International Journal of Applied Mathematics and Theoretical Physics
2021; 7(4): 105-111
http://www.sciencepublishinggroup.com/j/ijamtp
doi: 10.11648/j.ijamtp.20210704.13
ISSN: 2575-5919 (Print); ISSN: 2575-5927 (Online)

Theoretical Realization of a Two Qubit Quantum Controlled-NOT Logic Gate and a Single Qubit Quantum Hadamard Logic Gate in the Anti-Jaynes-Cummings Model

Christopher Mayero*, Joseph Akeyo Omolo, Onyango Stephen Okeyo

Department of Physics and Materials Science, Faculty of Science, Maseno University, Maseno, Kenya

Email address:
cmayero@yahoo.com (C. Mayero), ojakeyo04@yahoo.co.uk (J. A. Omolo), onyangostiv09@gmail.com (O. S. Okeyo)
*Corresponding author

To cite this article:
Christopher Mayero, Joseph Akeyo Omolo, Onyango Stephen Okeyo. Theoretical Realization of a Two Qubit Quantum Controlled-NOT Logic Gate and a Single Qubit Quantum Hadamard Logic Gate in the Anti-Jaynes-Cummings Model. *International Journal of Applied Mathematics and Theoretical Physics*. Vol. 7, No. 4, 2021, pp. 105-111. doi: 10.11648/j.ijamtp.20210704.13

Received: August 30, 2021; **Accepted**: September 22, 2021; **Published**: November 5, 2021

Abstract: Quantum gates are fundamental in Quantum computing for their role in manipulating elementary information carriers referred to as quantum bits. In this paper, a theoretical scheme for realizing a quantum Hadamard and a quantum controlled-NOT logic gates operations in the anti-Jaynes-Cummings interaction process is provided. Standard Hadamard operation for a specified initial atomic state is achieved by setting a specific sum frequency and photon number in the normalized anti-Jaynes-Cummings qubit state transition operation with the interaction component of the anti-Jaynes-Cummings Hamiltonian generating the state transitions. The quantum controlled-NOT logic gate is realized when a single atomic qubit defined in a two-dimensional Hilbert space is the control qubit and two non-degenerate and orthogonal polarized cavities defined in a two-dimensional Hilbert space make the target qubit. With precise choice of interaction time in the anti-Jaynes-Cummings qubit state transition operations defined in the anti-Jaynes-Cummings sub-space spanned by normalized but non-orthogonal basic qubit state vectors, ideal unit probabilities of success in the quantum controlled-NOT operations is determined.

Keywords: Anti-Jaynes-Cummings, Jaynes-Cummings, Hadamard, Controlled-NOT

1. Introduction

In quantum computers, quantum bits (qubits) [1, 2] are the elementary information carriers. In such a computer, quantum gates [1-3] can manipulate arbitrary multi-partite quantum states [4] including arbitrary superposition of the computational basis states, which are frequently also entangled. Thus the logic gates of quantum computation are considerably more varied than the logic gates of classical computation. In addition, a quantum computer can solve problems exponentially faster than any classical computer [5] because by exploiting superposition principle and entanglement allows the computer to manipulate and store more bits of information than a classical computer.

In this paper a theoretical approach of realizing Hadamard and controlled-NOT (C-NOT) quantum logic gates which form a universal set for quantum computation [6-8] is presented. The important discovery and proof of a conserved excitation number operator of the anti-Jaynes-Cummings (AJC) Hamiltonian [9] now means that dynamics generated by the AJC Hamiltonian is exactly solvable, as demonstrated in the polariton and anti-polariton qubit (photospin qubit) models in [10, 11]. The reformulation developed in [9-11], drastically simplifies exact solutions of the AJC model which is applied in the present work.

The quantum C-NOT gate is defined as that which affects the unitary operation on two qubits which in a chosen orthonormal basis in $\mathbb{C}^2$ gives the C-NOT operation obtained as

$$|a\rangle|b\rangle \rightarrow |a\rangle|a \oplus b\rangle \tag{1}$$

where $|a\rangle$ is the control qubit, $|b\rangle$ is the target qubit and $\oplus$ indicates addition modulo 2 [1, 2, 12]. The C-NOT gate transforms superposition into entanglement thus acts as a measurement gate [1, 2, 12] fundamental in performing

algorithms in quantum computers [13]. Transformation to a separable state (product state) is realized by applying the C-NOT gate again. In this case, it is used to implement Bell measurement on the two qubits [14].

The Jaynes-Cummings (JC) model has been applied extensively in implementing C-NOT and Hadamard gate operations. Domokos *et al* (1995) [15] showed that using induced transitions between dressed states, it is possible to implement a C-NOT gate in which a cavity containing at most one photon is the control qubit and the atom is the target qubit. Later, Vitali, D. *et al* (2001) [16] proposed a scheme of implementing a C-NOT gate between two distinct but identical cavities, acting as control and target qubits respectively. By passing an atom prepared initially in ground state consecutively between the two cavities a C-NOT ($cavity \to atom$) and a C-NOT ($atom \to cavity$) is realized with the respective classical fields (S). Saif, F. *et al* (2001) [17] presented a study of quantum computing by engineering non-local quantum universal gates based on interaction of a two-level atom with two modes of electromagnetic field in high Q superconducting cavity. The two-level atom acted as the control qubit and the two-mode electromagnetic field served as the target qubit.

In this letter, an approach similar to that in [17] is applied, where a quantum C-NOT gate operation between two cavities defined in a two-dimensional Hilbert space spanned by the state vectors $|\mu_1\rangle = |0_a, 1_b\rangle$ and $|\mu_2\rangle = |1_a, 0_b\rangle$ as target qubits is implemented. Here $|\mu_1\rangle$ indicates that mode a is in vacuum state and one photon is present in mode b and $|\mu_2\rangle$ expresses the presence of one photon in mode a, when there is no photon in mode b. The control qubit in this respect is a two-level atom. The important difference with the approach used in [17] is the model, i.e., while the initial absolute atom-field ground state $|g, 0\rangle$ in the AJC interaction is affected by atom-cavity coupling, the ground state $|g, 0\rangle$ in the JC model [17] is not affected by atom-cavity coupling. A similar result was determined independently in [11]. Further, with precise choice of interaction time in the AJC qubit state transition operations defined in the AJC qubit sub-space spanned by normalized but non-orthogonal basics qubit state vectors [10, 11], C-NOT gate operations are realized between the two cavities.

The Hadamard gate also known as the Walsh-Hadamard gate is a single qubit gate [1, 2]. The Hadamard transformation is defined as

$$\hat{H} = \frac{\hat{\sigma}_x + \hat{\sigma}_z}{\sqrt{2}} \tag{2}$$

where it transforms computational basis states $|e\rangle(|0\rangle), |g\rangle(|1\rangle)$ into diagonal basis states according to

$$\begin{aligned} \hat{H}|e\rangle &= \frac{|e\rangle + |g\rangle}{\sqrt{2}} \quad ; \quad \hat{H}|g\rangle = \frac{|e\rangle - |g\rangle}{\sqrt{2}} \\ \hat{H}|0\rangle &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad ; \quad \hat{H}|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \end{aligned} \tag{3}$$

Vitali, D. *et al* (2001) [16] showed that one qubit operation can be implemented on qubits represented by two internal atomic states because it amounts to applying suitable Rabi pulses. He demonstrated that the most practical solution on implementing one qubit operations on two Fock states is sending the atoms through the cavity. If the atom inside the cavity undergoes a $\frac{\pi}{2}$ pulse one realizes a Hadamard-phase gate. Saif, F. *et al* (2001) [17] also showed that it is possible to realize Hadamard operation by a controlled interaction between a two-mode high Q electromagnetic cavity field and a two-level atom. In his approach, the two-level atom is the control qubit, whereas the target qubit is made up of two modes of cavity field. Precision of the gate operations is realized by precise selection of interaction times of the two-level atom with the cavity mode. In this paper, Hadamard operation in the AJC interaction for a specified initial atomic state is determined by setting a specific sum frequency $\bar{\delta}$ and photon number n in the normalized AJC qubit state transition operation [10, 11], noting that the interaction components of the AJC Hamiltonian generates state transitions.

The content of this paper is therefore summarized as follows. Section 2 presents an overview of the theoretical model. In sections 3 and 4 respectively, implementation of quantum C-NOT and Hadamard gates in the AJC interaction are presented. Finally section 5 contains the conclusion.

2. The Model

The AJC Hamiltonian determined through symmetrization of the QRM in [9-11] is of the form

$$\begin{aligned} \hat{\bar{H}} &= \hbar\omega \hat{\bar{N}} + 2\hbar\lambda \hat{\bar{A}} - \frac{1}{2}\hbar\omega \quad ; \quad \hat{\bar{N}} = \hat{a}\hat{a}^\dagger + \hat{s}_- \hat{s}_+ \\ \hat{\bar{A}} &= \bar{\alpha}\hat{s}_z + \hat{a}\hat{s}_- + \hat{a}^\dagger \hat{s}_+ \quad ; \quad \bar{\alpha} = \frac{\bar{\delta}}{2\lambda} = \frac{\omega_0 + \omega}{2\lambda} \end{aligned} \tag{4}$$

Here, ω, $\hat{a}$, $\hat{a}^\dagger$ are quantized field mode angular frequency annihilation and creation operators while ω_0, $\hat{s}_z$, $\hat{s}_+$, $\hat{s}_-$ are atomic state transition angular frequency and atomic transition operators. $\bar{\delta}$ is the sum frequency parameter and $\hat{\bar{N}}$, $\hat{\bar{A}}$ are anti-polariton qubit conserved excitation number and state transition operator.

Applying the state transition operator $\hat{\bar{A}}$ from Eq. (4) to the initial atom-field n-photon ground state vector $|\psi_{gn}\rangle$ and initial n-photon excited state vector $|\psi_{en}\rangle$ respectively, the basic qubit state vectors satisfy the qubit state algebraic operations [10, 11]

$$\begin{aligned} \hat{\bar{A}}|\psi_{gn}\rangle &= \bar{A}_{gn}|\bar{\phi}_{gn}\rangle \quad ; \quad \hat{\bar{A}}|\bar{\phi}_{gn}\rangle = \bar{A}_{gn}|\psi_{gn}\rangle; \\ \hat{\bar{A}}|\psi_{en}\rangle &= \bar{A}_{en}|\bar{\phi}_{en}\rangle \quad ; \quad \hat{\bar{A}}|\bar{\phi}_{en}\rangle = \bar{A}_{en}|\psi_{en}\rangle. \end{aligned} \tag{5}$$

Here

$$|\psi_{en}\rangle = |e,n\rangle \;\; ; \;\; |\psi_{gn}\rangle = |g,n\rangle \tag{6}$$

are the initial uncoupled qubit state vectors while

$$\begin{aligned} |\bar{\phi}_{en}\rangle &= \bar{c}_{en}|\psi_{en}\rangle + \bar{s}_{en}|\psi_{g,n-1}\rangle \;; \\ |\bar{\phi}_{gn}\rangle &= -\bar{c}_{gn}|\psi_{gn}\rangle + \bar{s}_{gn}|\psi_{e,n+1}\rangle \end{aligned} \tag{7}$$

are the coupled transition qubit state vectors.

In the AJC subspace spanned by normalized but non-orthogonal basic qubit state vectors $|\psi_{en}\rangle$, $|\bar{\phi}_{en}\rangle$ and $|\psi_{gn}\rangle$, $|\bar{\phi}_{gn}\rangle$ respectively, the basic qubit normalized state transition operators $\hat{\bar{\varepsilon}}_g$, $\hat{\bar{\varepsilon}}_e$ and identity operators $\hat{\bar{I}}_e$, $\hat{\bar{I}}_g$ are introduced according to the definitions [11]

$$\begin{aligned} \hat{\bar{\varepsilon}}_g &= \frac{\hat{\bar{A}}}{\bar{A}_{gn}}, \;\; \hat{\bar{I}}_g = \frac{\hat{\bar{A}}^2}{\bar{A}_{gn}^{\;2}} \;\Rightarrow\; \hat{\bar{\varepsilon}}_g^{\;2} = \hat{\bar{I}}_g \;; \\ \hat{\bar{\varepsilon}}_e &= \frac{\hat{\bar{A}}}{\bar{A}_{en}}, \;\; \hat{\bar{I}}_e = \frac{\hat{\bar{A}}^2}{\bar{A}_{en}^{\;2}} \;\Rightarrow\; \hat{\bar{\varepsilon}}_e^{\;2} = \hat{\bar{I}}_e \;, \end{aligned} \tag{8}$$

with respective general algebraic properties

$$\begin{aligned} e^{i\varphi\hat{\bar{\varepsilon}}_e} &= \cos(\varphi)\hat{\bar{I}}_e + i\sin(\varphi)\hat{\bar{\varepsilon}}_e; \\ e^{-i\theta\hat{\bar{\varepsilon}}_g} &= \cos(\theta)\hat{\bar{I}}_g - i\sin(\theta)\hat{\bar{\varepsilon}}_g, \end{aligned} \tag{9}$$

which are useful in evaluating time-evolution operators.

The AJC qubit Hamiltonian defined within the qubit subspace spanned by the respective basic qubit state vectors $|\psi_{en}\rangle$, $|\bar{\phi}_{en}\rangle$ and $|\psi_{gn}\rangle$, $|\bar{\phi}_{gn}\rangle$ specified for initial atomic basis states $|e\rangle$, $|g\rangle$ in a n-photon field mode respectively are then expressed in terms of the normalized qubit state transition operators $\hat{\bar{\varepsilon}}_e$, $\hat{\bar{\varepsilon}}_g$ and identity operators $\hat{\bar{I}}_e$, $\hat{\bar{I}}_g$ in the forms

$$\begin{aligned} \hat{\bar{H}}_e &= \hbar\omega\left(n+\frac{1}{2}\right)\hat{\bar{I}}_e + \hbar\bar{R}_{en}\hat{\bar{\varepsilon}}_e \;\; ; \;\; \bar{R}_{en} = 2\lambda\bar{A}_{en} \;\; ; \\ \bar{A}_{en} &= \sqrt{n+\frac{\bar{\delta}^2}{16\lambda^2}} \;, \\ \hat{\bar{H}}_g &= \hbar\omega\left(n+\frac{3}{2}\right)\hat{\bar{I}}_g + \hbar\bar{R}_{gn}\hat{\bar{\varepsilon}}_g \;\; ; \;\; \bar{R}_{gn} = 2\lambda\bar{A}_{gn} \;\; ; \\ \bar{A}_{gn} &= \sqrt{(n+1)+\frac{\bar{\delta}^2}{16\lambda^2}}. \end{aligned} \tag{10}$$

3. Quantum C-NOT Gate Operations

In order to realize a C-NOT quantum gate operation in this context, the control qubit is a two-level atom which constitutes a two dimensional Hilbert space spanned by atomic excited $|e\rangle$ and ground $|g\rangle$ states as basis vectors. Two non-degenerate and orthogonal polarized cavity modes C_a and C_b make the target qubit. The target qubit is defined in two-dimensional Hilbert space spanned by the state vector $|\mu_1\rangle = |0_a, 1_b\rangle$ which indicates that mode a is in the vacuum state and one photon is present in mode b, and the state vector $|\mu_2\rangle = |1_a, 0_b\rangle$, which expresses the presence of one photon in mode a, when there is no photon in mode b. This means that the cavity modes should have a maximum of one photon.

It is important to note that in the AJC interaction [11], for a process starting from the state $|e,n\rangle$ where an atom in excited state $|e\rangle$ enters a n-photon field mode, the excited atom emits a positive energy photon, triggering the anti-rotating (counter rotating) negative frequency field mode to absorb a negative energy photon causing a transition $|e,n\rangle \to |g,n-1\rangle$. On the other hand, a process starting from the state $|g,n\rangle$ where an atom in ground state $|g\rangle$ is in a n-photon field mode, the anti-rotating negative frequency field mode emits a negative energy photon, triggering the atom to absorb a positive energy photon, causing the transition $|g,n\rangle \to |e,n+1\rangle$.

Now, with reference to the AJC qubit state transition operators in Eq. (9), let us first consider when an atom initially in ground state $|g\rangle$ enters an electromagnetic cavity with mode a in vacuum state and a single-photon in mode b. As stated, the atom couples to the anti-rotating negative frequency component of the field mode undergoing an AJC qubit state transition. This is because an atom in an initial ground state $|g\rangle$ couples to the anti-rotating negative frequency field mode in an initial vacuum state $|0\rangle$, spontaneously emitting a negative energy photon, thereby triggering Rabi oscillations at frequency $\bar{R}_{g0}$ between the atom (control qubit) and the field mode (target qubit).

After the atom interacts with mode a for a time $t = \dfrac{\pi}{\bar{R}_{g0}}$, equal to half Rabi oscillation time, the driving field is modulated such that

$$\theta = \bar{R}_{g0}t = 2\lambda\bar{A}_{g0}t = \pi. \tag{11}$$

During the AJC interaction, where the counter-rotating terms $\hat{a}^{\dagger}\hat{\sigma}_+$ and $\hat{a}\hat{\sigma}_-$ are the off-resonance driving field, we consider a case in which the atom-field coupling constant is far much greater than the sum frequency $(\lambda \gg \bar{\delta})$ so that the perturbative parameter $\eta = \dfrac{\lambda}{\bar{\delta}}$ defined within the perturbative region [18] of the ultrastrong coupling regime (USC) becomes

very large. With this, from Eq. (10), for a field mode in an initial vacuum state $n=0$ the Rabi frequency parameter $\bar{A}_{g0} \cong 1$. The qubit state transitions will now be driven by positive and negative energy photon absorption-emission process. It should be noted that if this condition was not met, in a completely off-resonant interaction the field mode frequency ω and the atomic transition frequency ω_0 would have been sufficiently detuned and so there would have been no transition between atomic basis states $|e\rangle$ and $|g\rangle$ during the interaction.

Now, since $\bar{A}_{g0} \cong 1$ Eq. (11) reduces to

$$\theta = \lambda t = \frac{\pi}{2} \tag{12}$$

where $\lambda t = \frac{\pi}{2}$ is the atomic velocity.

The evolution of this interaction determined by applying the AJC qubit state transition operation in Eq. (9) for an atom initially in ground state $|g\rangle$ noting the definitions of the normalized state transition operator $\hat{\bar{\varepsilon}}_g$ and the identity operator $\hat{\bar{I}}_g$ in Eq. (8) is of the form

$$e^{-i\theta\hat{\bar{\varepsilon}}_g}|g,0_a\rangle = \cos(\theta)|g,0_a\rangle - i\sin(\theta)|e,1_a\rangle \tag{13}$$

which reduces to

$$|g,0_a\rangle = -i|e,1_a\rangle \tag{14}$$

We observe that the atom interacted with mode a and completed half of the Rabi oscillation, as a result it contributed a photon to mode a and evolved to excited state $|e\rangle$.

At this point it is important to note that there is a clear simultaneous excitation of both atom and field states, as expected in an AJC interaction. After the interaction time, it enters mode b containing a single photon, interacting with the cavity mode as follows

$$-ie^{i\alpha\hat{\bar{\varepsilon}}_e}|e,1_b\rangle = -i\cos(\alpha)|e,1_b\rangle + \sin(\alpha)|g,0_b\rangle \tag{15}$$

where here, the qubit state transitions are driven by positive and negative photon emission-absorption process different from that observed in Eq. (13).

After an interaction with mode b for a time $t_1 = 2t$ such that $t_1 = \frac{\pi(\bar{R}_{g0} + \bar{R}_{e1})}{\bar{R}_{g0}\bar{R}_{e1}}$ the driving field modulation is of the form

$$\alpha = \left(\frac{\bar{R}_{g0}\bar{R}_{e1}}{\bar{R}_{g0} + \bar{R}_{e1}}\right)t = \frac{\pi}{2} \tag{16}$$

Considering once more the set condition $(\lambda \gg \bar{\delta})$, the Rabi frequency parameters in Eq. (10), first for an atom in an initial ground state $|g\rangle$ interacting with a field mode in an initial vacuum state $|0\rangle$ will assume a value $\bar{A}_{g0} \cong 1$ and the Rabi frequency parameter for an atom in an initial excited $|e\rangle$ state interacting with an initial single-photon field mode $|1\rangle$ will have a value $\bar{A}_{e1} \cong 1$, and so the Rabi frequencies will have the respective values $\bar{R}_{g0} = \bar{R}_{e1} = 2\lambda$.

The driving field modulation determined from Eq. (16) takes the form

$$\alpha = \lambda t = \frac{\pi}{2} \tag{17}$$

and the form of Eq. (15) results into an evolution

$$-i|e,1_b\rangle \to |g,0_b\rangle \tag{18}$$

The result in Eq. (18) shows that the atom evolves to ground state and absorbs a photon initially in mode b, a simultaneous de-excitation of atom and field states, as expected in an AJC process. The atom clearly performs a swapping of the electromagnetic field between the two field modes by controlled interaction.

When the atom in ground state $|g\rangle$, enters the electromagnetic cavity with mode b in vacuum state and a single photon in mode a in that order, qubit state transitions take the same form as in Eqs. (14) and (18). That is, there will be simultaneous excitation of atom and field states during the qubit state transition $|g,0_b\rangle \to -i|e,1_b\rangle$ followed by a qubit state transition that results into simultaneous de-excitation of the qubit states $-i|e,1_a\rangle \to |g,0_a\rangle$. This shows a clear swapping of the electromagnetic field between the two field modes a, b by controlled interaction when the atom is initially in ground state $|g\rangle$.

When the atom in excited state $|e\rangle$ enters mode a in vacuum state, that is, the target qubit $|\mu_1\rangle$, the atom propagates as a free wave without coupling to the field mode in vacuum state $|0\rangle$ [11], leaving the cavity without altering the state of the cavity-field mode. This is because an atom in an initial excited state $|e\rangle$ does not couple $(\lambda = 0)$ to a free anti-rotating negative frequency field mode in an initial vacuum state $|0\rangle$.

The atom in excited state $|e\rangle$ then interacts with mode b containing a single photon-field mode as follows

$$e^{i\beta\hat{\bar{\varepsilon}}_e}|e,1_b\rangle = \cos(\beta)|e,1_b\rangle + i\sin(\beta)|g,0_b\rangle \tag{19}$$

After completing one Rabi oscillation in a time $t = \frac{2\pi}{\bar{R}_{e1}}$ at

a Rabi frequency $\bar{R}_{e1} = 2\lambda\bar{A}_{e1} = 2\lambda$ since the Rabi frequency parameter $\bar{A}_{e1} \cong 1$ as determined from the initial approximation condition. This gives

$$\beta = \lambda t = \pi \tag{20}$$

with $\lambda t = \pi$ the atomic velocity. The form of Eq. (19) results into an evolution

$$|e,1_b\rangle \to -|e,1_b\rangle \tag{21}$$

We see in Eq. (21) that the atom in excited state $|e\rangle$ again leaves the cavity mode containing one photon without altering the state of the single-photon field mode.

When the atom in excited state $|e\rangle$ enters mode b in vacuum state followed by mode a containing a single-photon field mode, i.e., in the target qubit $|\mu_2\rangle$, we observe a similar qubit state transition where in mode b the atom propagates as a free wave without coupling to the field mode followed by the transition $|e,1_a\rangle \to -|e,1_a\rangle$ in mode a after one complete Rabi oscillation time.

These results, show that the target qubit made up of the electromagnetic field remains unchanged if the control qubit, that is, the two-level atom, is initially in excited state $|e\rangle$, while when the atom is in ground state $|g\rangle$, the cavity states $|0\rangle$ and $|1\rangle$ flip. We shall refer to this gate as the AJC C-NOT $(atom \to cavity)$.

Success probability for the C-NOT gate is given by

$$P_s = 1 - \left(\sin^2(\phi_a) + \cos^2(\phi_a)\sin^2(\phi_b)\right) \tag{22}$$

In terms of Rabi frequency we write Eq. (22) as

$$P_s = 1 - \left(\sin^2\left(\bar{R}_a\Delta t_a\right) + \cos^2\left(\bar{R}_a\Delta t_a\right)\sin^2\left(\bar{R}_b\Delta t_b\right)\right) \tag{23}$$

Now, for the atom in ground state $|g\rangle$ entering the electromagnetic cavity with modes a, b in vacuum state and single-photon field modes b, a, the former has interaction parameters

$$\begin{aligned} &\bar{R}_a = \bar{R}_{g0} = 2\lambda\ ;\ \Delta t_a = \frac{\pi}{\bar{R}_{g0}} = \frac{\pi}{2\lambda}\ ; \\ &\bar{R}_b = \bar{R}_{e1} = 2\lambda\ ;\ \Delta t_b = \frac{\pi}{2}\frac{\left(\bar{R}_{g0} + \bar{R}_{e1}\right)}{\bar{R}_{g0}\bar{R}_{e1}} = \frac{\pi}{2\lambda} \end{aligned} \tag{24}$$

and the latter

$$\begin{aligned} &\bar{R}_a = \bar{R}_{e1} = 2\lambda\ ;\ \Delta t_a = \frac{\pi}{2}\frac{\left(\bar{R}_{e1} + \bar{R}_{g0}\right)}{\bar{R}_{e1}\bar{R}_{g0}} = \frac{\pi}{2\lambda}\ ; \\ &\bar{R}_b = \bar{R}_{g0} = 2\lambda\ ;\ \Delta t_b = \frac{\pi}{\bar{R}_{g0}} = \frac{\pi}{2\lambda}, \end{aligned} \tag{25}$$

which results in a unit probability of success $P_s = 1$ for the respective transitions.

Finally when the atom initially in excited state $|e\rangle$ enters modes a, b in vacuum state and a single photon in modes b, a for target qubits $|\mu_1\rangle$, $|\mu_2\rangle$ respectively, the former has interaction parameters

$$\begin{aligned} &\bar{R}_a = \bar{R}_{e0} = 2\lambda\bar{A}_{e0} = 0 \quad ; \quad \lambda = 0 \quad ; \quad \bar{A}_{e0} = 0\ ; \\ &\Delta t_a = \infty\ ; \\ &\bar{R}_b = \bar{R}_{e1} = 2\lambda\bar{A}_{e1} = 2\lambda \quad ; \quad \bar{A}_{e1} \cong 1\ ; \\ &\Delta t_b = \frac{2\pi}{\bar{R}_{e1}} = \frac{\pi}{\lambda} \end{aligned} \tag{26}$$

and the latter

$$\begin{aligned} &\bar{R}_a = \bar{R}_{e1} = 2\lambda\bar{A}_{e1} = 2\lambda \quad ; \quad \bar{A}_{e1} \cong 1; \\ &\Delta t_a = \frac{2\pi}{\bar{R}_{e1}} = \frac{\pi}{\lambda} \\ &\bar{R}_b = \bar{R}_{e0} = 2\lambda\bar{A}_{e0} = 0 \quad ; \quad \bar{A}_{e0} = 0\ ;\ \lambda = 0\ ; \\ &\Delta t_b = \infty, \end{aligned} \tag{27}$$

which in both transitions respectively a unit probability of success is determined.

It is clear that success probabilities depend mainly upon the precise selection of interaction times of the control qubit (two-level atom) with the target qubit (successive cavity modes).

4. Hadamard Logic Gate Operation in the AJC Interaction Process

The normalized qubit state transition operators in Eq. (8) are the Hadamard gate operators for an atom in an initial excited $|e\rangle$ and ground $|g\rangle$ states respectively which takes the explicit forms

$$\begin{aligned} &\hat{\bar{\varepsilon}}_g = \frac{\bar{\alpha}\hat{s}_z + \hat{a}\hat{s}_- + \hat{a}^\dagger\hat{s}_+}{\bar{A}_{gn}}\ ; \\ &\hat{\bar{\varepsilon}}_e = \frac{\bar{\alpha}\hat{s}_z + \hat{a}\hat{s}_- + \hat{a}^\dagger\hat{s}_+}{\bar{A}_{en}}\ . \end{aligned} \tag{28}$$

Defining an initial state vector $|e,n\rangle$ in which an atom in an initial excited state $|e\rangle$ enters a field mode in a number state $|n\rangle$, by applying Eq. (8) (or Eq. (28)) a qubit state transition in the AJC interaction is determined in the form

$$\hat{\bar{\varepsilon}}_e|e,n\rangle = \bar{a}_{en}|e,n\rangle + \bar{b}_{en}|g,n-1\rangle = |\bar{\phi}_{en}\rangle\ ;\ \hat{\bar{\varepsilon}}_e|\bar{\phi}_{en}\rangle = |e,n\rangle \tag{29}$$

Similarly, defining an initial state vector $|g,n\rangle$ in which an atom in an initial ground state $|g\rangle$ is in a field mode in a

number state $|n\rangle$, by applying Eq. (8) (or Eq. (28) results into a qubit state transition in the AJC process of the form

$$\hat{\bar{\varepsilon}}_g |g,n\rangle = -\bar{p}_{gn}|g,n\rangle + \bar{q}_{gn}|e,n+1\rangle = |\bar{\phi}_{gn}\rangle \;\; ; \;\; \hat{\bar{\varepsilon}}_g |\bar{\phi}_{gn}\rangle = |g,n\rangle \quad (30)$$

In Eqs. (29) and (30) $|e,n\rangle$, $|g,n\rangle$ and $|\bar{\phi}_{en}\rangle$, $|\bar{\phi}_{gn}\rangle$ are the AJC initial and transition qubit state vectors while $\bar{a}_{en}$, $\bar{b}_{en}$, $\bar{p}_{gn}$, $\bar{q}_{gn}$ are the normalization parameters. In this respect, let us now consider two examples specifying Hadamard rotations for initial atomic basis states $|g\rangle$, $|e\rangle$ entering field modes in an initial vacuum and single-photon field mode states $|0\rangle$, $|1\rangle$ respectively.

The single qubit Hadamard operation at sum frequency $\bar{\delta} = 4\lambda$ and $n = 0$ specified for an atom in an initial ground state $|g\rangle$ is defined as

$$\hat{\bar{\varepsilon}}_g = \frac{\hat{\bar{A}}}{\bar{A}_{g0}} = \frac{1}{\bar{A}_{g0}}\left(2\hat{s}_z + \hat{a}\hat{s}_- + \hat{a}^\dagger \hat{s}_+\right) \quad (31)$$

where from Eq. (10)

$$\bar{A}_{g0} = \sqrt{2} \quad (32)$$

The qubit state transition takes the explicit form

$$\hat{\bar{\varepsilon}}_g |g,0\rangle = -\frac{1}{\sqrt{2}}|g,0\rangle + \frac{1}{\sqrt{2}}|e,1\rangle \quad (33)$$

As expected, the initial atomic state $|g\rangle$ is rotated to

$$|g\rangle \to \frac{1}{\sqrt{2}}\left(|e\rangle - |g\rangle\right) \quad (34)$$

Similarly, by applying Eq. (8) or (Eq. (28)), the Hadamard operation at sum frequency $\bar{\delta} = 4\lambda$ and $n = 1$ specified for an atom in an initial excited state $|e\rangle$ is defined as

$$\hat{\bar{\varepsilon}}_e = \frac{\hat{\bar{A}}}{\bar{A}_{e1}} = \frac{1}{\bar{A}_{e1}}\left(2\hat{s}_z + \hat{a}\hat{s}_- + \hat{a}^\dagger \hat{s}_+\right) \quad (35)$$

where from Eq.(10)

$$\bar{A}_{en} = \sqrt{2} \quad (36)$$

The final form of the qubit state transition is

$$\hat{\bar{\varepsilon}}_e |e,1\rangle = \frac{1}{\sqrt{2}}|e,1\rangle + \frac{1}{\sqrt{2}}|g,0\rangle \quad (37)$$

From Eq. (37), the initial atomic state $|e\rangle$ is rotated to

$$|e\rangle \to \frac{1}{\sqrt{2}}\left(|e\rangle + |g\rangle\right) \quad (38)$$

The Hadamard rotations in Eqs. (34) and (38) determined in the AJC process in Eqs. (31) and (35), agree precisely with the standard definition in Eq. (3) for initial atomic basis states $|e\rangle$, $|g\rangle$.

5. Conclusion

In this paper we have shown how to implement quantum C-NOT and quantum Hadamard logic gates in the AJC interaction mechanism where we observed that it is possible to achieve qubit state transitions starting with both the atom and the field mode in an absolute ground state which is not possible in the JC interaction mechanism. In the C-NOT gate operations we obtained ideal unit probabilities of success during the gate operation with a standard result of only flipping the target qubit when the control qubit is in an atomic ground state $|g\rangle$. The impressive unit success probabilities were achieved through accurate selection of interaction times during the C-NOT operations. In the Hadamard operations, standard Hadamard rotations of initial atomic basis states $|e\rangle$ and $|g\rangle$ into diagonal basis states respectively, were determined in the AJC interaction process.

These results will provide a good foundation in Quantum Optics where now the AJC interaction process can solely be implemented in QIP as opposed to the extensively applied numerical methods in the full QRM to probe its features characterized by blue-side band transitions. We note here that the advanced approaches do not give clearly the dynamical features generated solely by the AJC interaction. In the present work, we have demonstrated the dynamical effects of AJC model through qubit state transitions, determined from the reformulation of the QRM [9-11]. As part of the findings, it is clear that despite the coupling terms simultaneously exciting or de-exciting the atom and field states, C-NOT and Hadamard gate operations produce exactly the same results as those of the extensively applied JC model cited herein.

We propose practical implementation of these gates where the state of the cavity field can be described in terms of basis states $|0\rangle$ and $|1\rangle$ corresponding to the vacuum and one-photon states respectively but strictly in the AJC model. This means that for it to work, there should never be more than one photon in a cavity. The experimental implementation process should also comprise of a set of two-level atoms described by basis states $|e\rangle$ and $|g\rangle$. In addition, with reference to the C-NOT and Hadamard processes in Sec. 3 and 4 there should be two types of interaction; an almost resonant case in the C-NOT gate process when the perturbative parameter η is too large (i.e atom-field coupling strength λ far much greater than the sum frequency $\bar{\delta} = \omega_0 + \omega$) and an off-resonant interaction in the Hadamard gate operation with the advantage of also starting off with the field mode in the vacuum state and atom in the ground state. The experimental processes and challenges can then be addressed based on the scheme chosen.

Acknowledgements

We thank Maseno University, Department of Physics and Materials Science for providing a conducive environment to do this work.

References

[1] Nielsen, M. A. and Chuang, I. L. (2011) Quantum Computation and Quantum Information, Cambridge University Press.

[2] W. Scherer, W. (2019) Mathematics of Quantum Computing: An Introduction. Springer Nature.

[3] Deutsch, D. E. (1989) Quantum computational networks. Proc R. Soc. London A, 425 (1868): 73–90.

[4] Raussendorf, R. and Briegel, H. J. (2001) A one-way quantum computer. Phys. Rev. Lett., 86 (22): 5188.

[5] Deutsch, D. and Jozsa, R. (1992) Rapid solution of problems by quantum computation. Proc. R. Soc. London A, 439 (1907): 553–558.

[6] Barenco, A., Bennett, C. H., Cleve, R., DiVincenzo, D. P., Margolus, N., Shor, P., Sleator, T., Smolin, J. A., and Weinfurter, H. (1995) Elementary gates for quantum computation. Phys. Rev. A, 52 (5): 3457.

[7] Shi, Y. (2001) Both Toffoli and controlled-NOT need little help to do universal quantum computation. arXiv preprint quant-ph/0205115.

[8] Boykin, P. O., Mor, T., Pulver, M., Roychowdhury, V. and Vatan, F. (2000) A new universal and fault-tolerant quantum basis. Inf. Process. Lett., 75 (3): 101–107.

[9] Omolo, J. A. (2021) Conserved excitation number and U (1) -symmetry operator for the anti-rotating (anti-Jaynes-Cummings) term of the Rabi Hamiltonian. arXiv preprint arXiv: 2103.06577 (Unpublished).

[10] Omolo, J. A. (2017) Polariton and anti-polariton qubits in the Rabi model. Preprint: Research Gate, DOI: 10.13140/ RG.2.2. 11833.67683 (Unpublished).

[11] Omolo, J. A. (2019) Photospins in the quantum Rabi model. Preprint: Research Gate, DOI: 10.13140/RG.2.2.27331.96807 (Unpublished).

[12] Barenco, A., Deutsch, D., Ekert, A., and Richard Jozsa, R. (1995) Conditional quantum dynamics and logic gates. Phys. Rev. Lett., 74 (20): 4083.

[13] Knill, E., Laflamme, R., Barnum, H., Dalvit, D., Dziarmaga, J., Gubernatis, J., Gurvits, L., Ortiz, G., Viola, L., and Zurek, W. H. (2002) Introduction to quantum information processing. arXiv preprint quant-ph/0207171.

[14] Braunstein, S. L., Mann, A., and Revzen, M. (1992) Maximal violation of Bell inequalities for mixed states. Phys. Rev. Lett., 68 (22): 3259.

[15] Domokos, P., Raimond, J.-M. Brune, M., and Haroche, S. (1995) Simple cavity-QED two-bit universal quantum logic gate: The principle and expected performances. Phys. Rev. A, 52 (5): 3554.

[16] Vitali, D., Giovannetti, V., and Tombesi. P. (2001) Quantum Gates and Networks with Cavity QED Systems. In Macroscopic Quantum Coherence and Quantum Computing, pages 235–244. Springer.

[17] Saif, F., Ul Islam, R. and Javed, M. (2007) Engineering quantum universal logic gates in electromagnetic-field modes. J. Russ. Laser Res., 28 (5): 529–534.

[18] Rossatto, D. Z., Villas-Boas, C. J., Sanz, M., and Solano, E. (2017) Spectral classification of coupling regimes in the quantum Rabi model. Phys. Rev. A 96, 013849.