Control Science and Engineering
2024; 8(1): 13-28
http://www.sciencepublishinggroup.com/j/cse
doi: 10.11648/j.cse.20240801.12
ISSN: 2994-7421 (Online)

Near-optimality Conditions for Relaxed and Strict Mean-field Singular FBSDEs

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To cite this article:
Ruijing Li. (2024). Near-optimality Conditions for Relaxed and Strict Mean-field Singular FBSDEs. *Control Science and Engineering, 8*(1), 13-28. https://doi.org/10.11648/j.cse.20240801.12

Received: January 4, 2024; **Accepted:** January 17, 2024; **Published:** January 24, 2024

Abstract: In this paper, we investigate the relaxed and strict near-optimality conditions for mean-field singular FBSDEs, where the coefficients depend on the state of the solution process as well as of its expected value. Moreover, the cost functional is also of mean-field type. This makes the control problem time inconsistent in the sense that the Bellman's optimality principle does not hold. The purpose of this paper is to establish necessary and sufficient conditions of near-optimality for relaxed and strict mean-field singular controls. For strict mean-field singular FBSDEs, whose wellposedness is ensured under the twice continuously differentiable assumptions of coefficients. Then, the moment estimations of variational processes as well as first-order and second-order adjoint processes are presented by using Burkholder-Davis-Gundy inequality. Further, by introducing Hamiltonian function via Ekeland's variational principle, the necessary near-optimality conditions are established. For relaxed mean-field singular FBSDEs, we first give the definition of admissible set of relaxed singular controls, then use the mapping defined by Dirac measure, we prove that the near-optimal problem of strict singular controls is a particular case of the near-optimal problem of relaxed singular ones. Further, a well known chattering lemma is introduced. By virtue of this famous lemma addition with the stability of trajectories with respect to the control variable and dominated convergence theorem, necessary as well as sufficient near-optimality conditions for relaxed controls are established.

Keywords: Near-optimal Singular Control, Mean-field SDE, Relaxed and Strict Control, Adjoint Equation, Ekeland's Variational Principle

1. Introduction

We are interested in the mean-field singular stochastic control problems of systems governed by the following stochastic differential equation,

$$\begin{cases} dx_t^{(u,\xi)} = b(t, x_t^{(u,\xi)}, Ex_t^{(u,\xi)}, u_t)dt + \sigma(t, x_t^{(u,\xi)}, Ex_t^{(u,\xi)}, u_t)dW_t + G_t d\xi_t, \\ x_s^{(u,\xi)} = y, \end{cases} \tag{1}$$

where b, σ and G are given deterministic functions, $(W_t)_{t\geq 0}$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.

The cost functional associated with (1) is given by

$$J(s, y, u, \xi) = E[g(x_T^{(u,\xi)}, Ex_T^{(u,\xi)}) + \int_s^T l(t, x_t^{(u,\xi)}, Ex_t^{(u,\xi)}, u_t)dt + \int_s^T k_t d\xi_t],$$

with the value function

$$V(s,y)=\inf_{(u,\xi)\in\mathcal{U}}J(s,y,u,\xi),$$

where $\mathcal{U}$ is the set of admissible controls.

Singular stochastic control problems have been studied extensively. The key approachs are the dynamic programming and the Pontryagin's maximum principle. By using the second approach, the first version of stochastic maximum principle are obtained under the condition of linear dynamics, convex cost functional and convex state constraints [9]. For nonlinear singular SDEs with controlled diffusion coefficient, necessary optimal conditions were established by [5]. And we refer the reader to [3, 24] for more information.

As is well known, for the strict control problem, the optimal solution may not exist even in the simplest situations. However if the strict control domain $\mathcal{U}$ is embedded into a larger space $P(U)$, which is the space of probability measures on U equipped with the topology of stable convergence of measures, the existence of optimal solution can be ensured. For example, in 1995, using compactification techniques, the existence of an optimal singular control is derived [17]. And on the database of the above result, a relaxed singular stochastic maximum principle is established in the case of uncontrolled diffusion coefficient [4]. For the controlled diffusion term, refer to [2].

It needs to be emphasized that, it is enough to find near-optimal solutions with a more practical problem. For the purpose of practical use, near-optimal control problems of the forward stochastic differential equation were firstly studied [28]. Furthermore, in 2012, necessary and sufficient conditions of near-optimality for singular stochastic systems with controlled diffusion coefficient were derived [16]. While this paper mainly studies mean-field singular stochastic systems. It is obtained as the mean-square limit of a system of interacting particles

$$dx_t^{i,n}=b(t,x_t^{i,n},\frac{1}{n}\Sigma_{j=1}^n x_t^{i,n},u_t)dt+\sigma(t,x_t^{i,n},\frac{1}{n}\Sigma_{j=1}^n x_t^{i,n},u_t)dW_t^i+G_t d\xi_t,$$

when $n\to\infty$, which is the classical McKean-Vlasov model (see [20]). So it is meaningful to study the near-optimality of the above system both from theoretical aspects and applicative ones.

On the other hand, it is well known that the existence of optimal strict control is ensured by the Filippov condition. That is to say, without the Filippov condition, the optimal strict control may not exist. To overcome this difficulty, the technique can be refined to introduce a bigger class equipped with a richer topological structure, which can ensure the existence of the optimal solution. And call this class the relaxed control set, whose elements are at time t probability measures $q_t(da)$ defined on U. Using compactification techniques, the first result of the existence of optimal relaxed control was derived [15], and then a relaxed stochastic maximum principle with controlled diffusion coefficient was established [4]. More versions of relaxed stochastic maximum principle refer to [1, 10, 11, 13, 18, 23, 25]. Further, notice the fact that the cost functional J may be nonlinear with respect to the expectation, makes the control problem time inconsistent in the sense that Bellman's optimality principle does not hold. To solve this problem, by using Malliavin calculus, a stochastic maximum principle of mean-field type with convex control domain was obtained [21]. However, when referring to non-convex control domain, in this paper, we adopt the following specific method to study the problem. Roughly speaking, a double perturbation of the optimal control is adopted, the convex perturbation to the singular part and the spike perturbation to the absolutely continuous part. Then we introduce the first order adjoint equation, which is a linear mean-field backward SDE, and the second order adjoint equation, which remains the same as in [22]. By virtue of Ekeland's variational principle and some stabilities of trajectories with respect to the control variable, necessary near-optimal conditions are established. Furthermore, under some additional assumptions, we prove that obtained necessary conditions are also sufficient.

The paper is organized as follows: Some statements of strict singular control problems are in Section 2. In Section 3, we give adjoint processes and some prior estimates. By virtue of these estimates, necessary as well as sufficient near-optimal conditions are established for strict mean-field singular controls in Section 4. Finally, in Section 5 and 6, we state relaxed singular control problems and get the corresponding results based on the obtained conclusions in Section 4.

2. The Strict Mean-field Singular Control Problem

Assume a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},P)$ satisfies the usual condition, on this space we define a R^d-valued standard Brownian motion $(W_t)_{t\geq0}$, and the natural filtration $\{\mathcal{F}_t\}_{t\geq0}$ generated by $(W_t)_{t\geq0}$.

Let A_1 be a nonempty compact subset of R^k, $A_2=([0,\infty))^m$ and U_1 the class of measurable, $\mathcal{F}_t$-adapted processes valued in A_1, U_2 the class of measurable, $\mathcal{F}_t$-adapted processes ξ valued in A_2 such that ξ is nondecreasing, left continuous with right limits and $\xi_s=0$.

For any given $s\in[0,T)$, a pair of $\mathcal{F}_t$-adapted processes $(u,\xi)\in U_1\times U_2$ is called admissible, if it satisfies $E[\sup_{s\leq t\leq T}|u_t|^2+|\xi_T|^2]<\infty$. We denote by $\mathcal{U}=\mathcal{U}_1\times\mathcal{U}_2$ the set of all admissible controls.

For any $(u,\xi)\in\mathcal{U}$, we consider the following mean-field stochastic control system,

$$\begin{cases} dx_t^{(u,\xi)} = b(t, x_t^{(u,\xi)}, Ex_t^{(u,\xi)}, u_t)dt + \sigma(t, x_t^{(u,\xi)}, Ex_t^{(u,\xi)}, u_t)dW_t + G_t d\xi_t, \\ x_s^{(u,\xi)} = y, \end{cases} \tag{2}$$

where mappings $b : [s,T] \times R^n \times R^n \times A_1 \to R^n$; $\sigma : [s,T] \times R^n \times R^n \times A_1 \to M_{n\times d}(R)$; $G : [s,T] \to M_{n\times m}(R)$ are given deterministic functions.

The cost functional to be minimized over $\mathcal{U}$ is given by

$$J(s,y,u,\xi) = E[g(x_T^{(u,\xi)}, Ex_T^{(u,\xi)}) + \int_s^T l(t, x_t^{(u,\xi)}, Ex_t^{(u,\xi)}, u_t)dt + \int_s^T k_t d\xi_t], \tag{3}$$

where mappings $l : [s,T] \times R^n \times R^n \times A_1 \to R$, $g : R^n \times R^n \to R$, $k : [s,T] \to A_2$ are determinate functions, and we define the value function by

$$V(s,y) = \inf_{(u,\xi)\in\mathcal{U}} J(s,y,u,\xi). \tag{4}$$

Throughout this paper, we make use of the following notations:

f_x : the gradient or Jacobian of a function f with respect to x;

f_{xx} : the Hessian of a scalar function f with respect to x;

$|\cdot|$: the norm of an Euclidean space;

χ_A : the indicator function of a set A;

$M_{n\times d}(R)$: the space of $n \times d$ real matrices;

C, C_i, $i = 1, 2, \cdots$: multiplicative constants required in the analysis.

Since the objective of this paper is to study near-optimality, we give the precise definition as in [28].

Definition 1 Both a family of admissible pairs $\{(u^\varepsilon, \xi^\varepsilon)\}$ parameterized by $\varepsilon > 0$ and any element $(u^\varepsilon, \xi^\varepsilon)$ in the family are called near-optimal if

$$|J(s,y,u^\varepsilon,\xi^\varepsilon) - V(s,y)| \le r(\varepsilon)$$

holds for sufficiently small $\varepsilon > 0$, where r is a function of ε satisfying $\lim_{\varepsilon\to 0} r(\varepsilon) = 0$. The estimate $r(\varepsilon)$ is called an error bound. If $r(\varepsilon) = C\varepsilon^\delta$ for some $\delta > 0$ independent of the constant C, then $(u^\varepsilon, \xi^\varepsilon)$ is called near-optimal with order ε^δ. Especially when $r(\varepsilon) = \varepsilon$, the admissible control $(u^\varepsilon, \xi^\varepsilon)$ is called ε-optimal.

Next, we make some assumptions.

(A1) b, σ, l are linear growth with $(x, \tilde{x}, u)$, they and their first-order derivative are Lipschitz continuous in $x, \tilde{x}$.

(A2) g and its first-order derivative are linear growth in $(x, \tilde{x}, u)$ and Lipschitz continuous in $x, \tilde{x}$.

(A3) G and k are continuous and G is bounded.

Under above assumptions, $\forall\ (u,\xi) \in \mathcal{U}$, the system (2) and the cost functional J are all with wellposedness. Furthermore, $\forall\ q > 0$, there exists a constant C such that

$$E(\sup_{t\in[s,T]} |x_t^{(u,\xi)}|^q) < C(q).$$

Now let us recall the definition of the Clarke generalized gradient as well as Ekeland's principle, which will be used in the sequel.

Definition 2 ([12]) Let X be a convex set in R^d and let $\eta(\cdot) : X \to R$ be a locally Lipschitz function. The generalized gradient of η at $\bar{x} \in X$, denoted by $\partial_x \eta(\bar{x})$, is a set defined by

$$\partial_x \eta(\bar{x}) = \{p \in R^d | p\xi \le \eta^0(\bar{x};\xi), for\ any\ \xi \in R^d\},$$

where

$$\eta^0(\bar{x};\xi) = \limsup_{x\in X, x+h\xi\in X, x\to\bar{x}, h\to 0^+} \frac{\eta(x+h\xi) - \eta(x)}{h}.$$

Lemma 3 ([14]) Let (S,d) be a complete metric space and $\rho(\cdot) : S \to R$ be lower-semicontinuous and bounded from below. For a given $\varepsilon > 0$, suppose $v^\varepsilon \in S$ satisfying $\rho(v^\varepsilon) \le \inf_{v\in S} \rho(v) + \varepsilon$. Then for any $\lambda > 0$, there exists a $v^\lambda \in S$ such that

$$\rho(v^\lambda) \le \rho(v^\varepsilon),$$
$$d(v^\lambda, v^\varepsilon) \le \lambda,$$
$$\rho(v^\lambda) \le \rho(v) + \frac{\varepsilon}{\lambda} d(v, v^\lambda), \quad \forall v \in S.$$

To apply Ekeland's variational principle, we endow the set $\mathcal{U}$ with an appropriate metric. More precisely, for any $(u,\xi), (v,\eta) \in \mathcal{U}$, we define

$$d_1(u,v) = P \otimes dt\{(w,t) \in \Omega \times [s,T] : u(w,t) \ne v(w,t)\};$$

$$d_2(\xi,\eta) = [E(\sup_{t\in[s,T]} |\xi_t - \eta_t|^2)]^{\frac{1}{2}};$$

$$d((u,\xi),(v,\eta)) = d_1(u,v) + d_2(\xi,\eta);$$

where $P \otimes dt$ is the product measure of P with the Lebesgue measure dt.

Remark 4 According to Lemma 4.5 of [4], $(\mathcal{U}, d)$ is a complete metric space and the cost functional J is continuous from $\mathcal{U}$ into R.

3. Adjoint Equations and Some Prior Estimates

This section is devoted to give the first order adjoint equation of mean-field type, and the second order adjoint equation remaining the same as in [22], and some prior estimates for later use.

Lemma 5 Let (A1), (A3) hold, for any $(u,\xi), (v,\eta) \in \mathcal{U}$ and $0 < \alpha < 1$, $\beta > 0$ satisfying

$\alpha\beta < 1$, we have

$$E(\sup_{t\in[s,T]} |x_t^{(u,\xi)} - x_t^{(v,\eta)}|^{2\beta}) \leq Cd^{\alpha\beta}((u,\xi),(v,\eta)),$$

where $C > 0$ is a constant, $x_t^{(u,\xi)}$ and $x_t^{(v,\eta)}$ are trajectories corresponding to (u,ξ) and (v,η) respectively.

Proof Let $\beta \geq 1$. $\forall\, r \geq s$, via BDG inequality, we deduce

$$\begin{aligned}
& E\sup_{s\leq t\leq r} |x_t^{(u,\xi)} - x_t^{(v,\eta)}|^{2\beta} \\
\leq\ & C_1 E\int_s^r |b(t,x_t^{(u,\xi)},Ex_t^{(u,\xi)},u_t) - b(t,x_t^{(v,\eta)},Ex_t^{(v,\eta)},v_t)|^{2\beta} \\
& +|\sigma(t,x_t^{(u,\xi)},Ex_t^{(u,\xi)},u_t) - \sigma(t,x_t^{(v,\eta)},Ex_t^{(v,\eta)},v_t)|^{2\beta} dt + C_1 E|\xi_T - \eta_T|^{2\beta} \\
\leq\ & C_2 E\int_s^r |b(t,x_t^{(u,\xi)},Ex_t^{(u,\xi)},u_t) - b(t,x_t^{(v,\eta)},Ex_t^{(v,\eta)},u_t)|^{2\beta} \\
& +|b(t,x_t^{(v,\eta)},Ex_t^{(v,\eta)},u_t) - b(t,x_t^{(v,\eta)},Ex_t^{(v,\eta)},v_t)|^{2\beta}\chi_{u_t\neq v_t} dt \\
& +C_2 E\int_s^r |\sigma(t,x_t^{(u,\xi)},Ex_t^{(u,\xi)},u_t) - \sigma(t,x_t^{(v,\eta)},Ex_t^{(v,\eta)},u_t)|^{2\beta} \\
& +|\sigma(t,x_t^{(v,\eta)},Ex_t^{(v,\eta)},u_t) - \sigma(t,x_t^{(v,\eta)},Ex_t^{(v,\eta)},v_t)|^{2\beta}\chi_{u_t\neq v_t} dt \\
& +C_2 E|\xi_T - \eta_T|^{2\beta}.
\end{aligned} \tag{5}$$

Taking $p = \frac{1}{\alpha\beta} > 1$, there exists a $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Noting (A1), the definition of d, d_1 and d_2, (5) becomes

$$\begin{aligned}
& E\sup_{s\leq t\leq r} |x_t^{(u,\xi)} - x_t^{(v,\eta)}|^{2\beta} \\
\leq\ & C_3 E\int_s^r \sup_{s\leq t\leq \tau} |x_t^{(u,\xi)} - x_t^{(v,\eta)}|^{2\beta} d\tau \\
& +C_3\{E\int_s^T [1+|x_t^{(v,\eta)}| + |Ex_t^{(v,\eta)}| + |u_t| + |v_t|]^{2\beta q} dt\}^{\frac{1}{q}} \{E\int_s^r \chi_{u_t\neq v_t} dt\}^{\frac{1}{p}} \\
& +C_3\{E|\xi_T - \eta_T|^{4\beta p}\}^{\frac{1}{2p}} \\
\leq\ & C_4 E\int_s^r \sup_{s\leq t\leq \tau} |x_t^{(u,\xi)} - x_t^{(v,\eta)}|^{2\beta} d\tau + C_4 d_1^{\alpha\beta}(u,v) + C_4 d_2^{\alpha\beta}(\xi,\eta) \\
\leq\ & C_5 E\int_s^r \sup_{s\leq t\leq \tau} |x_t^{(u,\xi)} - x_t^{(v,\eta)}|^{2\beta} d\tau + C_5 d^{\alpha\beta}((u,\xi),(v,\eta)).
\end{aligned} \tag{6}$$

By using Gronwall's inequality to (6), the estimation follows.

For $0 < \beta < 1$, we can get the following estimation by Cauchy-Schwartz inequality,

$$E\sup_{s\leq t\leq T} |x_t^{(u,\xi)} - x_t^{(v,\eta)}|^{2\beta} \leq \{E\sup_{s\leq t\leq T} |x_t^{(u,\xi)} - x_t^{(v,\eta)}|^2\}^\beta \leq Cd^{\alpha\beta}((u,\xi),(v,\eta)).$$

Thus, we complete the proof.

Define the Hamiltonian associated with the random variable X as follows:

$$H(t,X,u,p,q) = b(t,X,E[X],u)p + \sigma(t,X,E[X],u)q - l(t,X,E[X],u), \tag{7}$$

where (p,q) is the solution of (8).

Next, we introduce the following adjoint equations:

$$\begin{cases} -dp_t^{(u,\xi)} = \{b_x^{(u,\xi)} p_t^{(u,\xi)} + \sigma_x^{(u,\xi)} q_t^{(u,\xi)} - l_x^{(u,\xi)} + E[b_{\tilde{x}}^{(u,\xi)} p_t^{(u,\xi)}] + E[\sigma_{\tilde{x}}^{(u,\xi)} q_t^{(u,\xi)}] - El_{\tilde{x}}^{(u,\xi)}\}dt - q_t^{(u,\xi)} dW_t, \\ p_T^{(u,\xi)} = -g_x^{(u,\xi)}(T) - Eg_{\tilde{x}}^{(u,\xi)}(T), \end{cases} \tag{8}$$

and

$$\begin{cases} -dP_t^{(u,\xi)} = \{2b_x^{(u,\xi)} P_t^{(u,\xi)} + [\sigma_x^{(u,\xi)}]^2 P_t^{(u,\xi)} + 2\sigma_x^{(u,\xi)} Q_t^{(u,\xi)} + H_{xx}^{(u,\xi)}\}dt - Q_t^{(u,\xi)} dW_t, \\ P_T^{(u,\xi)} = -g_{xx}^{(u,\xi)}(T), \end{cases} \tag{9}$$

where $H_{xx}^{(u,\xi)} = b_{xx}^{(u,\xi)}p_t^{(u,\xi)} + \sigma_{xx}^{(u,\xi)}q_t^{(u,\xi)} - l_{xx}^{(u,\xi)}$, $\rho^{(u,\xi)} = \rho(t, x_t^{(u,\xi)}, Ex_t^{(u,\xi)}, u_t)$, $\rho = b, \sigma, l$, and $g^{(u,\xi)}(T) = g(x_T^{(u,\xi)}, Ex_T^{(u,\xi)})$.

Under assumptions (A1) and (A2), according to Theorem 3.1 of [8], (8) and (9) admit unique $\mathcal{F}_t$-adapted solutions (p,q) and (P,Q) respectively, such that

$$E[\sup_{t\in[s,T]}|p_t|^2 + \int_s^T |q_t|^2 dt] < \infty. \tag{10}$$

$$E[\sup_{t\in[s,T]}|P_t|^2 + \int_s^T |Q_t|^2 dt] < \infty. \tag{11}$$

In the following lemma, the continuity associated with adjoint states with respect to the metric d is given.

Lemma 6 For any $0 < \alpha < 1$ and $1 < \beta < 2$ satisfying $(1+\alpha)\beta < 2$, there exists a constant $C > 0$, such that for (u,ξ), $(v,\eta) \in \mathcal{U}$, the corresponding solutions of adjoint equations (8) and (9) satisfy

$$E\int_s^T |p_t^{(u,\xi)} - p_t^{(v,\eta)}|^\beta + |q_t^{(u,\xi)} - q_t^{(v,\eta)}|^\beta dt \le C d^{\frac{\alpha\beta}{2}}((u,\xi),(v,\eta)),$$

and

$$E\int_s^T |P_t^{(u,\xi)} - P_t^{(v,\eta)}|^\beta + |Q_t^{(u,\xi)} - Q_t^{(v,\eta)}|^\beta dt \le C d^{\frac{\alpha\beta}{2}}((u,\xi),(v,\eta)).$$

Proof First, we set $\bar{p}_t = p_t^{(u,\xi)} - p_t^{(v,\eta)}$, $\bar{q}_t = q_t^{(u,\xi)} - q_t^{(v,\eta)}$, then $\bar{p}_t$, $\bar{q}_t$ satisfy the following backward SDE:

$$\begin{cases} -d\bar{p}_t = \{b_x^{(u,\xi)}\bar{p}_t + \sigma_x^{(u,\xi)}\bar{q}_t + E[b_{\tilde{x}}^{(u,\xi)}\bar{p}_t] + E[\sigma_{\tilde{x}}^{(u,\xi)}\bar{q}_t] + \bar{h}_t\}dt - \bar{q}_t dW_t, \\ \bar{p}_T = -g_x^{(u,\xi)}(T) + g_x^{(v,\eta)}(T) - Eg_{\tilde{x}}^{(u,\xi)}(T) + Eg_{\tilde{x}}^{(v,\eta)}(T), \end{cases}$$

where

$$\begin{aligned} \bar{h}_t = & (b_x^{(u,\xi)} - b_x^{(v,\eta)})p_t^{(v,\eta)} + (\sigma_x^{(u,\xi)} - \sigma_x^{(v,\eta)})q_t^{(v,\eta)} \\ & + E[(b_{\tilde{x}}^{(u,\xi)} - b_{\tilde{x}}^{(v,\eta)})p_t^{(v,\eta)}] + E[(\sigma_{\tilde{x}}^{(u,\xi)} - \sigma_{\tilde{x}}^{(v,\eta)})q_t^{(v,\eta)}] \\ & - l_x^{(u,\xi)} + l_x^{(v,\eta)} - El_{\tilde{x}}^{(u,\xi)} + El_{\tilde{x}}^{(v,\eta)}. \end{aligned}$$

Assume ψ is the following SDEs' solution:

$$\begin{cases} d\psi_t = [b_x^{(u,\xi)}\psi_t + b_{\tilde{x}}^{(u,\xi)}E\psi_t + |\bar{p}_t|^{\beta-1}sgn(\bar{p}_t)]dt + [\sigma_x^{(u,\xi)}\psi_t + \sigma_{\tilde{x}}^{(u,\xi)}E\psi_t + |\bar{q}_t|^{\beta-1}sgn(\bar{q}_t)]dW_t, \\ \psi_s = 0, \end{cases} \tag{12}$$

By using Hölder's inequality and (10), we have

$$E\int_s^T ||\bar{p}_t|^{\beta-1}sgn(\bar{p}_t)|^2 + ||\bar{q}_t|^{\beta-1}sgn(\bar{q}_t)|^2 dt < \infty.$$

Noting (A1), the existence and uniqueness of the solution of (12) are ensured.

On the other hand, since $1 < \beta < 2$, there exists $\gamma > 2$, such that $\frac{1}{\beta} + \frac{1}{\gamma} = 1$. From (12), we can get

$$\begin{aligned} E\sup_{t\in[s,T]}|\psi_t|^\gamma & \le C_1 E\int_s^T |\bar{p}_t|^{\beta\gamma-\gamma} + |\bar{q}_t|^{\beta\gamma-\gamma}dt \\ & = C_1 E\int_s^T |\bar{p}_t|^\beta + |\bar{q}_t|^\beta dt. \end{aligned}$$

Noting (10), we have $E\sup_{t\in[s,T]}|\psi_t|^\gamma < \infty$.

Applying Itô's formula to $\bar{p}_t\psi_t$ and taking expectations, we get

$$E\int_s^T \bar{p}_t|\bar{p}_t|^{\beta-1}sgn(\bar{p}_t) + \bar{q}_t|\bar{q}_t|^{\beta-1}sgn(\bar{q}_t)dt$$

$$\begin{aligned}
&= E[\bar{p}_T\psi_T] + E\int_s^T \psi_t\bar{h}_t dt \\
&\le \{E|\bar{p}_T|^\beta\}^{\frac{1}{\beta}}\{E|\psi_T|^\gamma\}^{\frac{1}{\gamma}} + \{E\int_s^T |\bar{h}_t|^\beta dt\}^{\frac{1}{\beta}}\{E\int_s^T |\psi_t|^\gamma dt\}^{\frac{1}{\gamma}} \\
&\le C_2\{E\int_s^T |\bar{p}_t|^\beta + |\bar{q}_t|^\beta dt\}^{\frac{1}{\gamma}}\{[E|\bar{p}_T|^\beta]^{\frac{1}{\beta}} + [E\int_s^T |\bar{h}_t|^\beta dt]^{\frac{1}{\beta}}\}. \qquad (13)
\end{aligned}$$

The left side of (13) equals to

$$E\int_s^T |\bar{p}_t|^\beta + |\bar{q}_t|^\beta dt. \qquad (14)$$

Furthermore, noting $\frac{\alpha\beta}{2} < 1 - \frac{\beta}{2} < 1$, (A2) and Lemma 5, we have

$$\begin{aligned}
E|\bar{p}_T|^\beta &= E|-g_x^{(u,\xi)}(T) + g_x^{(v,\eta)}(T) - Eg_{\tilde{x}}^{(u,\xi)}(T) + Eg_{\tilde{x}}^{(v,\eta)}(T)|^\beta \\
&\le C_3E|x_T^{(u,\xi)} - x_T^{(v,\eta)}|^\beta \\
&\le C_4d^{\frac{\alpha\beta}{2}}((u,\xi),(v,\eta)). \qquad (15)
\end{aligned}$$

and

$$\begin{aligned}
& E\int_s^T |b_x^{(u,\xi)} - b_x^{(v,\eta)}|^\beta|p_t^{(v,\eta)}|^\beta dt + E\int_s^T |E[(b_{\tilde{x}}^{(u,\xi)} - b_{\tilde{x}}^{(v,\eta)})p_t^{(v,\eta)}]|^\beta dt \\
\le\ & C_5E\int_s^T |b_x(t,x_t^{(u,\xi)},Ex_t^{(u,\xi)},u_t) - b_x(t,x_t^{(v,\eta)},Ex_t^{(v,\eta)},u_t)|^\beta|p_t^{(v,\eta)}|^\beta \\
& +|b_x(t,x_t^{(v,\eta)},Ex_t^{(v,\eta)},u_t) - b_x(t,x_t^{(v,\eta)},Ex_t^{(v,\eta)},v_t)|^\beta\chi_{u_t\neq v_t}|p_t^{(v,\eta)}|^\beta dt \\
& +C_5E\int_s^T |b_{\tilde{x}}(t,x_t^{(u,\xi)},Ex_t^{(u,\xi)},u_t) - b_{\tilde{x}}(t,x_t^{(v,\eta)},Ex_t^{(v,\eta)},u_t)|^\beta|p_t^{(v,\eta)}|^\beta \\
& +|b_{\tilde{x}}(t,x_t^{(v,\eta)},Ex_t^{(v,\eta)},u_t) - b_{\tilde{x}}(t,x_t^{(v,\eta)},Ex_t^{(v,\eta)},v_t)|^\beta\chi_{u_t\neq v_t}|p_t^{(v,\eta)}|^\beta dt \\
\le\ & C_6E\int_s^T |x_t^{(u,\xi)} - x_t^{(v,\eta)}|^\beta|p_t^{(v,\eta)}|^\beta + \chi_{u_t\neq v_t}|p_t^{(v,\eta)}|^\beta dt \\
\le\ & C_7\{E\int_s^T |x_t^{(u,\xi)} - x_t^{(v,\eta)}|^{\frac{2\beta}{2-\beta}}dt\}^{\frac{2-\beta}{2}}\{E\int_s^T |p_t^{(v,\eta)}|^2dt\}^{\frac{\beta}{2}} \\
& +C_7\{E\int_s^T |p_t^{(v,\eta)}|^2dt\}^{\frac{\beta}{2}}\{E\int_s^T \chi_{u_t\neq v_t}dt\}^{\frac{2-\beta}{2}} \\
\le\ & C_8\{d^{\frac{\alpha\beta}{2-\beta}}((u,\xi),(v,\eta))\}^{\frac{2-\beta}{2}} + C_8d^{\frac{2-\beta}{2}}((u,\xi),(v,\eta)) \\
\le\ & C_9d^{\frac{\alpha\beta}{2}}((u,\xi),(v,\eta)). \qquad (16)
\end{aligned}$$

Using the similar procedure to σ, we can get

$$\begin{aligned}
& E\int_s^T |\sigma_x^{(u,\xi)} - \sigma_x^{(v,\eta)}|^\beta|q_t^{(v,\eta)}|^\beta dt + E\int_s^T |E[(\sigma_{\tilde{x}}^{(u,\xi)} - \sigma_{\tilde{x}}^{(v,\eta)})q_t^{(v,\eta)}]|^\beta dt \\
& \le C_{10}d^{\frac{\alpha\beta}{2}}((u,\xi),(v,\eta)). \qquad (17)
\end{aligned}$$

and

$$E\int_s^T |l_x^{(u,\xi)} - l_x^{(v,\eta)}|^\beta + |E(l_{\tilde{x}}^{(u,\xi)} - l_{\tilde{x}}^{(v,\eta)})|^\beta dt \le C_{11}d^{\frac{\alpha\beta}{2}}((u,\xi),(v,\eta)) \qquad (18)$$

Combining (16)-(18), we can obtain

$$E\int_s^T |\bar{h}_t|^\beta dt \le C_{12}d^{\frac{\alpha\beta}{2}}((u,\xi),(v,\eta)) \qquad (19)$$

The desired result follows from (13), (14), (15) and (19).
Similarly, we can prove the second inequality.

4. Necessary and Sufficient Near-optimality Conditions for Strict Controls

In this section, we give the main result of this paper. First, we define the $\mathcal{H}$-function as follows:

$$\mathcal{H}(t,X,u)=H(t,X,u,p,q+P\sigma(t,x_t,Ex_t,u_t))-\frac{1}{2}\sigma^2(t,X,E[X],u)P. \tag{20}$$

Theorem 7 For any $\delta\in(0,\frac{1}{3}]$ and any ε-optimal singular control $(u^\varepsilon,\xi^\varepsilon)$, there exists a constant $C=C(\delta)>0$ such that for each $\varepsilon>0$,

$$E\int_s^T\mathcal{H}(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},u_t^\varepsilon)dt\geq\sup_{v\in\mathcal{U}_1}E\int_s^T\mathcal{H}(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},v_t)dt-C\varepsilon^\delta,$$

and

$$-C\varepsilon^\delta\leq E\int_s^T(k_t+G_t^Tp_t^{(u^\varepsilon,\xi^\varepsilon)})d(\eta_t-\xi_t^\varepsilon).$$

Proof According to Remark 4, via Ekeland's variational principle with $\lambda=\varepsilon^{\frac{2}{3}}$, there exists an admissible pair $(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)$, such that $\forall\ (u,\xi)\in\mathcal{U}$,

$$d((u^\varepsilon,\xi^\varepsilon),(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon))\leq\varepsilon^{\frac{2}{3}},$$

and

$$J^\varepsilon(s,y,\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)\leq J^\varepsilon(s,y,u,\xi), \tag{21}$$

where $J^\varepsilon(s,y,u,\xi)=J(s,y,u,\xi)+\varepsilon^{\frac{1}{3}}d((u,\xi),(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon))$. It means that the control pair $(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)$ is optimal for the system (2) with the new cost functional J^ε. Next, we use a double perturbation of the control $(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)$ to derive the variational inequality. More precisely, for $\tau\in[s,T)$, $v\in\mathcal{U}_1$, $\eta\in\mathcal{U}_2$ and sufficiently small $\theta>0$, we define

$$(u_t^\theta,\xi_t^\theta)=\begin{cases}(v,\bar{\xi}_t^\varepsilon+\theta(\eta_t-\bar{\xi}_t^\varepsilon)), & t\in[\tau,\tau+\theta],\\ (\bar{u}_t^\varepsilon,\bar{\xi}_t^\varepsilon+\theta(\eta_t-\bar{\xi}_t^\varepsilon)), & \text{otherwise},\end{cases}$$

Since $J^\varepsilon(s,y,\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)\leq J^\varepsilon(s,y,u^\theta,\bar{\xi}^\varepsilon)$ and $d((\bar{u}^\varepsilon,\bar{\xi}^\varepsilon),(u^\theta,\bar{\xi}^\varepsilon))\leq\theta$, we have

$$-\theta\varepsilon^{\frac{1}{3}}\leq J(s,y,u^\theta,\bar{\xi}^\varepsilon)-J(s,y,\bar{u}^\varepsilon,\bar{\xi}^\varepsilon). \tag{22}$$

We can see that the right side of (22) is independent of the singular part, by [7], (22) becomes

$$\begin{aligned}-\theta\varepsilon^{\frac{1}{3}}\ \leq\ & -E\int_\tau^{\tau+\theta}[b(t,x_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},v)-b(t,x_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},\bar{u}_t^\varepsilon)]p_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)}\\ &+[\sigma(t,x_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},v)-\sigma(t,x_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},\bar{u}_t^\varepsilon)]q_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)}\\ &-l(t,x_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},v)+l(t,x_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},\bar{u}_t^\varepsilon)\\ &+\frac{1}{2}P_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)}[\sigma(t,x_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},v)-\sigma(t,x_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},\bar{u}_t^\varepsilon)]^2dt+o(\theta).\end{aligned} \tag{23}$$

Dividing (23) by θ and sending $\theta\to 0$, we derive

$$\begin{aligned}-\varepsilon^{\frac{1}{3}}\ \leq\ & -E\{[b(\tau,x_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},v)-b(\tau,x_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},\bar{u}_\tau^\varepsilon)]p_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)}\\ &+[\sigma(\tau,x_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},v)-\sigma(\tau,x_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},\bar{u}_\tau^\varepsilon)]q_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)}\\ &-l(\tau,x_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},v)+l(\tau,x_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},\bar{u}_\tau^\varepsilon)\\ &+\frac{1}{2}P_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)}[\sigma(\tau,x_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},v)-\sigma(\tau,x_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_\tau^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},\bar{u}_\tau^\varepsilon)]^2\}.\end{aligned} \tag{24}$$

Furthermore, we give some estimations of the following difference by using similar methods as in [28],

$$\begin{aligned}C\varepsilon^\delta\ \geq\ & E\int_s^T[b(t,x_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},\bar{u}_t^\varepsilon)-b(t,x_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},v)]p_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)}\\ &-[b(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},Ex_t^{(u^\varepsilon,\xi^\varepsilon)},u_t^\varepsilon)-b(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},Ex_t^{(u^\varepsilon,\xi^\varepsilon)},v)]p_t^{(u^\varepsilon,\xi^\varepsilon)}dt,\end{aligned} \tag{25}$$

$$\begin{aligned} C\varepsilon^{\delta} \quad \geq \quad & E\int_s^T [\sigma(t,x_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},\bar{u}_t^\varepsilon)-\sigma(t,x_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},v)]q_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)} \\ & -[\sigma(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},Ex_t^{(u^\varepsilon,\xi^\varepsilon)},u_t^\varepsilon)-\sigma(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},Ex_t^{(u^\varepsilon,\xi^\varepsilon)},v)]q_t^{(u^\varepsilon,\xi^\varepsilon)}dt, \end{aligned} \tag{26}$$

$$\begin{aligned} C\varepsilon^{\delta} \quad \geq \quad & E\int_s^T l(t,x_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},v)-l(t,x_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},\bar{u}_t^\varepsilon) \\ & -l(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},Ex_t^{(u^\varepsilon,\xi^\varepsilon)},v)+l(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},Ex_t^{(u^\varepsilon,\xi^\varepsilon)},u_t^\varepsilon)dt, \end{aligned} \tag{27}$$

and

$$\begin{aligned} C\varepsilon^{\delta} \quad \geq \quad & E\int_s^T -\frac{1}{2}P_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)}[\sigma(t,x_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},v)-\sigma(t,x_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},Ex_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)},\bar{u}_t^\varepsilon)]^2 \\ & +\frac{1}{2}P_t^{(u^\varepsilon,\xi^\varepsilon)}[\sigma(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},Ex_t^{(u^\varepsilon,\xi^\varepsilon)},v)-\sigma(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},Ex_t^{(u^\varepsilon,\xi^\varepsilon)},u_t^\varepsilon)]^2dt. \end{aligned} \tag{28}$$

Combining (24)-(28) together, we obtain

$$\begin{aligned} & E\int_s^T H(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},u_t^\varepsilon,p_t^{(u^\varepsilon,\xi^\varepsilon)},q_t^{(u^\varepsilon,\xi^\varepsilon)})-H(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},v,p_t^{(u^\varepsilon,\xi^\varepsilon)},q_t^{(u^\varepsilon,\xi^\varepsilon)}) \\ & -\frac{1}{2}P_t^{(u^\varepsilon,\xi^\varepsilon)}[\sigma(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},Ex_t^{(u^\varepsilon,\xi^\varepsilon)},v)-\sigma(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},Ex_t^{(u^\varepsilon,\xi^\varepsilon)},u_t^\varepsilon)]^2dt \leq C\varepsilon^\delta. \end{aligned} \tag{29}$$

Noting the Hamiltonian (20), the first variational inequality immediately follows from (29).
Furthermore, from (21) we have

$$J^\varepsilon(s,y,\bar{u}^\varepsilon,\bar{\xi}^\varepsilon) \leq J^\varepsilon(s,y,\bar{u}^\varepsilon,\xi^\theta),$$

Further, we can deduce

$$J(s,y,\bar{u}^\varepsilon,\xi^\theta)-J(s,y,\bar{u}^\varepsilon,\bar{\xi}^\varepsilon) \geq -C\theta\varepsilon^\delta. \tag{30}$$

Finally, according to Lemma 5 of [18], we have

$$\lim_{\theta\to 0}\frac{J(s,y,\bar{u}^\varepsilon,\xi^\theta)-J(s,y,\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)}{\theta} = E\int_s^T (k_t+G_t^Tp_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)})d(\eta_t-\bar{\xi}_t^\varepsilon). \tag{31}$$

Combining (30) and (31) together, we get

$$-C\varepsilon^\delta \leq E\int_s^T (k_t+G_t^Tp_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)})d(\eta_t-\bar{\xi}_t^\varepsilon). \tag{32}$$

Moreover,

$$\begin{aligned} & E\int_s^T (k_t+G_t^Tp_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)})d(\eta_t-\bar{\xi}_t^\varepsilon)-E\int_s^T (k_t+G_t^Tp_t^{(u^\varepsilon,\xi^\varepsilon)})d(\eta_t-\xi_t^\varepsilon) \\ = \quad & E\int_s^T (k_t+G_t^Tp_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)})d(\xi_t^\varepsilon-\bar{\xi}_t^\varepsilon)+E\int_s^T G_t^T(p_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)}-p_t^{(u^\varepsilon,\xi^\varepsilon)})d(\eta_t-\xi_t^\varepsilon). \end{aligned}$$

Noting (A3), (10) and Lemma 6, by using Cauchy-Schwartz inequality to the above equality, we can finally get

$$C\varepsilon^\delta \geq E\int_s^T (k_t+G_t^Tp_t^{(\bar{u}^\varepsilon,\bar{\xi}^\varepsilon)})d(\eta_t-\bar{\xi}_t^\varepsilon)-E\int_s^T (k_t+G_t^Tp_t^{(u^\varepsilon,\xi^\varepsilon)})d(\eta_t-\xi_t^\varepsilon). \tag{33}$$

This completes the proof by combining (32) and (33).

Remarks 8 (1) If $\varepsilon = 0$, it is just the necessary condition of exact optimality of singular mean-field SDE.

(2) If $G = k = 0$, we can obtain the necessary near-optimality conditions for mean-field stochastic systems with controlled diffusion coefficient.

(3) If $\varepsilon = 0$ and $G = k = 0$, the maximum principle for mean-field SDEs in [7] is presented.

In the following theorem, we prove that under some additional assumptions, necessary conditions obtained above are also sufficient.

(A4) b, σ, l are differentiable in u, and there exists a constant $C > 0$ such that for $\rho = b, \sigma, l$,

$$|\rho(t,x,\tilde{x},u_1)-\rho(t,x,\tilde{x},u_2)|+|\rho_u(t,x,\tilde{x},u_1)-\rho_u(t,x,\tilde{x},u_2)|\le C|u_1-u_2|.$$

Theorem 9 Assume the Hamiltonian $H(t,\cdot,\cdot,p_t^{(u^\varepsilon,\xi^\varepsilon)},q_t^{(u^\varepsilon,\xi^\varepsilon)})$ is concave for a.e. $t\in[s,T]$, P-a.s. and $g(\cdot,\cdot)$ is convex. Let $(p_t^{(u^\varepsilon,\xi^\varepsilon)},q_t^{(u^\varepsilon,\xi^\varepsilon)})$ be the solution of the adjoint equation (8) controlled by $(u^\varepsilon,\xi^\varepsilon)$. If for any $(u,\xi)\in\mathcal{U}$ and some $\varepsilon>0$,

$$E\int_s^T \mathcal{H}(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},u_t^\varepsilon)dt\ge\sup_{u\in\mathcal{U}_1}E\int_s^T \mathcal{H}(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},u_t)dt-\varepsilon \tag{34}$$

and

$$E\int_s^T k_t d(\xi_t-\xi_t^\varepsilon)\ge -C\varepsilon^{\frac{1}{2}} \tag{35}$$

hold, then we have

$$J(s,y,u^\varepsilon,\xi^\varepsilon)\le\inf_{(u,\xi)\in\mathcal{U}}J(s,y,u,\xi)+C\varepsilon^{\frac{1}{2}},$$

with $C>0$ a constant independent of ε.

Proof First, we rewrite the cost functional J as

$$J(s,y,u,\xi)=J_1(s,y,u)+J_2(s,y,\xi), \tag{36}$$

where

$$J_1(s,y,u)=E[g(x_T^{(u,\xi)},Ex_T^{(u,\xi)})+\int_s^T l(t,x_t^{(u,\xi)},Ex_t^{(u,\xi)},u_t)dt],$$

and

$$J_2(s,y,\xi)=E\int_s^T k_t d\xi_t.$$

Fix $\varepsilon>0$, for any (u,ξ), $(v,\eta)\in\mathcal{U}$, a new metric $\tilde{d}$ on $\mathcal{U}$ is defined by:

$$\tilde{d}((u,\xi),(v,\eta))=\tilde{d}_1(u,v)+d_2(\xi,\eta),$$

where

$$\tilde{d}_1(u,v)=E\int_s^T\lambda_t^\varepsilon|u_t-v_t|dt,$$

and

$$\lambda_t^\varepsilon=1+|p_t^{(u^\varepsilon,\xi^\varepsilon)}|+|q_t^{(u^\varepsilon,\xi^\varepsilon)}|+|P_t^{(u^\varepsilon,\xi^\varepsilon)}|+|P_t^{(u^\varepsilon,\xi^\varepsilon)}||x_t^{(u^\varepsilon,\xi^\varepsilon)}|.$$

It is easy to see that $\tilde{d}_1$ is a complete metric on $(\mathcal{U}_1,\tilde{d}_1)$, Hence $(\mathcal{U},\tilde{d})$ is also a complete metric space as the product of two complete spaces under the metric $\tilde{d}$.

A functional $\bar{J}$ on $\mathcal{U}_1$ is defined by

$$\bar{J}(u)=E\int_s^T\mathcal{H}(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},u_t)dt.$$

Noting (A4), we can easily check that $\bar{J}$ is continuous on $\mathcal{U}_1$ with respect to $\tilde{d}_1$. Further, with the help of (34) as well as Ekeland's variational principle, there exists a $\bar{u}^\varepsilon\in\mathcal{U}_1$ such that

$$\tilde{d}_1(\bar{u}^\varepsilon,u^\varepsilon)\le\varepsilon^{\frac{1}{2}},$$

and

$$E\int_s^T\tilde{\mathcal{H}}(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},\bar{u}_t^\varepsilon)dt=\max_{u\in\mathcal{U}_1}E\int_s^T\tilde{\mathcal{H}}(t,x_t^{(u^\varepsilon,\xi^\varepsilon)},u_t)dt,$$

where

$$\tilde{\mathcal{H}}(t,x,u)=\mathcal{H}(t,x,u)-\varepsilon^{\frac{1}{2}}\lambda_t^\varepsilon|u-\bar{u}^\varepsilon|.$$

The integral-form maximum condition implies a pointwise maximum condition, that is, for a.e. $t \in [s,T]$ and P-a.s.,

$$\tilde{\mathcal{H}}(t, x_t^{(u^\varepsilon,\xi^\varepsilon)}, \bar{u}_t^\varepsilon) = \max_{u \in \mathcal{U}_1} \tilde{\mathcal{H}}(t, x_t^{(u^\varepsilon,\xi^\varepsilon)}, u_t).$$

Then, by Lemma 2.3 of [26], we have

$$0 \in \partial_u \tilde{\mathcal{H}}(t, x_t^{(u^\varepsilon,\xi^\varepsilon)}, \bar{u}_t^\varepsilon).$$

While

$$\partial_u \tilde{\mathcal{H}}(t, x_t^{(u^\varepsilon,\xi^\varepsilon)}, \bar{u}_t^\varepsilon) \subseteq \partial_u \mathcal{H}(t, x_t^{(u^\varepsilon,\xi^\varepsilon)}, \bar{u}_t^\varepsilon) + [-\varepsilon^{\frac{1}{2}}\lambda_t^\varepsilon, \varepsilon^{\frac{1}{2}}\lambda_t^\varepsilon].$$

Furthermore, $\mathcal{H}$ is differentiable in u, so there exists a $\beta_t^\varepsilon \in [-\varepsilon^{\frac{1}{2}}\lambda_t^\varepsilon, \varepsilon^{\frac{1}{2}}\lambda_t^\varepsilon]$, such that

$$\beta_t^\varepsilon = -\mathcal{H}_u(t, x_t^{(u^\varepsilon,\xi^\varepsilon)}, \bar{u}_t^\varepsilon),$$

i.e.

$$\begin{aligned}
& H_u(t, x_t^{(u^\varepsilon,\xi^\varepsilon)}, \bar{u}_t^\varepsilon, p_t^{(u^\varepsilon,\xi^\varepsilon)}, q_t^{(u^\varepsilon,\xi^\varepsilon)}) \\
= \; & -\beta_t^\varepsilon - \sigma_u(t, x_t^{(u^\varepsilon,\xi^\varepsilon)}, Ex_t^{(u^\varepsilon,\xi^\varepsilon)}, \bar{u}_t^\varepsilon) P_t^{(u^\varepsilon,\xi^\varepsilon)} \sigma(t, x_t^{(u^\varepsilon,\xi^\varepsilon)}, Ex_t^{(u^\varepsilon,\xi^\varepsilon)}, u_t^\varepsilon) \\
+ \; & \sigma_u(t, x_t^{(u^\varepsilon,\xi^\varepsilon)}, Ex_t^{(u^\varepsilon,\xi^\varepsilon)}, \bar{u}_t^\varepsilon) P_t^{(u^\varepsilon,\xi^\varepsilon)} \sigma(t, x_t^{(u^\varepsilon,\xi^\varepsilon)}, Ex_t^{(u^\varepsilon,\xi^\varepsilon)}, \bar{u}_t^\varepsilon).
\end{aligned}$$

Applying the similar method as in [28] for the rest of the proof, we can finally obtain

$$J_1(s,y,u^\varepsilon) \le \inf_{u \in \mathcal{U}_1} J_1(s,y,u) + C\varepsilon^{\frac{1}{2}}. \tag{37}$$

From (35), we have

$$J_2(s,y,\xi^\varepsilon) \le J_2(s,y,\xi) + C\varepsilon^{\frac{1}{2}}. \tag{38}$$

Combining (36), (37) and (38), we can arrive at the conclusion.

Remark 10 Under the assumption of Theorem 9, a sufficient condition for an admissible control $(u^\varepsilon, \xi^\varepsilon)$ to be ε-optimal is

$$E\Big[\int_s^T \mathcal{H}(t, x_t^{(u^\varepsilon,\xi^\varepsilon)}, u_t^\varepsilon)dt - \int_s^T k_t d\xi_t^\varepsilon\Big] \ge \sup_{(u,\xi)\in\mathcal{U}} \Big\{E\int_s^T \mathcal{H}(t, x_t^{(u^\varepsilon,\xi^\varepsilon)}, u_t)dt - \int_s^T k_t d\xi_t\Big\} - C_1\varepsilon,$$

where $C_1 > 0$ is a constant only depending on C.

5. The Relaxed Mean-field Singular Control Problem

In this section, relaxed singular control problems are studied. First, we give the definition of admissible set of relaxed singular controls.

Definition 11 An admissible relaxed singular control is a pair (q,η) of progresses such that
(i)q is a $P(A_1)$-valued process, progressively measurable with respect to $(\mathcal{F}_t)_t$ and such that for each t, $I_{(0,t]}q$ is $\mathcal{F}_t$-measurable and $E[\sup_{t\in[s,T]} |q_t|^2] < \infty$.
(ii) $\eta \in \mathcal{U}_2$.

We denote by $\mathcal{R} = \mathcal{R}_1 \times \mathcal{U}_2$ the set of all admissible relaxed singular controls.

For any $(q,\eta) \in \mathcal{R}$, we consider the following stochastic system:

$$\begin{cases} dx_t^{(q,\eta)} = \int_{A_1} b(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, a) q_t(da)dt + \int_{A_1} \sigma(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, a) q_t(da)dW_t + G_t d\eta_t, \\ x_s^{(q,\eta)} = y. \end{cases} \tag{39}$$

The cost functional associated with (39) is given by

$$\mathcal{J}(q,\eta) = E[g(x_T^{(q,\eta)}, Ex_T^{(q,\eta)}) + \int_s^T \int_{A_1} l(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, a) q_t(da)dt + \int_s^T k_t d\eta_t], \tag{40}$$

and the value function by

$$\mathcal{V}(s,y) = \inf_{(q,\eta)\in\mathcal{R}} \mathcal{J}(s,y,q,\eta). \tag{41}$$

A relaxed singular control $(\bar{q}, \bar{\eta})$ is called ε-optimal if it satisfies

$$|\mathcal{J}(s, y, \bar{q}, \bar{\eta}) - \mathcal{V}(s, y)| \leq \varepsilon.$$

Remark 12 If we put $\bar{\rho}(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, q_t) = \int_{A_1} \rho(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, a) q_t(da)$, where $\rho = b, \sigma, l$, then (5.1) becomes

$$\begin{cases} dx_t^{(q,\eta)} = \bar{b}(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, q_t)dt + \bar{\sigma}(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, q_t)dW_t + G_t d\eta_t, \\ x_s^{(q,\eta)} = y, \end{cases} \tag{42}$$

with the cost functional given by

$$\mathcal{J}(q, \eta) = E[g(x_T^{(q,\eta)}, Ex_T^{(q,\eta)}) + \int_s^T \bar{l}(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, q_t)dt + \int_s^T k_t d\eta_t]. \tag{43}$$

It is clear that $P(A_1)$ is compact and convex, furthermore, the coefficients of (42)and (43) check the same assumptions as those of (2) and (3). Therefore, for every $(q, \eta) \in \mathcal{R}$, (42) admits a unique solution and the cost functional $\mathcal{J}$ is well defined from $\mathcal{R}$ into R.

Remark 13 The set of strict singular controls $\mathcal{U}$ is embedded into the set of relaxed singular controls $\mathcal{R}$ by the mapping

$$f : u \in \mathcal{U} \mapsto f_u(dt, da) = dt\delta_{u_t}(da) \in \mathcal{R},$$

where δ_u is the Dirac measure concentrated at a single point u. Furthermore, if $q_t = \delta_{u_t}$, then for each $t \in [s, T]$, we have

$$\int_{A_1} \rho(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, a) q_t(da) = \int_{A_1} \rho(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, a) \delta_{u_t}(da) = \rho(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, u_t),$$

where $\rho = b, \sigma, l$.

In this case $x_t^{(q,\eta)} = x_t^{(u,\eta)}$, $Ex_t^{(q,\eta)} = Ex_t^{(u,\eta)}$, $\mathcal{J}(q, \eta) = J(u, \eta)$, then we get a strict singular control problem. That is to say, the problem of strict singular controls is a particular case of the problem of relaxed singular ones.

Before we draw the desired conclusions, we make another assumption and introduce the well known chattering lemma, which will play an important role in the following demonstration.

(A5) b, σ, l are bounded.

Lemma 14 ([19]) Let (q_t) be a predictable process with values in the space of probability measures on A_1. Then there exists a sequence of predictable processes (u^n) with values in A_1 such that the sequence of random measures $(\delta_{u_t^n}(da)dt)$ converges weakly to $q_t(da)$, P-a.s.

Lemma 15 Let (A1)-(A3), (A5) hold, for any $(q, \eta) \in \mathcal{R}$, there exists a sequence $(u^n, \eta)_{n \geq 1} \subset \mathcal{U}$ such that

$$\lim_{n \to \infty} E(\sup_{t \in [s,T]} |x_t^{(u^n,\eta)} - x_t^{(q,\eta)}|^2) = 0,$$

and

$$\lim_{n \to \infty} J(u^n, \eta) = \mathcal{J}(q, \eta),$$

where $x_t^{(u^n,\eta)}$, $x_t^{(q,\eta)}$ are trajectories associated with (u^n, η) and (q, η) respectively.

Proof By using (A1) and Burkholder-Davis-Gundy inequality, from (2) and (39), we have

$$\begin{aligned} & E(\sup_{t \in [s,T]} |x_t^{(u^n,\eta)} - x_t^{(q,\eta)}|^2) \\ \leq\ & C_1 E \int_s^T |b(t, x_t^{(u^n,\eta)}, Ex_t^{(u^n,\eta)}, u_t^n) - b(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, u_t^n)|^2 dt \\ & + C_1 E \int_s^T |\int_{A_1} b(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, a)\delta_{u_t^n}(da) - \int_{A_1} b(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, a) q_t(da)|^2 dt \\ & + C_1 E \int_s^T |\sigma(t, x_t^{(u^n,\eta)}, Ex_t^{(u^n,\eta)}, u_t^n) - \sigma(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, u_t^n)|^2 dt \\ & + C_1 E \int_s^T |\int_{A_1} \sigma(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, a)\delta_{u_t^n}(da) - \int_{A_1} \sigma(t, x_t^{(q,\eta)}, Ex_t^{(q,\eta)}, a) q_t(da)|^2 dt \\ \leq\ & C_2 E \int_s^T |x_t^{(u^n,\eta)} - x_t^{(q,\eta)}|^2 dt + C_2 \alpha_t^n, \end{aligned} \tag{44}$$

where

$$\begin{aligned}\alpha_t^n &= E\int_s^T |\int_{A_1} b(t,x_t^{(q,\eta)},Ex_t^{(q,\eta)},a)\delta_{u_t^n}(da)-\int_{A_1} b(t,x_t^{(q,\eta)},Ex_t^{(q,\eta)},a)q_t(da)|^2dt\\ &\quad +E\int_s^T |\int_{A_1} \sigma(t,x_t^{(q,\eta)},Ex_t^{(q,\eta)},a)\delta_{u_t^n}(da)-\int_{A_1} \sigma(t,x_t^{(q,\eta)},Ex_t^{(q,\eta)},a)q_t(da)|^2dt.\end{aligned}$$

Noting (A5), by using Lemma 14 and dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\alpha_t^n=0. \tag{45}$$

From (44) and (45), the conclusion is drawn by using Gronwall's inequality to (44).

On the other hand, noting (A1) and (A2), we have

$$\begin{aligned}&|J(u^n,\eta)-\mathcal{J}(q,\eta)|\\ \le\ & E|g(x_T^{(u^n,\eta)},Ex_T^{(u^n,\eta)})-g(x_T^{(q,\eta)},Ex_T^{(q,\eta)})|\\ &+E\int_s^T |l(t,x_t^{(u^n,\eta)},Ex_t^{(u^n,\eta)},u_t^n)-l(t,x_t^{(q,\eta)},Ex_t^{(q,\eta)},u_t^n)|dt\\ &+E\int_s^T |\int_{A_1} l(t,x_t^{(q,\eta)},Ex_t^{(q,\eta)},a)\delta_{u_t^n}(da)-\int_{A_1} l(t,x_t^{(q,\eta)},Ex_t^{(q,\eta)},a)q_t(da)|dt\\ \le\ & C_3E|x_T^{(u^n,\eta)}-x_T^{(q,\eta)}|+C_3E\int_s^T|x_t^{(u^n,\eta)}-x_t^{(q,\eta)}|dt+C_3\beta_t^n,\end{aligned} \tag{46}$$

where

$$\beta_t^n=E\int_s^T |\int_{A_1} l(t,x_t^{(q,\eta)},Ex_t^{(q,\eta)},a)\delta_{u_t^n}(da)-\int_{A_1} l(t,x_t^{(q,\eta)},Ex_t^{(q,\eta)},a)q_t(da)|dt.$$

By using (A5), Lemma 14 and dominated convergence theorem, we also obtain that

$$\lim_{n\to\infty}\beta_t^n=0.$$

By virtue of Cauchy-Schwartz inequality to (46), noting the first drawn conclusion, we can easily get the second result.

Remark 16 Assume that $(\mu^\varepsilon,\eta^\varepsilon)$ is the ε-optimal control pair, according to Lemma 15, there exist $(u^{n,\varepsilon},\eta^\varepsilon)\in\mathcal{U}$ and a positive sequence (ε_n) with $\lim_{n\to\infty}\varepsilon_n=0$, such that

$$J(u^{n,\varepsilon},\eta^\varepsilon)\le\mathcal{J}(\mu^\varepsilon,\eta^\varepsilon)+\varepsilon_n\le\inf_{(u^n,\eta)\in\mathcal{U}}J(u^n,\eta)+\varepsilon_n.$$

and it is easy to see that $(u^{n,\varepsilon},\eta^\varepsilon)$ is an ε-optimal control pair.

Next, we give adjoint equations corresponding to $(u^{n,\varepsilon},\eta^\varepsilon)$ and $(\mu^\varepsilon,\eta^\varepsilon)$ respectively.

$$\begin{cases}-dp_t^{n,\varepsilon}=\{b_x^{n,\varepsilon}p_t^{n,\varepsilon}+\sigma_x^{n,\varepsilon}q_t^{n,\varepsilon}-l_x^{n,\varepsilon}+E[b_{\tilde{x}}^{n,\varepsilon}p_t^{n,\varepsilon}]+E[\sigma_{\tilde{x}}^{n,\varepsilon}q_t^{n,\varepsilon}]-El_{\tilde{x}}^{n,\varepsilon}\}dt-q_t^{n,\varepsilon}dW_t,\\ p_T^{n,\varepsilon}=-g_x^{n,\varepsilon}(T)-Eg_{\tilde{x}}^{n,\varepsilon}(T),\end{cases} \tag{47}$$

$$\begin{cases}-dP_t^{n,\varepsilon}=\{2b_x^{n,\varepsilon}P_t^{n,\varepsilon}+[\sigma_x^{n,\varepsilon}]^2P_t^{n,\varepsilon}+2\sigma_x^{n,\varepsilon}Q_t^{n,\varepsilon}+H_{xx}^{n,\varepsilon}\}dt-Q_t^{n,\varepsilon}dW_t,\\ P_T^{n,\varepsilon}=-g_{xx}^{n,\varepsilon}(T),\end{cases} \tag{48}$$

and

$$\begin{cases}-dp_t^{\mu,\varepsilon}=\{b_x^{\mu,\varepsilon}p_t^{\mu,\varepsilon}+\sigma_x^{\mu,\varepsilon}q_t^{\mu,\varepsilon}-l_x^{\mu,\varepsilon}+E[b_{\tilde{x}}^{\mu,\varepsilon}p_t^{\mu,\varepsilon}]+E[\sigma_{\tilde{x}}^{\mu,\varepsilon}q_t^{\mu,\varepsilon}]-El_{\tilde{x}}^{\mu,\varepsilon}\}dt-q_t^{\mu,\varepsilon}dW_t,\\ p_T^{\mu,\varepsilon}=-g_x^{\mu,\varepsilon}(T)-Eg_{\tilde{x}}^{\mu,\varepsilon}(T),\end{cases} \tag{49}$$

$$\begin{cases}-dP_t^{\mu,\varepsilon}=\{2b_x^{\mu,\varepsilon}P_t^{\mu,\varepsilon}+[\sigma_x^{\mu,\varepsilon}]^2P_t^{\mu,\varepsilon}+2\sigma_x^{\mu,\varepsilon}Q_t^{\mu,\varepsilon}+H_{xx}^{\mu,\varepsilon}\}dt-Q_t^{\mu,\varepsilon}dW_t,\\ P_T^{\mu,\varepsilon}=-g_{xx}^{\mu,\varepsilon}(T),\end{cases} \tag{50}$$

where

$$\rho^{n,\varepsilon} = \rho(t, x_t^{n,\varepsilon}, Ex_t^{n,\varepsilon}, u_t^{n,\varepsilon}), \;\; g^{n,\varepsilon}(T) = g(x_T^{n,\varepsilon}, Ex_T^{n,\varepsilon})$$

and

$$\rho^{\mu,\varepsilon} = \int_{A_1} \rho(t, x_t^{\mu,\varepsilon}, Ex_t^{\mu,\varepsilon}, a)\mu_t^{\varepsilon}(da), \;\; g^{\mu,\varepsilon}(T) = g(x_T^{\mu,\varepsilon}, Ex_T^{\mu,\varepsilon}),$$

$$H(t, x_t^{\mu,\varepsilon}, \mu_t^{\varepsilon}, p_t^{\mu,\varepsilon}, q_t^{\mu,\varepsilon}) = \int_{A_1} H(t, x_t^{\mu,\varepsilon}, a, p_t^{\mu,\varepsilon}, q_t^{\mu,\varepsilon})\mu_t^{\varepsilon}(da),$$

$x_t^{n,\varepsilon} := x_t^{(u^{n,\varepsilon},\eta^{\varepsilon})}$, $x_t^{\mu,\varepsilon} := x_t^{(\mu^{\varepsilon},\eta^{\varepsilon})}$, $\rho = b, \sigma, l$.

Lemma 17 Let $(p_t^{n,\varepsilon}, q_t^{n,\varepsilon})$ and $(p_t^{\mu,\varepsilon}, q_t^{\mu,\varepsilon})$ be solutions of (47) and (49), $(P_t^{n,\varepsilon}, Q_t^{n,\varepsilon})$ and $(P_t^{\mu,\varepsilon}, Q_t^{\mu,\varepsilon})$ be solutions of (48) and (50)respectively, we have

$$\lim_{n\to\infty}(\sup_{t\in[s,T]} E|p_t^{n,\varepsilon} - p_t^{\mu,\varepsilon}|^2 + E\int_s^T |q_t^{n,\varepsilon} - q_t^{\mu,\varepsilon}|^2 dt) = 0,$$

and

$$\lim_{n\to\infty}(\sup_{t\in[s,T]} E|P_t^{n,\varepsilon} - P_t^{\mu,\varepsilon}|^2 + E\int_s^T |Q_t^{n,\varepsilon} - Q_t^{\mu,\varepsilon}|^2 dt) = 0.$$

Proof Applying Itô's formula to $(p_t^{\mu,\varepsilon} - p_t^{n,\varepsilon})^2$ on $[s, T]$, we have

$$\begin{aligned} & E|p_s^{\mu,\varepsilon} - p_s^{n,\varepsilon}|^2 + E\int_s^T |q_t^{\mu,\varepsilon} - q_t^{n,\varepsilon}|^2 dt \\ = \;& E|g_x^{\mu,\varepsilon}(T) + Eg_{\tilde{x}}^{\mu,\varepsilon}(T) - g_x^{n,\varepsilon}(T) - Eg_{\tilde{x}}^{n,\varepsilon}(T)|^2 + 2E\int_s^T (p_t^{\mu,\varepsilon} - p_t^{n,\varepsilon})(I_t^{\mu,\varepsilon} - I_t^{n,\varepsilon})dt, \end{aligned}$$

where

$$I_t^{\mu,\varepsilon} = b_x^{\mu,\varepsilon} p_t^{\mu,\varepsilon} + \sigma_x^{\mu,\varepsilon} q_t^{\mu,\varepsilon} - l_x^{\mu,\varepsilon} + E[b_{\tilde{x}}^{\mu,\varepsilon} p_t^{\mu,\varepsilon}] + E[\sigma_{\tilde{x}}^{\mu,\varepsilon} q_t^{\mu,\varepsilon}] - El_{\tilde{x}}^{\mu,\varepsilon},$$

and

$$I_t^{n,\varepsilon} = b_x^{n,\varepsilon} p_t^{n,\varepsilon} + \sigma_x^{n,\varepsilon} q_t^{n,\varepsilon} - l_x^{n,\varepsilon} + E[b_{\tilde{x}}^{n,\varepsilon} p_t^{n,\varepsilon}] + E[\sigma_{\tilde{x}}^{n,\varepsilon} q_t^{n,\varepsilon}] - El_{\tilde{x}}^{n,\varepsilon}.$$

By using Young's inequality $|ab| \le \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2$, we obtain

$$\begin{aligned} & E|p_s^{\mu,\varepsilon} - p_s^{n,\varepsilon}|^2 + E\int_s^T |q_t^{\mu,\varepsilon} - q_t^{n,\varepsilon}|^2 dt \\ \le \;& E|g_x^{\mu,\varepsilon}(T) + Eg_{\tilde{x}}^{\mu,\varepsilon}(T) - g_x^{n,\varepsilon}(T) - Eg_{\tilde{x}}^{n,\varepsilon}(T)|^2 + E\int_s^T \frac{1}{\varepsilon}|p_t^{\mu,\varepsilon} - p_t^{n,\varepsilon}|^2 dt + \varepsilon E\int_s^T |I_t^{\mu,\varepsilon} - I_t^{n,\varepsilon}|^2 dt \\ \le \;& (\frac{1}{\varepsilon} + 24C_1\varepsilon)E\int_s^T |p_t^{\mu,\varepsilon} - p_t^{n,\varepsilon}|^2 dt + 24C_1\varepsilon E\int_s^T |q_t^{\mu,\varepsilon} - q_t^{n,\varepsilon}|^2 dt + \varepsilon\phi_t^n, \end{aligned}$$

where

$$\begin{aligned} \phi_t^n \;=\; & \frac{1}{\varepsilon}E|g_x^{\mu,\varepsilon}(T) + Eg_{\tilde{x}}^{\mu,\varepsilon}(T) - g_x^{n,\varepsilon}(T) - Eg_{\tilde{x}}^{n,\varepsilon}(T)|^2 \\ & +12E\int_s^T |(b_x^{\mu,\varepsilon} - b_x^{n,\varepsilon})p_t^{n,\varepsilon}|^2 dt + 12E\int_s^T |(b_{\tilde{x}}^{\mu,\varepsilon} - b_{\tilde{x}}^{n,\varepsilon})p_t^{n,\varepsilon}|^2 dt \\ & +12E\int_s^T |(\sigma_x^{\mu,\varepsilon} - \sigma_x^{n,\varepsilon})q_t^{n,\varepsilon}|^2 dt + 12E\int_s^T |(\sigma_{\tilde{x}}^{\mu,\varepsilon} - \sigma_{\tilde{x}}^{n,\varepsilon})q_t^{n,\varepsilon}|^2 dt \\ & +6E\int_s^T |l_x^{\mu,\varepsilon} - l_x^{n,\varepsilon}|^2 dt + 6E\int_s^T |l_{\tilde{x}}^{\mu,\varepsilon} - l_{\tilde{x}}^{n,\varepsilon}|^2 dt. \end{aligned}$$

Taking $\varepsilon = \frac{1}{48C_1}$, we get

$$E|p_s^{\mu,\varepsilon} - p_s^{n,\varepsilon}|^2 + \frac{1}{2}E\int_s^T |q_t^{\mu,\varepsilon} - q_t^{n,\varepsilon}|^2 dt \leq C_2 E\int_s^T |p_t^{\mu,\varepsilon} - p_t^{n,\varepsilon}|^2 dt + C_2\phi_t^n. \quad (51)$$

Noting (A1), (A2), (A5), we have

$$\begin{aligned}
& E\int_s^T |(b_x^{\mu,\varepsilon} - b_x^{n,\varepsilon})p_t^{n,\varepsilon}|^2 dt \\
\leq\ & C_3 E\int_s^T |\int_{A_1} b_x(t, x_t^{\mu,\varepsilon}, Ex_t^{\mu,\varepsilon}, a)\mu_t^\varepsilon(da) - \int_{A_1} b_x(t, x_t^{\mu,\varepsilon}, Ex_t^{\mu,\varepsilon}, a)\delta_{u_t^n}(da)|^2 |p_t^{n,\varepsilon}|^2 dt \\
& + C_3 E\int_s^T |\int_{A_1} b_x(t, x_t^{\mu,\varepsilon}, Ex_t^{\mu,\varepsilon}, a)\delta_{u_t^n}(da) - \int_{A_1} b_x(t, x_t^{n,\varepsilon}, Ex_t^{n,\varepsilon}, a)\delta_{u_t^n}(da)|^2 |p_t^{n,\varepsilon}|^2 dt \\
\leq\ & C_4 E\int_s^T |\int_{A_1} b_x(t, x_t^{\mu,\varepsilon}, Ex_t^{\mu,\varepsilon}, a)\mu_t^\varepsilon(da) - \int_{A_1} b_x(t, x_t^{\mu,\varepsilon}, Ex_t^{\mu,\varepsilon}, a)\delta_{u_t^n}(da)|^2 |p_t^{n,\varepsilon}|^2 dt \\
& + C_4 E\int_s^T |x_t^{\mu,\varepsilon} - x_t^{n,\varepsilon}|^2 |p_t^{n,\varepsilon}|^2 dt.
\end{aligned}$$

By using Lemma 14 and Lemma 15 to the right side of the above inequality, we obtain

$$\lim_{n\to\infty} E\int_s^T |(b_x^{\mu,\varepsilon} - b_x^{n,\varepsilon})p_t^{n,\varepsilon}|^2 dt = 0.$$

Using the same procedure, we can also get

$$\lim_{n\to\infty} E\int_s^T |(b_{\tilde{x}}^{\mu,\varepsilon} - b_{\tilde{x}}^{n,\varepsilon})p_t^{n,\varepsilon}|^2 dt = 0,$$

$$\lim_{n\to\infty} E\int_s^T |(\sigma_x^{\mu,\varepsilon} - \sigma_x^{n,\varepsilon})q_t^{n,\varepsilon}|^2 dt = 0,$$

$$\lim_{n\to\infty} E\int_s^T |(\sigma_{\tilde{x}}^{\mu,\varepsilon} - \sigma_{\tilde{x}}^{n,\varepsilon})q_t^{n,\varepsilon}|^2 dt = 0,$$

$$\lim_{n\to\infty} E\int_s^T |l_x^{\mu,\varepsilon} - l_x^{n,\varepsilon}|^2 dt = 0,$$

$$\lim_{n\to\infty} E\int_s^T |l_{\tilde{x}}^{\mu,\varepsilon} - l_{\tilde{x}}^{n,\varepsilon}|^2 dt = 0.$$

On the other hand, since

$$E|g_x^{\mu,\varepsilon}(T) + Eg_{\tilde{x}}^{\mu,\varepsilon}(T) - g_x^{n,\varepsilon}(T) - Eg_{\tilde{x}}^{n,\varepsilon}(T)|^2 \leq C_5 E|x_T^{\mu,\varepsilon} - x_T^{n,\varepsilon}|^2,$$

it follows that

$$\lim_{n\to\infty} E|g_x^{\mu,\varepsilon}(T) + Eg_{\tilde{x}}^{\mu,\varepsilon}(T) - g_x^{n,\varepsilon}(T) - Eg_{\tilde{x}}^{n,\varepsilon}(T)|^2 = 0.$$

Through above arguments, we obtain

$$\lim_{n\to\infty} \phi_t^n = 0.$$

Then the first result follows by using Gronwall's inequality to (51).

Similarly, we can prove the second equality.

6. Necessary and Sufficient Near-optimality Conditions for Relaxed Controls

In this section, we study the mean-field problem of relaxed singular controls. Necessary as well as sufficient conditions of near-optimality are established.

Theorem 18 For any $\delta \in (0, \frac{1}{3}]$ and any ε-optimal relaxed singular control $(\mu_t^\varepsilon, \eta^\varepsilon)$, there exists a constant $C = C(\delta) >$

0 such that for each $\varepsilon > 0$

$$E\int_s^T \mathcal{H}(t, x_t^{\mu,\varepsilon}, \mu_t^\varepsilon)dt \geq \sup_{\nu\in\mathcal{R}_1} E\int_s^T \mathcal{H}(t, x_t^{\mu,\varepsilon}, \nu_t)dt - C\varepsilon^\delta,$$

and

$$-C\varepsilon^\delta \leq E\int_s^T (k_t + G_t^T p_t^{\mu,\varepsilon})d(\eta_t - \eta_t^\varepsilon),$$

where

$$\mathcal{H}(t, x_t^{\mu,\varepsilon}, \mu_t^\varepsilon) = \int_{A_1} \mathcal{H}(t, x_t^{\mu,\varepsilon}, a)\mu_t^\varepsilon(da).$$

Proof For the ε-optimal control pair $(\mu_t^\varepsilon, \eta^\varepsilon)$, according to Lemma 14, there exists a strict control pair $(u_t^{n,\varepsilon}, \eta^\varepsilon)$ such that $\delta_{u_t^{n,\varepsilon}}(da)dt$ converges weakly to $\mu_t^\varepsilon(da)dt$, and from Remark 16, we know that $(u_t^{n,\varepsilon}, \eta^\varepsilon)$ is also ε-optimal. So by virtue of Theorem 7, we have

$$E\int_s^T \mathcal{H}(t, x_t^{n,\varepsilon}, u_t^{n,\varepsilon})dt \geq \sup_{v\in\mathcal{U}_1} E\int_s^T \mathcal{H}(t, x_t^{n,\varepsilon}, v_t)dt - C\varepsilon^\delta,$$

and

$$-C\varepsilon^\delta \leq E\int_s^T (k_t + G_t^T p_t^{n,\varepsilon})d(\eta_t - \eta_t^\varepsilon).$$

Taking limits on both sides to above two inequalities, by Fatou's Lemma, Lemma 15 and 17, we can get the final results.

Theorem 19 Assume the Hamiltonian $H(t, \cdot, \cdot, p_t, q_t)$ is concave for a.s. $t \in [s, T]$, P-a.s. and $g(\cdot, \cdot)$ is convex. Let $(p_t^{\mu,\varepsilon}, q_t^{\mu,\varepsilon})$ be the solution of the adjoint equation (49) controlled by $(\mu^\varepsilon, \eta^\varepsilon)$. If for some $\varepsilon > 0$ and any $(\nu, \eta) \in \mathcal{R}$,

$$E\int_s^T \mathcal{H}(t, x_t^{\mu,\varepsilon}, \mu_t^\varepsilon)dt \geq \sup_{\nu\in\mathcal{R}_1} E\int_s^T \mathcal{H}(t, x_t^{\mu,\varepsilon}, \nu_t)dt - \varepsilon$$

and

$$E\int_s^T k_t d(\eta_t - \eta_t^\varepsilon) \geq -C\varepsilon^{\frac{1}{2}}$$

hold, then we have

$$\mathcal{J}(s, y, \mu^\varepsilon, \eta^\varepsilon) \leq \inf_{(\nu,\eta)\in\mathcal{R}} \mathcal{J}(s, y, \nu, \eta) + C\varepsilon^{\frac{1}{2}},$$

Proof By applying the same arguments as in the proof of Theorem 9, the conclusion is drawn.

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Funding

This work is supported by Guangzhou Science and Technology Program Project (Grant No. 202201011057).

Conflicts of Interest

The author declares no conflicts of interest.

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