\int_0^t \Big(\big(u(x,t)\big) + K\big(u(x,t)\big) + f(x,t)\Big) dt\, dt,\ A + B\,t + \int_0^t\int_0^t \Big(L\big(u^*(x,t)\big) + K\big(u^*(x,t)\big) + f(x,t)\Big)\, dt\ \ dt\Big)$$

$$= d\Big(\int_0^t\int_0^t \Big(L\big(u(x,t)\big) + K\big(u(x,t)\big)\Big) dt\, dt,\ \int_0^t\int_0^t \Big(L\big(u^*(x,t)\big) + K\big(u^*(x,t)\big)\Big)\, dt\ \ dt\Big)$$

$$\le\ d\Big(\int_0^t\int_0^t \Big(L\big(u(x,t)\big) + L\big(u^*(x,t)\big)\Big)\, dt\, dt, \int_0^t\int_0^t \Big(K\big(u(x,t)\big) + K\big(u^*(x,t)\big)\Big)\, dt\, dt\ \Big)$$

$$\le \frac{(M_1+M_2)\,t^2}{2} = \mathrm{m}\, d(u,u^*)$$

Accordingly, one gets $(1-m)\, d(u,u^*) \leq 0$, since, $0<m<1$, then $d(u,u^*)=0$, which implies that $u=u^*$.
This result proves that equation (4) has a unique solution.

3. The Algorithm of the Solution

This section starts with the basic concepts of the homotopy technique for Eq. (1). It may be decomposed into two parts; namely, L and N which are known as linear and nonlinear parts, respectively, as follows:

$$L(u(x,t)) + N(u(x,t)) + f(x,t) = 0, \tag{6}$$

then a homotopy equation is constructed in the following form:

$$H(u,p) = L(u(x,t)) - L(U(x,t)) + \rho\big[L(U(x,t)) + N\big(u(x,t)\big) + f(x,t)\big] = 0, \rho \in [0,1], \tag{7}$$

where U is an initial guess, sometimes called a trial function, for the solution of Eq. (1) with the use of the artificial homotopy parameter ρ to expand $u(x,t,\rho)$ as

$$u(x,t,\rho) = u_0(x,t) + \rho u_1(x,t) + \rho^2 u_2(x,t) + \ldots, \tag{8}$$

take the initial guess as a power series function in x and t as

$$U(x,t) = \textstyle\sum_{n=0}^{\infty} a_n(x) t^n. \tag{9}$$

Now, consider

$$L(u) = u^{(k)}(x,t) \tag{10}$$

Combining equations (8-10) and Eq. (7), one finds

$$u_0^{(k)}(x,t) + \rho u_1^{(k)}(x,t) + \ldots - U^{(k)}(x,t) + \rho\big[U^{(k)}(x,t) - f(x,t) - N(u(x,t))\big] = 0, \tag{11}$$

Equating the coefficients of like powers of ρ, one gets

$$\rho^0 : u_0^{(k)}(x,t) = \textstyle\sum_{n=0}^{\infty} a_n(x) t^n, \tag{12}$$

integrating both sides of Eq. (12), k −times, one gets

$$u_0(x,t) = \textstyle\sum_{n=k}^{\infty} a_n(x) t^n + \ldots + \beta x + \alpha, \tag{13}$$

$$\rho^1 : u_1^{(k)}(x,t) = -U^{(k)}(x,t) - f(x,t) - N(u(x,t)), \tag{14}$$

$$\vdots$$

Set $u_1(x) = 0$ in Eq. (14), then compare the coefficients of like powers of x to get the undetermined coefficients $a_n's$.

At this stage, substitute from the coefficients of $a_n's$ into Eq. (13) to obtain the required exact solution.

It should be mentioned here that

The selection of the guessing function in the classical HPM is rather difficult. Now, this new technique introduces a simple way of selecting a proper initial approximation. Therefore, this new technique gives a general method. It is a rapid convergence to the exact solution with simpler calculations.

The exact solutions need a convergent series as given in Eq. (13). Otherwise, only an approximate solution of the given differential equation is obtained.

The above method is still valid in the case of the linear as well as nonlinear.

4. Illustrated Examples

Duffing equation is one of the most significant and classical nonlinear differential equations because of its wide applications in science and engineering. Duffing equation is a result of potential applications in physics, biology, and communication theory. With initial conditions and bounded periodic solutions, Salas and Castillo [17] discovered an exact solution to the cubic Duffing oscillator problem. The Jacobi elliptic function is used to express their solution.cn.

Example 1

In this example, the applicability of the previous method in solving a Duffing equation is demonstrated. Consider the following cubic Duffing oscillator equation:

$$u''(x) + u(x) - u^3 = 0, \tag{15}$$

subjected to the initial conditions: $u(0) = 0$ and $u'(0) = \frac{1}{\sqrt{2}}$.

On using the new modified method as given in our previous paper [1], the trial function of Eq. (15) may be written as the following power series:

$$U(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{16}$$

where $a's$ are arbitrary constants to be determined.

The homotopy equation of Eq. (15) may be written as:

$$H(u,\rho) = \frac{d^2}{dx^2}(u - U) + \rho\left\{\frac{d^2U}{dx^2} + u - u^3\right\} = 0. \tag{17}$$

Following the homotopy perturbation, the solution of the dependent function $u(x)$ as given in the homotopy equation (17), with the aid of the artificial parameter ρ, may be expanded as a power series as follows:

$$u(x,\rho) = u_0(x) + \rho u_1(x) + \rho^2 u_2(x) + \ldots \tag{18}$$

Substituting from Eq. (18) into Eq. (17) and equating the coefficients of like powers of ρ of both sides, one gets the following equations:

$$\rho^0 : u_0''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}, \tag{19}$$

and

$$\rho : u_1''(x) = u_0^3(x) - u_0(x) - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}. \tag{20}$$

The special solution of Eq. (19) will be found directly as

$$u_0(x) = \frac{x}{\sqrt{2}} + \sum_{n=2}^{\infty} a_n x^n. \tag{21}$$

Substituting Eq. (21) into Eq. (20) and then integrating twice with respect to x, it follows that the special solution of Eq. (20) may be written as:

$$u_1(x) = -a_2 x^2 - \left(a_3 + \frac{1}{6\sqrt{2}}\right)x^3 - \left(a_4 + \frac{a_2}{12}\right)x^4 + \frac{1}{80}\left(\sqrt{2} - 4a_3 - 80a_5\right)x^5 + \frac{1}{60}\left(3a_2 - 2(a_4 + 30a_6)\right)x^6 + \ldots \tag{22}$$

The cancellation of the first-order solution as given by Eq. (22) yields the following unknown coefficients $a's$:

The even-order terms are all of zero values, meanwhile, the odd-order terms give

$$a_3 = -\frac{1}{6\sqrt{2}}, a_5 = \frac{1}{30\sqrt{2}}, a_7 = -\frac{17}{2520\sqrt{2}}, a_9 = \frac{31}{22680\sqrt{2}} \ldots\text{..etc.} \tag{23}$$

Substituting from Eq. (23) into Eq. (21), it follows that the closed form of the given cubic Duffing oscillator equation is given by

$$u(x) = tanh\left(\frac{x}{\sqrt{2}}\right). \tag{24}$$

On using similar arguments as given above, one can find the exact solution of the following cubic Duffing oscillator equation:

$$u''(x) + u(x) + u^3 = 0, \tag{25}$$

subject to the initial conditions: $u(0) = 0$, and $u'(0) = \frac{i}{\sqrt{2}}$, in the form

$$u(x) = i\, tanh\left(\frac{x}{\sqrt{2}}\right). \tag{26}$$

It should be noted that similar solutions of the previous Duffing equations were obtained by Moatimid [18].

The nonlinear Schrödinger equation can be used to modulate a wide range of physical problems. The Schrödinger equation is a nonlinear partial differential equation that explains the evolution equation of surface waves in hydrodynamic stability [19]. The coupled Schrödinger equations are used to simulate a wide range of physical processes, including solid-state physics, plasma waves, and so on [12]. He's HPM can be described as a broad technique for solving nonlinear functional equations of various types. It's been used to solve nonlinear Schrödinger problems [13], nonlinear heat transfer equations [14], and the quadratic Riccati differential equation [15]. Aminikhah et al. [16] present a novel effective technique for systems of coupled Schrödinger equations. They used various examples to demonstrate the efficacy of their method.

The main goal of this work is to extend our previous method [1] to obtain exact solutions for the different ordinary as well as partial differential equations. These equations acquire their importance in accordance with the physical situations.

Example 2 (Schrödinger equation)

$$iu_t + \frac{1}{2}u_{xx} - u\cos^2 x - u^2\overline{u} = 0, I.C.\, u(0,t) = 0, u_x(0,t) = e^{-3it/2} \tag{27}$$

Consider the two parts L(u) and N(u) as follows:

$$L(u) = u_{xx} \text{ and } N(u) = 2iu_t - 2u\cos^2 x - 2u^2\overline{u} \tag{28}$$

The initial guess, in this case, may be written as:

$$U(x,t) = \sum_{n=0}^{\infty} a_n(t)x^n \tag{29}$$

The homotopy function may be written as

$$H(u,\rho) = u_{xx} - U + \rho[U + 2iu_t - 2u\cos^2 x - 2u^2\overline{u}], \tag{30}$$

According to the homotopy perturbation, the function u(x, t) may be written as:

$$u(x,t;\rho) = u_0(x,t) + \rho u_1(x,t) + \rho^2 u_2(x,t) + \ldots \tag{31}$$

Therefore, the given partial differential equation is written as

$$u_{0xx} + \rho u_{1xx} + \ldots - \sum_{n=0}^{\infty} a_n(t)x^n + \rho\left(\sum_{n=0}^{\infty} a_n(t)x^n + 2iu_{0t} - 2u_0\cos^2 x - 2u_0^2\overline{u_0} + \ldots\right) = 0 \tag{32}$$

Equating the coefficients of like powers of ρ by zero gives

$$\rho^0 : u_{0xx} = \sum_{n=0}^{\infty} a_n(t)x^n, I.C.: u_0(0,t) = 0, u_{0x}(0,t) = e^{-3it/2} \tag{33}$$

$$\rho : u_{1xx} = 2u_0\cos^2 x + 2u_0^2\overline{u_0} - 2iu_{0t} - \sum_0^{\infty} a_n(t)x^n \;\; I.C.: u_1(0,t) = 0, u_{1x}(0,t) = 0 \tag{34}$$

The special solution of Eq. (33) is given by

$$u_0 = xe^{-3it/2} + \sum_{n=0}^{\infty} \frac{a_n(t)x^{n+2}}{(n+1)(n+2)} \tag{35}$$

Substituting Eq. (35) into Eq. (34), integrating twice with respect to x, and also applying the corresponding I. C., one finds

$$u_1 = \int_0^x \left\{\int_0^x \left\{2\left(xe^{-3it/2} + \sum_{n=0}^{\infty} \frac{a_n(t)x^{n+2}}{(n+1)(n+2)}\right)\cos^2 x\right\}dx\right\}dx + \int_0^x \left\{\int_0^x \left\{2\left(xe^{-3it/2} + \sum_{n=0}^{\infty} \frac{a_n(t)x^{n+2}}{(n+1)(n+2)}\right)^2 \left(xe^{-3it/2} + \sum_{n=0}^{\infty} \frac{a_n(t)x^{n+2}}{(n+1)(n+2)} \sum_{n=0}^{\infty} \frac{\overline{a_n}(x)t^{n+1}}{n+1}\right)\right\}dx\right\}dx - \int_0^x \left\{\int_0^x \left\{\sum_{n=0}^{\infty} a_n(t)x^n\right\}dx\right\}dx \tag{36}$$

The expansion of Eq. (36) may be rewritten as:

$$u_1 = -\frac{a_0}{2}x^2 - \frac{1}{6}\left(a_1 + e^{-3it/2}\right)x^3 + \frac{1}{12}(a_0 - a_2 - ia_0')x^4 + \frac{1}{60}(a_1 - 3a_3 ia_1')x^5 + \ldots \tag{37}$$

The basic idea in this approach is to cancel the first-order term, $u_1 = 0$, It follows that the remaining terms in the power series of the perturbed solution, as given by Eq. (28), will vanish.

Now, the coefficients of the like power of x will be equated to zero. This leads to obtaining the coefficients of the power series of the zero-order solution given by Eq. (33). These coefficients may be written as follows:

$$a_0 = 0, a_1 = -e^{-3it/2}, a_2 = 0, \text{ and } a_3 = \frac{1}{6}e^{-3it/2} \ldots \tag{38}$$

The zero-order solution then becomes

$$u_0 = \left\{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \ldots\ldots\right\}e^{-3it/2} = e^{-3it/2}\sin x \tag{39}$$

Eq. (39) gives the required exact solution for the given partial differential equation.

The Klein-Gordon (KG) equations play a significant role in relativistic quantum physics. Many investigations have been done on KG equations with various types of potentials, utilising a variety of methods to characterise and solve the related relativistic physical systems, including the asymptotic iteration method, the formal variable separation method, and supersymmetric quantum mechanics [18]. The following examples deal with some inhomogeneous linear and as nonlinear KG problems.

Example 3

Consider the following homogeneous sine-Gordon equation:

$$u_{tt} - u_{xx} + \sin u = 0, \tag{40}$$

with the initial conditions $u(x,0) = 0$ and $u_t(x,0) = 4Sechx$, where $u = u(x,t)$ is a function of the variables x and t.

On using the new introduced method, the trial function may be written as given by

$$U(x,t) = \sum_{n=0}^{\infty} a_n(x)t^n \tag{41}$$

The homotopy equation may be written as:

$$H(u,\rho) = \frac{\partial^2}{\partial t^2}(u-U) + \rho\left\{U - \frac{\partial^2 u}{\partial x^2} + \sin u\right\} = 0 \quad (42)$$

In accordance with the homotopy perturbation, the solution $u(x,t)$ of the homotopy equation (42), with the aid of the artificial parameter ρ, may be expanded as before.

Substituting from the regular perturbation of the dependent function (u) into Eq. (42) and equating the coefficients of like powers of ρ in both sides, one gets the following equations:

$$\rho^0: u_{0tt} = \sum_{n=2}^{\infty} n(n-1)a_n(x)t^{n-2} \quad (43)$$

The special solution of Eq. (43) is directly

$$u_0 = 4tSechx + \sum_{n=2}^{\infty} a_n(x)t^n \quad (44)$$

The first-order equation may be written as

$$u_{1tt} = u_{0xx} - \sum_{n=2}^{\infty} n(n-1)a_n(x)t^{n-2} - \sin u_0 \quad (45)$$

Substituting Eq. (44) into Eq. (45), then integrating twice with respect to t, it follows, after lengthy but straightforward calculations, that the special solution of Eq. (45) may be written as:

$$u_1 = -a_2 t^2 - \left(a_3 + \frac{4}{3}Sech^3 x\right)t^3 + \left(\frac{a_2'' - a_2}{12} - a_4\right)t^4 + \left(a_5 - \frac{4}{5}Sech^5 x\right)t^5$$

$$+\left(\frac{a_4''-a_4}{30} - a_6 + \frac{16a_2 Sinh(2x)}{15(1+Cosh(2x))Sinh(2x))^2}\right)t^6 - \left(a_7 + \frac{4}{7}Sech^7 x\right)t^7 + \cdots \ldots \quad (46)$$

The cancellation of the first-order solution as given by Eq. (46) yields the following undetermined coefficients $a_n(x)'s$.

$$a_n(x) = \begin{cases} \frac{4(-)^{(n-1)/2}Sech^n x}{n}, \text{ for } n \text{ odd} \\ 0, \text{ for } n \text{ even} \end{cases} \quad (47)$$

substituting Eq. (47) into Eq. (44), it follows that the closed form of the solution given sine-Gordon equation is given as

$$u(x,t) = 4\sum_{n=1,3,5}^{\infty} \frac{(-)^{(n-1)/2}}{nCosh^n x}. \quad (48)$$

Therefore, the closed form of Eq. (48) is

$$u(x,t) = 4Arc\tan(tSechx), \quad (49)$$

which is the exact solution of the initial-value sine-Gordon equation that is given in Eq. (41).

To this end, the present novel method is a powerful mathematical tool to solve the sine-Gordon equation. It is also a promising method to solve other nonlinear equations given in our previous work [1]. This method solves the problem without any need for the discretization of the variables. The Mathematica software is used to calculate the series obtained. It should be noted that the previous same solution is obtained by the reduced differential transform method (RDTM) by Keskin et al. [20].

Example 4

Through this example, we demonstrate the applicability of the previous method to the following partial non-homogeneous linear Klein-Gordon equation:

$$u_{tt} - u_{xx} - 2u = -2\sin x \sin t, \quad (50)$$

subjected to the initial conditions: $u(x,0) = 0$ and $u_t(x,0) = \sin x$.

On using the newly introduced method as given in our previous paper [1], the trial function of Eq. (50) as the following power series:

$$U(x,t) = \sum_{n=0}^{\infty} a_n(x)t^n, \quad (51)$$

where $a's$ are arbitrary constants to be determined.

The homotopy equation of Eq. (50) may be written as:

$$H(u,\rho) = u_{tt} - u_{xx} - U_{tt} + \rho\{U_{tt} - 2u + 2\sin x \sin t\} = 0. \quad (52)$$

In accordance with the homotopy perturbation, the solution of the dependent function $u(x,t,\rho)$ as given in the homotopy equation (52), with the aid of the artificial parameter ρ, may be expanded as a power series as follows

$$u(x,t,\rho) = u_0(x,t) + \rho u_1(x,t) + \rho^2 u_2(x,t) + \ldots \quad (53)$$

Substituting Eq. (53) into Eq. (52) and equating the coefficients of like powers of ρ of both sides, one gets the following equations:

$$\rho^0: u_{0tt} = \sum_{n=2}^{\infty} n(n-1)a_n(x)\, t^{n-2}, \quad (54)$$

and

$$\rho: u_{1tt} = u_{0xx} + 2u_0 - 2\sin x \sin t - \sum_{n=2}^{\infty} n(n-1)a_n(x)\, t^{n-2} \quad (55)$$

The special solution of Eq. (54) is directly

$$u_0(x,t) = t\sin x + \sum_{n=2}^{\infty} a_n(x)t^n. \quad (56)$$

Substituting Eq. (56) into Eq. (55) and then integrating partially twice with respect to t, it follows that the special solution of

Eq. (54) may be written as:

$$u_1(x,t) = -a_2t^2 - \left(a_3 + \frac{\sin x}{6}\right)t^3 + \frac{1}{12}(2a_2 - 12a_4 + a_2'')x^4 + \frac{1}{60}(\sin x + 6a_3 - 60a_5 + 3a_3'')t^5$$
$$+ \tfrac{1}{30}(2a_4 - 30a_6 + a_4''))t^6 + \left(-\tfrac{\sin x}{2520} + \tfrac{1}{42}(2a_5 - 42a_7 + a_5'')\right)t^7 + \cdots \quad (57)$$

The cancellation of the first-order solution as given by Eq. (57) yields the following unknown coefficients $a's$.

The even-order terms are all of zero values, meanwhile, the odd-order terms give

$$a_3 = -\tfrac{\sin x}{3!}, a_5 = \tfrac{\sin x}{5!}, a_7 = -\tfrac{\sin x}{7!}, a_9 = \tfrac{\sin x}{9!}, \ldots \text{ etc.} \quad (58)$$

Substituting Eq. (58) into Eq. (56), it follows that the closed form of the given cubic Duffing oscillator equation is given by

$$u(x,t) = \sin x \sin t. \quad (59)$$

The well-known sine-Gordon equation is one of the most important classes in all partial differential equations that appears in a wide range of applied mathematics. Partial differential equations can also be found in a variety of scientific domains, such as the motion of a stiff pendulum coupled to a stretched wire [21-24]. It also happens in solid-state physics, nonlinear optics, and fluid motion instability. There are numerous numerical approaches for obtaining numerical answers for problems of this nature [22].

One of the sine-Gordon equations will be solved in the following example using the newly introduced method, which is offered to address the disadvantage of the other methods' extensive calculations. As a result, the method's fundamental advantage is that it offers the user with an analytical approximation, in many cases an exact solution, in a swiftly convergent sequence with elegantly computed terms.

Example 5

This example investigates the possibility of our method to solve the following partial non-homogeneous nonlinear Klein-Gordon equation:

$$u_{tt} - u_{xx} + u^2 = f(x,t),\ f(x,t) = -x\cos t + x^2\cos^2 t, \quad (60)$$

subjected to the initial conditions: $u(x,0) = x$ and $u_t(x,0) = 0$.

On using the newly introduced method as before the trial function of Eq. (60) may be written as given by Eq. (16)

$$U(x,t) = \textstyle\sum_{n=0}^{\infty} a_n(x)t^n, \quad (61)$$

The homotopy equation of Eq. (61) may be written as:

$$H(u,\rho) = u_{tt} - u_{xx} - U_{tt} + \rho\{U_{tt} + u^2 + x\cos t - x^2\cos^2 t\} = 0. \quad (62)$$

In accordance with the homotopy perturbation, the solution of the dependent function $u(x,t,\rho)$ as given in the homotopy equation (62), with the aid of the artificial parameter ρ, may be written as given by Eq. (62).

Substituting Eq. (61) into Eq. (62) and equating the coefficients of like powers of ρ of both sides, one gets the following equations:

$$\rho^0: u_{0tt} = \textstyle\sum_{n=2}^{\infty} n(n-1)a_n(x)\,t^{n-2}, \quad (63)$$

and

$$\rho: u_{1tt} = u_{0xx} - u_0^2 - x\cos t + x^2\cos^2 t - \textstyle\sum_{n=2}^{\infty} n(n-1)a_n(x)\,t^{n-2} \quad (64)$$

The special solution of Eq. (63) is directly

$$u_0(x,t) = x + \textstyle\sum_{n=2}^{\infty} a_n(x)t^n. \quad (65)$$

Substituting Eq. (65) into Eq. (64) and then integrating partially twice with respect to t, it follows that the special solution of Eq. (64) may be written as:

$$u_1(x,t) = -\left(a_2 + \tfrac{x}{2}\right)t^2 - a_3t^3 + \tfrac{1}{24}(x - 2x^2 - 4xa_2 - 24a_4 + 2a_2'')t^4 + \tfrac{1}{20}(-2xa_3 - 20a_5 + a_3'')t^5 + \tfrac{1}{720}(-x + 8x^2 -$$
$$24a_2^2 - 48xa_4 - 720a_6 + 24a_4'')t^6 + \tfrac{1}{21}\left(\tfrac{a_5''}{2} - a_2a_3 - xa_5 - 21a_7\right)t^7 + \ldots\ldots \quad (66)$$

The cancellation of the first-order solution as given by Eq. (66) yields the following unknown coefficients $a's$ as follows: the odd-order terms are all of zero values, meanwhile, the even-order terms are given by

$$a_2 = -\tfrac{x}{2!}, a_4 = \tfrac{x}{4!}, a_6 = -\tfrac{x}{6!}, a_8 = \tfrac{x}{8!}, a_{10} = -\tfrac{x}{10!}, \ldots \text{ etc.} \quad (67)$$

Substituting Eq. (67) into Eq. (65), it follows that the closed form of the given cubic Duffing oscillator equation is given by

$$u(x,t) = x\cos t. \quad (68)$$

Example 6

Consider now the Klein-Gordon equation, but for $f(x,t) = 6xt(x^2 - t^2) + x^6t^6$, i.e

$$u_{tt} - u_{xx} + u^2 = 6xt(x^2 - t^2) + x^6 t^6, \tag{69}$$

subjected to the initial conditions: $u(x,0) = 0$ and $u_t(x,0) = 0$.

On using the newly introduced method as before, the trial function of Eq. (69) may be written as given by Eq. (20).

The homotopy equation of Eq. (69) may be written as:

$$H(u,\rho) = u_{tt} - u_{xx} - U_{tt} + \rho\{U_{tt} + u^2 - 6xt(x^2 - t^2) - x^6 t^6\} = 0. \tag{70}$$

In accordance with the homotopy perturbation, the solution of the dependent function $u(x,t,\rho)$ as given in the homotopy equation (17), with the aid of the artificial parameterρ, may be written as given by Eq. (70)

Substituting Eq. (17) into Eq. (70) and equating the coefficients of like powers of ρ of both sides, one gets the following equations:

$$\rho^0:\ u_{0tt} = \sum_{n=2}^{\infty} n(n-1)a_n(x)\, t^{n-2}, \tag{71}$$

and

$$\rho:\ u_{1tt} = u_{0xx} - u_0^2 + 6xt(x^2 - t^2) + x^6 t^6 - \sum_{n=2}^{\infty} n(n-1)a_n(x)\, t^{n-2} \tag{72}$$

The special solution of Eq. (71) is directly

$$u_0(x,t) = \sum_{n=2}^{\infty} a_n(x) t^n. \tag{73}$$

Substituting Eq. (73) into Eq. (74) and then integrating partially twice with respect to t, it follows that the special solution of Eq. (72) may be written as:

$$u_1(x,t) = (-a_3 + x^3)t^3 + \left(a_4 - \tfrac{1}{12}a_2''\right)t^4 + \left(\tfrac{3}{10}x + a_5 - \tfrac{1}{20}a_3''\right)t^5$$
$$+\left(\tfrac{1}{30}a_2^2 + a_6 - \tfrac{1}{30}a_4''\right)t^6 + \left(\tfrac{1}{21}a_2 a_3 + a_7 - \tfrac{1}{42}a_5''\right)t^7 + \ldots \tag{74}$$

The cancellation of the first-order solution as given by Eq. (74) yields the following unknown coefficients $a's$ as follows: The odd-order terms are all of zero values, meanwhile, the even-order terms give

$$a_3 = x^3, a_4 = a_5 = a_6 = a_7 = a_8 = \ldots = 0 \ldots \text{etc.} \tag{75}$$

Substituting Eq. (75) into Eq. (73), it follows that the closed form of the given cubic Duffing oscillator equation is given by

$$u(x,t) = x^3 t^3 \tag{76}$$

Table 1. *The absolute error of a 4-term approximation of Example 4.*

x_i	t_i	$\lvert u_{exact} - u_{app.4(Pic.M)}\rvert$	$\lvert u_{exact} - u_{app4(H.P.M)}\rvert$ [25]	$\lvert u_{exact} - u_{app(F.H.P.M)}\rvert$
0.1	0.1	2.755 x 10^{-16}	2.724 x 10^{-16}	0.0
0.2	0.2	2.802 x 10^{-13}	2.802 x 10^{-13}	1.041 x 10^{-16}
0.3	0.3	1.602 x 10^{-11}	1.602x 10^{-11}	1.311 x 10^{-14}
0.4	0.4	2.809 x 10^{-10}	2.809 x 10^{-10}	4.088 x 10^{-13}
0.5	0.5	2.575 x 10^{-9}	2.575x 10^{-9}	5.855 x 10^{-12}

Table 2. *The absolute error of a 4-term approximation of Example 5.*

x_i	t_i	$\lvert u_{exact} - u_{app.4(Pic.M)}\rvert$	$\lvert u_{exact} - u_{app4(H.P.M)}\rvert$ [25]	$\lvert u_{exact} - u_{app(F.H.P.M)}\rvert$
0.1	0.1	1.141 x 10^{-17}	1.242 x 10^{-16}	0.0
0.2	0.2	1.86747 x 10^{-13}	2.044 x 10^{-12}	0.0
0.3	0.3	5.451 x 10^{-11}	5.968 x 10^{-10}	0.0
0.4	0.4	3.059 x 10^{-9}	3.349 x 10^{-8}	0.0
0.5	0.5	6.951 x 10^{-8}	7.615 x 10^{-7}	0.0

Table 3. *The absolute error of a 4-term approximation of Example 6.*

x_i	t_i	$\lvert u_{exact} - u_{app.4(Pic.M)}\rvert$	$\lvert u_{exact} - u_{app4(H.P.M)}\rvert$ [25]	$\lvert u_{exact} - u_{app(F.H.P.M)}\rvert$
0.1	0.1	7.911 x 10^{-13}	3.749 x 10^{-12}	0.0
0.2	0.2	4.015 x 10^{-10}	1.128 x 10^{-9}	0.0
0.3	0.3	1.518x 10^{-8}	3.175 x 10^{-8}	3.331 x 10^{-16}
0.4	0.4	1.969x 10^{-7}	3.229 x 10^{-7}	1.411 x 10^{-14}
0.5	0.5	1.413 x 10^{-6}	1.781 x 10^{-6}	2.545 x 10^{-13}

In Table 1, Table 2, and Table 3, the absolute errors for three methods have been computed: the Picard method, the

Homotopy perturbation method, and the Frobenius homotopy perturbation method [1]. These results shows the ease and accuracy of the outcomes of the last method after a few steps.

Remark:

The Absolute error for the Frobenius homotopy perturbation method is computed for only finite numbers of terms of the expansion (62), (65), and (73).

Example 7

Consider the following coupled Klein-Gordon equations:

$$u_{tt} - u_{xx} + sin\, u = 0, \tag{77}$$

with the initial conditions $u(A,t) = Sech(A-t)^2$ and $u(B,t) = Sech(B-t)^2$

$$u_t(y,0) = 2Sech(y)^2\, tanh\, y \text{ and } v_t(y,0) = 2Sech(y)^2\, tanh\, y,$$

where $u = u(y,t)$ and $v = v(y,t)$ are functions of the variables y and t.

On using the newly introduced method, the trial function may be written as follows

$$U(y,t) = \sum_{n=0}^{\infty} a_n(t) y^n \text{ and } V(y,t) = \sum_{n=0}^{\infty} b_n(t) y^n \tag{78}$$

The homotopy equation may be written as:

$$H_1(u,\rho) = \frac{\partial^2}{\partial y^2}(u - U) + \rho\{U_{yy} - u_{tt} - u + v\} = 0$$

$$H_2(v,\rho) = \frac{\partial^2}{\partial y^2}(v - V) + \rho\{V_{yy} - u_{tt} + u - v\} = 0 \tag{79}$$

In accordance with the homotopy perturbation, the solution $u(x,t)$ of the homotopy equation (79), with the aid of the artificial parameter ρ, may be expanded as before.

Substituting from the regular perturbation of the dependent function (u) into Eq. (79) and equating the coefficients of like powers of ρ in both sides, one gets the following equations:

$$\text{For H}_1\text{: } \rho^0\text{: } u_{0yy} = \sum_{n=2}^{\infty} n(n-1) a_n(t) y^{n-2}$$

$$\text{For H2: } \rho^0\text{: } v_{0yy} = \sum_{n=2}^{\infty} n(n-1) a_n(t) y^{n-2} \tag{80}$$

The special solution of Eq. (80) is directly

$$u_0 = Sech(t)^2 + \sum_{n=2}^{\infty} a_n(t) y^n - 2ySech(t)^2\, tanh\, t$$

$$v_0 = Sech(t)^2 + \sum_{n=2}^{\infty} b_n(t) y^n - 2ySech(t)^2\, tanh\, t \tag{81}$$

The first-order equation may be written as

$$u_{1yy} = u_{0tt} - \sum_{n=2}^{\infty} n(n-1) a_n(t) y^{n-2} + u_0 - v_0$$

$$v_{1yy} = v_{0tt} - \sum_{n=2}^{\infty} n(n-1) b_n(t) y^{n-2} - u_0 + v_0 \tag{82}$$

Substituting Eq. (81) into Eq. (82), then integrating twice with respect to y, it follows, after lengthy but straightforward calculations, and after canceling the first-order solution the undetermined coefficients $a_n(t)'s$ and $b_n(t)'s$ of Eq. (82) are as follows:

$$a_2(t) = (-Sech(A-t)^2 + 3\, Sech(A-t)^2\ tanh(A-t)^2)$$

$$a_3(t) = -\frac{4}{3}(-2\, Sech(A-t)^2\ tanh(A-t) + 3\, Sech(A-t)^2\ tanh(A-t)^3)$$

$$a_4(t) = \frac{1}{3}(2\, Sech(A-t)^2 - 15\, Sech(A-t)^2\ tanh(A-t)^2 + 15\, Sech(A-t)^2\ tanh(A-t)^4)$$

$$a_5(t) = \frac{2}{15}(17\, Sech(A-t)^2\ tanh(A-t)\text{-}60\, Sech(A-t)^2\ tanh(A-t)^3 + 45\, Sech(A-t)^2\ tanh(A-t)^5) \tag{83}$$

$$\vdots$$

and

$$b_2(t) = (-Sech(A-t)^2 + 3\, Sech(A-t)^2\ tanh(A-t)^2)$$

$$b_3(t) = -\frac{4}{3}(-2\, Sech(A-t)^2\ tanh(A-t) + 3\, Sech(A-t)^2\ tanh(A-t)^3)$$

$$b_4(t) = \frac{1}{3}\left(2\,Sech(A-t)^2 - 15\,Sech(A-t)^2\ tanh(A-t)^2 + 15\,Sech(A-t)^2\ tanh(A-t)^4\right)$$

$$b_5(t) = \frac{2}{15}\left(17\,Sech(A-t)^2\ tanh(A-t) - 60\,Sech(A-t)^2\ tanh(A-t)^3 + 45\,Sech(A-t)^2\ tanh(A-t)^5\right) \quad (84)$$

:

Substituting Eq. (83) and Eq. (84) into Eq. (81), it follows that the closed form of the solution given equation is given as

$$u(y,t) = Sech(A+y-t)^2$$

$$v(y,t) = Sech(A+y-t)^2, \quad (85)$$

which is the exact solution of the coupled Klein-Gordon equation that is given in Eq. (81). This is the same problem that was solved in [26] by DOHA et al. in 2014 using Jacobi-Gauss-Lobatto collocation, but they only got an approximate solution to the problem.

To this end, the present unique method demonstrates that it is a powerful mathematical instrument for solving Klein-Gordon equations of various categories, as well as a potential way for solving additional nonlinear equations as described in our prior work [1]. This solution solves the problem without requiring the variables to be discretized. The obtained series is calculated using the Mathematica software. It should be mentioned that Keskin et al. [20] used the reduced differential transform method (RDTM) to obtain the prior solution.

Example 8

Consider the following nonlinear homogeneous partial differential equation:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u - u\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial x} = 0, \quad (86)$$

subjected to the initial conditions: $u(x,0) = cos\,x$, $u_t(x,0) = -sin\,x$. and $u_{tt}(x,0) = -cos\,x$.

On using the newly introduced method as before, the trial function of Eq. (84) may be written as given by Eq. (4.16).

The homotopy equation of Eq. (86) may be written as:

$$H(u,\rho) = u_{ttt} - U_{tt} + \rho\{U_{tt} - u_{xxt} - u_{xxtt} + u_{xxxx} + uu_t - u_{tt}u_x\} = 0. \quad (87)$$

In accordance with the homotopy perturbation, the solution of the dependent function $u(x,t,\rho)$ as given in the homotopy equation (87), with the aid of the artificial parameter ρ, may be written as given by Eq. (87).

Substituting Eq. (44) into Eq. (87) and equating the coefficients of like powers of ρ of both sides, one gets the following equations:

$$\rho^0:\ u_{0ttt} = \textstyle\sum_{n=3}^{\infty} n(n-1)(n-2)a_n(x)\,t^{n-3}, \quad (88)$$

And

$$\rho:\ u_{1ttt} = u_{0xxt} + u_{0xxtt} - u_{0xxxx} + u_0u_{0t} + u_{0tt}u_{0x} - \textstyle\sum_{n=3}^{\infty} n(n-1)(n-2)a_n(x)\,t^{n-3} \quad (89)$$

The special solution of Eq. (88) is directly

$$u_0(x,t) = cos\,x - \frac{t^2}{2}cos\,x - t\,sin\,x + \textstyle\sum_{n=3}^{\infty} a_n(x)t^n. \quad (90)$$

Substituting Eq. (88) into Eq. (89) and then integrating partially triple with respect to t, it follows that the special solution of Eq. (89) may be written as:

$$u_1(x,t) = \frac{1}{6}(sin\,x - 6a_3)t^3 + \frac{1}{48}(1 + 2\,cos\,x - cos\,2\,x + 2\,sin\,x - 12a_3\,sin\,x - 48a_4 + 12a_3'')t^4$$
$$+\frac{1}{120}(cos\,x + sin\,2\,x - 6a_3\,cos\,x - 24a_4\,sin\,x - 120a_5 + 6a_3'' + 24a_4'')t^5$$
$$+\frac{1}{480}\left(1 + cos\,2\,x - 4(a_3 + 20a_5)\,sin\,x - 480a_6 - 4\,cos\,x\,(8a_4 + a_3') + 16a_4'' + 80a_5'' - 4a_3^{(4)}\right) + \ldots \quad (91)$$

The cancellation of the first-order solution as given by Eq. (52), yields the following unknown coefficients $a's$ as follows:

$$a_3 = \frac{sin\,x}{3!}, a_4 = \frac{cos\,x}{4!}, a_5 = -\frac{sin\,x}{5!}, a_6 = -\frac{cos\,x}{6!}, a_7 = \frac{sin\,x}{7!}, a_8 = \frac{cos\,x}{8!}, a_9 = -\frac{sin\,x}{9!}, a_{10} = -\frac{cos\,x}{10!} .. \quad (92)$$

Substituting Eq. (92) into Eq. (91), it follows that the closed form of the given cubic Duffing oscillator equation is given by

$$u(x,t) = cos(x+t). \quad (93)$$

5. Conclusion

This paper modifies the well-known homotopy perturbation approach to provide analytical solutions to some initial-value ordinary and partial differential equations. The solution's unity has been proved. The Picard method and the Homotopy perturbation approach, which were utilised in the study, were compared for a few problems. The results reveal that this method stands out for its accuracy and simplicity of application, as it requires fewer steps than others. Furthermore, unlike most previous methods, the suggested method finds forms of analytical solutions more readily and fast.

6. Future Work

The same method might be modified to solve fractional differential equations. Authors recommends that the reader might find this work important to expand it to work on differential equations of fractional order.

References

[1] Y. O. El-Dib, and G. M. Moatimid, On the coupling of the homotopy perturbation and Frobenius method for exact solutions of singular nonlinear differential equations, Nonlinear Science Letters A, 9 (3), 220-230 (2018).

[2] J. H. He, Homotopy perturbation technique, Computer Methods in Applied Mechanics and Engineering, 178 (3-4), 257-262 (1999).

[3] J. H. He, Recent development of the homotopy perturbation method, Topological methods in nonlinear analysis, 31 (2), 205-209 (2008).

[4] J. H. He, Homotopy perturbation method with an auxiliary term, Abstract and Applied Analysis, Volume 2012 |Article ID 857612 (2012).

[5] J. H. He, Homotopy perturbation method with two expanding parameters, Indian journal of Physics, 88 (2), 193-196 (2014).

[6] J. H. He, Amplitude-frequency relationship for conservative nonlinear oscillators with odd nonlinearities, International Journal of Applied and Computational Mathematics, 3 (2), 1557-1560 (2017).

[7] J. H. He, Taylor series solution for a third order boundary value problem arising in architectural engineering, Ain Shams Engineering Journal, 11 (4), 1411-1414 (2020).

[8] J. H. He, G. M. Moatimid, and D. M., Mostapha, Nonlinear instability of two streaming-superposed magnetic Reiner-Rivlin Fluids by He-Laplace method, Journal of Electroanalytical Chemistry, 895, 115388 (2021) (11 pages).

[9] A. M. Siddiqui, T. Haroon, and S. Irum, the Torsional flow of third grade fluid using the modified homotopy perturbation method, Computers & Mathematics with Applications, 58 (11-12), 2274-2285 (2009).

[10] A. M. Wazwaz, A new method for solving initial value problems in the second-order ordinary differential equations. Applied Mathematics and Computation, 128 (1), 45-57 (2002).

[11] M. S. H. Chowdhury, and I. Hashim, Application of homotopy-perturbation method to Klein–Gordon and sine-Gordon equations, Chaos Solitons & Fractals 39 (4), 1928-1935 (2009).

[12] L. Wei, X. Zhang, S. Kumar, and A. Yildirim, A numerical study based on an implicit fully discrete local discontinuous Galerkin method for the time-fractional coupled Schrödinger system Computers & Mathematics with Applications, 64, 2603-2615 (2012).

[13] J. Biazar, and H. Ghazvini, Exact solution of nonlinear Schrödinger equations by He's homotopy perturbation method, Physics Letters A, 366 (1-2), 79-84 (2007).

[14] D. D. Ganji, The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer, Physics Letters A, 355 (4-5), 337- 341 (2006).

[15] A. M. Odibat, and S. Momani, Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order, Chaos, Solitons and Fractals, 36, 167-147 (2008).

[16] H. Aminikhah, F. Pournasiri, and F. Mehrdoust, A novel effective approach for systems of coupled Schrödinger equation, Pramana-Journal of Physics, 86 (1), 19-30 (2016).

[17] A. H. Salas, and J. E. Castillo, Exact solution to Duffing equation and the pendulum equation, Applied Mathematical Sciences, 8 (176), 8781-8789 (2014).

[18] Moatimid, G. M., Stability Analysis of a Parametric Duffing Oscillator, Journal of Engineering Mechanics, 146 (5): 05020001 (2020) (13 pages).

[19] A. R. Seadawy, and K. El-Rashidy, Nonlinear Rayleigh-Taylor instability of the cylindrical fluid with mass and heat transfer, Pramana - Journal of Physics, 87: 20 (2016).

[20] Y. Keskin, I. Caglar, and A. B. Koc, Numerical solution of sine-Gordon equation by reduced differential transform method, Proceedings of the World Congress on Engineering 2011 Vol I WCE 2011, July 6 - 8, 2011, London, U.K. (2011).

[21] E. Olğar, and H. Mutaf, Bound-states of s-wave Klein-Gordon equation for Wood-Saxon potential, Advances in Mathematical Physics, Volume 2015, Article ID 923076, (5 pages) (2015).

[22] D. Kaya, An application of the modified decomposition method for two-dimensional sine-Gordon using the modified decomposition method, Applied Mathematics and Computation, 159, 1-9 (2004).

[23] A. M. Wazwaz, A variable separated ODE method for solving the triple sine-Gordon and triple sinh-Gordon equations, Chaos, Solitons & Fractals, 33, 703-710 (2007).

[24] S. S. Ray, A numerical solution of the coupled sine-Gordon equation using the modified decomposition method, Applied Mathematics and Computation, 175, 1064-1054 (2006).

[25] E. Babolian, A. Azizi, J. Saeidian, Some notes on using the homotopy perturbation for solving time-dependent differential equations, Mathmatical and Computer Modeling, 50, 213-224, (2009).

[26] E. H. Doha, A. H. Bhrawy, D. Baleanu, M. A. Abdelkawy, Numerical treatment of coupled nonlinear hyperbolic Klein-Gordon equations, Romanian Journal of Physics, 59, 247-264, (2014).