International Journal of Statistical Distributions and Applications
2022; 8(1): 1-13
http://www.sciencepublishinggroup.com/j/ijsda
doi: 10.11648/j.ijsd.20220801.11
ISSN: 2472-3487 (Print); ISSN: 2472-3509 (Online)

Weibull Log Logistic {Exponential} Distribution: Some Properties and Application to Survival Data

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To cite this article:

Obalowu Job, Adeyinka Solomon Ogunsanya. Weibull Log Logistic {Exponential} Distribution: Some Properties and Application to Survival Data. *International Journal of Statistical Distributions and Applications.* Vol. 8, No. 1, 2022, pp. 1-13.
doi: 10.11648/j.ijsd.20220801.11

Received: February 13, 2022; **Accepted**: March 8, 2022; **Published**: March 15, 2022

Abstract: A four-parameter continuous probability model called the Weibull log-logistic {Exponential} distribution (WLLED) was introduced and studied in this research using T-log-logistic {Exponential} distribution via T-R{Y} framework to extend the two-parameter log-logistic distribution. The objective of this research is to explore the versatility and flexibility of the log-logistic and Weibull distributions in modeling lifetime data. Some basic structural properties which include the reliability measures and hazard function, cumulative hazard function, Moment, Quantile, skewness, kurtosis, mixture representation, order statistics and asymptotic behavior of the WLLED were obtained and established. The shape of the new four parameter distribution is also investigated. A simulation study was conducted to evaluate the MLE estimates, bias, and standard error for various parameter combinations and different sample sizes. The efficiency of the WLLE distribution was compared with other related distribution from the literature using five goodness-of-fit statistics: AIC, CAIC and BIC, Anderson-Darling A* and Cramér-Von Mises W*, methods of comparison. The method of maximum likelihood estimation was proposed in estimating its parameters. An application to the survival times of 121 patients with breast cancer dataset was provided and the WLLED displays a good fit. Finally, it is recommended that the WLLED can be used for modeling positively skewed real-life data.

Keywords: Log-logistic Distribution, Censored Data, Lifetime Data, Mathematical Statistics, Maximum Likelihood Estimation

1. Introduction

Standard distributions have been used extensively over the years in describing real life events and for modeling in fields like actuarial, biology, engineering, environmental sciences, medicine and many others. However, there is a need to extend these distributions for better modeling capability especially in applied fields of study. Therefore, methods for generating new families of distributions have been introduced in the literature and several families of distribution have also been studied. These attempts were aimed at providing greater flexibility in modeling data in real life practice. A list of these families of distributions are contained in Oguntunde P. E *et al* [1] and the references therein.

However, the interest of this research is to extend the log-logistic distribution following the concept of Aljarrah, M. A. et al [2]. Examples of some extended forms of log-logistic models are: The Type I half-logistic family of distributions [3], Zografos-Balakrishnan odd log-logistic family of distributions [4], generalized odd log-logistic family of distribution [5], beta odd log-logistic generalized family of distributions [6], Log-logistic Weibull Distribution [7] and many more.

The motivation for developing the WLLED in this article is the advantages presented by these generalized distributions; for instance, they have a hazard function that exhibits increasing, decreasing and bathtub shapes. The versatility and flexibility of the log-logistic and Weibull distributions in modeling lifetime data is also another motivation for this study.

The structure of the remaining part of this paper is as follows: in section 2, the WLLED is derived and its special models are identified. A linear representation for the WLLE density is derived and some of its mathematical properties including ordinary moments and order statistics are obtained in section 3. In section 4, estimation of model parameters using the method of maximum likelihood is provided. A real data set to establish the empirical flexibility of the new model is provided in section 5.

2. Derivation of T-log Logistic {Exponetial} Distribution

The log logistic distribution is the probability distribution of a random variable whose logarithm has a logistic distribution. Authors like [8-11] have studied the log-logistic distribution in some details. It has a cumulative distribution function of the form:

$$F_R = \frac{\left(\frac{x}{\alpha}\right)^{\beta}}{1+\left(\frac{x}{\alpha}\right)^{\beta}} \tag{1}$$

The corresponding probability density function (pdf) is:

$$f_R = \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta}}{\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^2} \tag{2}$$

where α>0 and β>0 are scale and shape parameter respectively. This distribution is used in survival analysis as a parametric model for events whose hazard rate increases initially and decreases later.

However, the new distribution with T-R{Y} framework is given as:

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt = F_T\big(Q_Y(F_R(x))\big)$$

The corresponding pdf is:

$$f_X(x) = f_R(x) \times \frac{f_T\big(Q_Y(F_R(x))\big)}{f_Y\big(Q_Y(F_R(x))\big)}$$

Alternatively,

$$f_X(x) = f_T\big(Q_Y(F_R(x))\big) \times Q\,'_Y(F_R(x)) \times f_R(x)$$

Ogunnsanya et al. [12] explored the properties of Odd Lomax-Exponential (type III) (OLE) distribution. Alzaatreh et al., ([13-16]) studied respectively, the T-R{exponential}, T-normal{Y}, T-gamma{Y} and T-Cauchy{Y} families of distributions with specific examples. Weibull Inverse Rayleigh Distribution [17]. Some general properties and application of the T-R{Y} family was studied by [2].

2.1. T-log Logistic {Expontential) Distribution

A new family of generalized log-logistic distribution based on the quantile function of standard exponential distribution with a support of the random variable T is $(0, \infty)$ is derived in this section as follows:

Let $F_X(x)$ be any random variable X and the pdf of any random variable Y defined on $(0, \infty]$,

$$G(x) = \int_0^{Q_Y(F_R(x))} r(t)dt = R\{Q_Y(F_R(x))\}$$

$$F_X(x) = F_T\big(Q_Y(F_R(x))\big)$$

$$F_X(x) = F_T(-log[1 - F_R(x)])$$

where $F_R(x)$ and $F_Y(x)$ are the log-logistic and exponential distributions. $F_T(x)$ is any lifetime distribution (T). This newly developed method is used to define the T-Log logistic{Y} family of distributions by making T the Weibull random variable.

For Log logistic {Exponential} distribution,

$$F_X(x) = \int_0^{-log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}} r(t)dt = F_T\left(-log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}\right)$$

The corresponding pdf is:

$$f_X(x) = f_R(x) \times \frac{f_T(-log[1 - F_R(x)])}{f_Y(-log[1 - F_R(x)])}$$

Alternatively,

$$f_X(x) = \frac{f_R(x)}{1 - F_R(x)} \times f_T(-log[1 - F_R(x)])$$

Therefore, the of generalized T-Log Logistic {Exponential} distribution is:

$$F_X(x) = F_T\left\{-log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}\right\} \tag{3}$$

The corresponding pdf is:

$$f_X(x) = \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta}}{\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^2} \times \frac{f_T\left\{-log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}\right\}}{f_Y\left\{-log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}\right\}} \tag{4}$$

Alzaatreh et al. [14], the T-Log Logistic{Exponential} can be interpreted as; "let Y be a lifetime random variable having Log logistic distribution and the cumulative hazard function that an individual (or component) following the lifetime Y will die (fail) at a time x is $-log[1 - F_R(x)]$ being the quantile function of a standard exponential distribution".

Alzaatreh et al [13] proposed the T-X method of generating distribution using: $W\big(F(x)\big) = Q_Y(F_R(x)) = -log[1 - F_R(x)]$, where W(F(x)) is the quantile function of the standard exponential distribution and the T-Log logistic{exponential} is a family of distribution from the hazard function and the cumulative hazard function of the

Log logistic distribution as:

$$f_X(x) = \frac{f_R(x)}{1-F_R(x)} \times f_T(-log[1-F_R(x)])$$

Alternatively,

$$f_X(x) = h_R(x) \times f_T\big(H_R(x)\big)$$

Survival Function.

The survival function of T-LLED is given as:

$$S_X(x) = 1 - F_T\left\{-log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}\right\} \quad (5)$$

For $\alpha, \beta > 0$ and $x \geq 0$.

Hazard Function.

The hazard function of the T-LLE distribution is derived from this definition:

$$h_X(x) = \frac{\frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta}}{\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^2} \times \frac{f_T\left\{-log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}\right\}}{f_Y\left\{-log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}\right\}}}{1-F_T\left\{-log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}\right\}} \quad (6)$$

For $\alpha, \beta > 0$ and $x \geq 0$.

2.2. Weibull-log Logistic {Exponential} Distribution

Let us consider a lifetime two-parameter Weibull distribution with pdf and given by:

$$f_T(\mathrm{t};\lambda,\mathrm{k}) = \left\{\frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1} e^{-\left(\frac{t}{\lambda}\right)^{k}}\right\};\ t\geq 0$$

where $\lambda, k \geq 0$ and $F_T(x) = 1 - e^{-\left(\frac{t}{\lambda}\right)^{k}}$ and the of Log-Logistic distribution as given in section 2 of this article.

Then the proposed WLLE distribution from (3) is:

$$F_X(x) = 1 - e^{,-\left(\frac{-log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}}{\lambda}\right)^{k}} \quad (7)$$

While the corresponding pdf is derived as follows from (4) is:

$$f_X(x) = \frac{\left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta}}{\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^2} \times \frac{f_T\left\{-log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}\right\}}{f_Y\left\{-log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}\right\}}$$

$$f_X(x) = \frac{1}{\beta} e^{-\left(\frac{x-\theta}{\beta}\right)} \times \frac{\frac{k}{\lambda}\left(\frac{b\left(\frac{x-\theta}{\beta}\right)}{\lambda}\right)^{k-1} e^{-\left(\frac{b\left(\frac{x-\theta}{\beta}\right)}{\lambda}\right)^{k}}}{\frac{1}{b} e^{\frac{-b\left(\frac{x-\theta}{\beta}\right)}{b}}}$$

Therefore, the pdf of the WLLE Distribution is given by:

$$f_X(x) = \frac{k\beta}{\lambda^k \alpha^{\beta}} x^{\beta-1}\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}\left(log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)\right)^{k-1} e^{-(1/\lambda)^k\left(log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)\right)^{k}} \quad (8)$$

where $\lambda, \alpha > 0\ and\ k, \beta, x \geq 0$ are parameters of WLLE distribution.

The plots for the pdf and of the WLLED are displayed in Figures 1and 2 respectively.

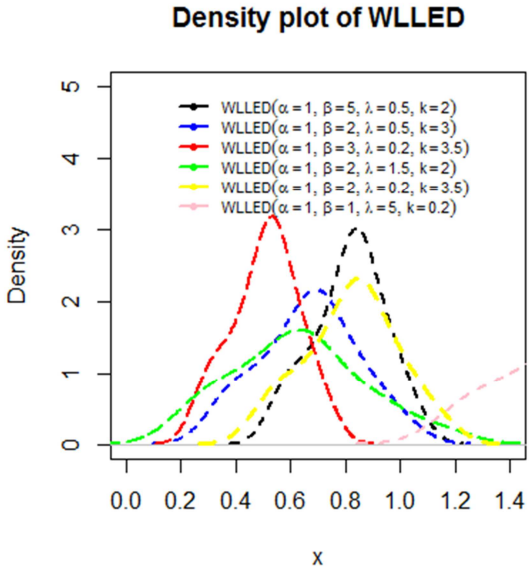

Figure 1. *Plot for the pdf of the WLLED.*

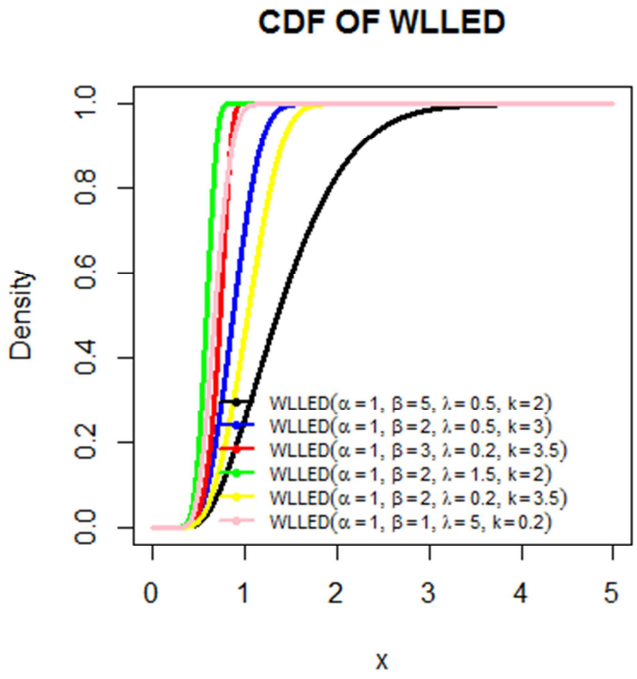

Figure 2. *CDF of the WLLE distribution.*

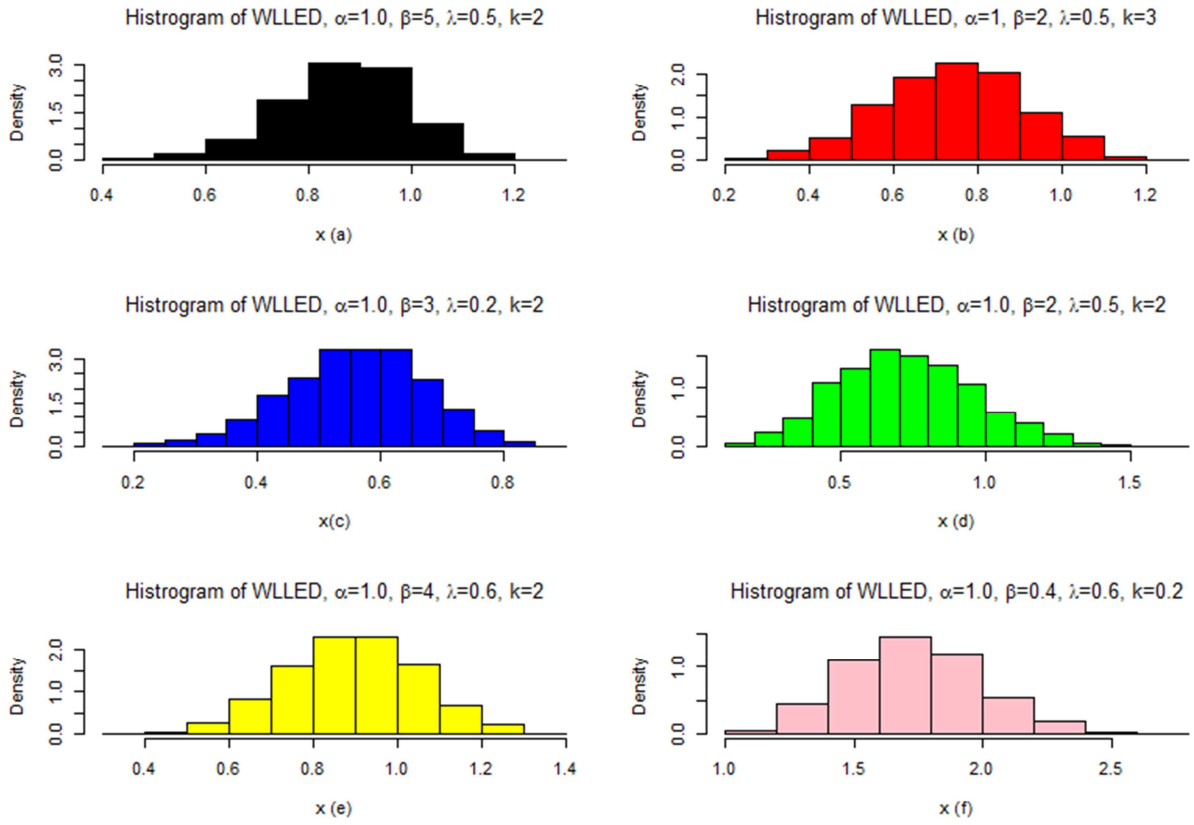

Figure 3. *Histogram of different combination of parameters.*

Figure 3 shows that a-c and e are negatively skewed while d and f are positively skewed this is an indication that the new model can fit both negatively and positively skewed data.

Survival Function

The survival function of the WLLE distribution is derived from this definition:

$$S_X(x) = 1 - F_X(x)$$

where $F_X(x)$ is the of the WLLE distribution. The survival function S(x) can be written as:

$$S_X(x) = e^{-(1/\lambda)^k log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^k} \quad (9)$$

For x>0, λ, β, k>0 and t>0. The probability that a system having age x units of time will survive up to x+t units of time is given by:

$$S(t \mid x) = \frac{S(x+t)}{S(x)}$$

$$S(t \mid x) = \frac{e^{-(1/\lambda)^k log\left(1+\left(\frac{x+t}{\alpha}\right)^\beta\right)^k}}{e^{-(1/\lambda)^k log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^k}}$$

Hazard Function

The hazard function of the WLLE distribution is derived from this definition:

$$h_X(x) = \frac{f_X(x)}{1-F_X(x)}$$

where $f_X(x)$ and $F_X(x)$ are the PDF and CDF WLLE distribution. Therefore, the hazard function h(x) of the WLLED can be written as:

$$h_X(x) = \frac{\frac{k\beta}{\lambda^k\alpha^\beta}x^{\beta-1}\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{-1} log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{k-1} e^{-(1/\lambda)^k log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^k}}{1-\left[1-e^{-(1/\lambda)^k log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^k}\right]}$$

$$h_X(x) = \frac{k\beta}{\lambda^k\alpha^\beta}x^{\beta-1}\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{-1} log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{k-1} \quad (10)$$

The corresponding plot for the survival function and hazard function of the WLLE distribution are displayed in Figures 4 and 5 respectively.

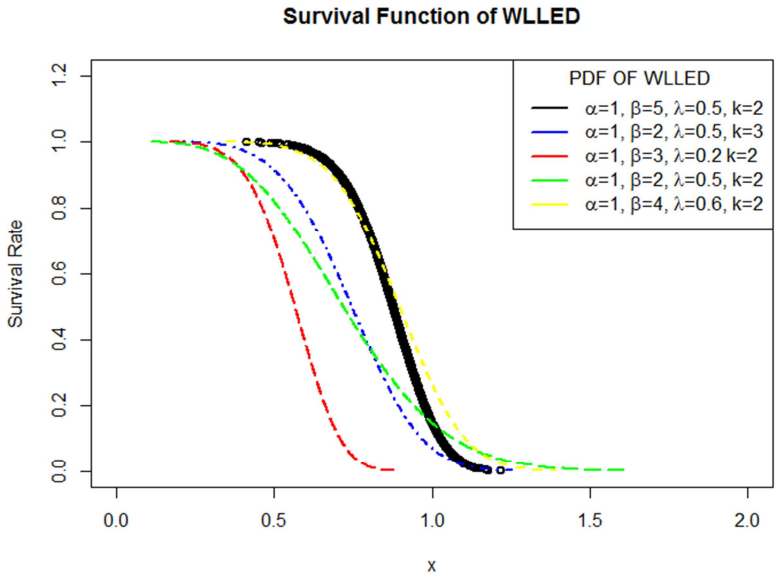

Figure 4. *Survival rate plot of WLLE distribution for sample size=1000 and for various values of k, λ when α=1 and k=2.*

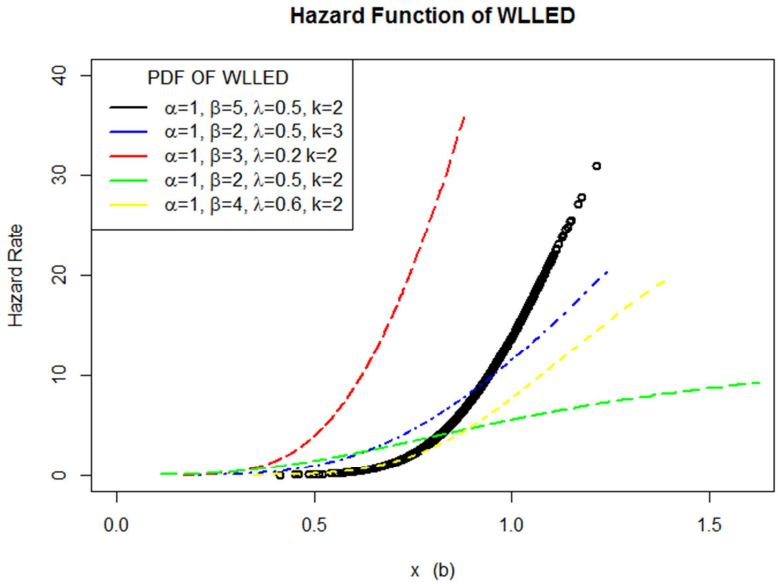

Figure 5. *Hazard rate plot of WLLE distribution for sample size=1000 and for various values of k, λ when α=1 and k=2.*

The log hazard of the WLLED which is frequently used in modeling is given by:

$$\lambda_X(t) = log(h_X(x)$$

$$\lambda_X(t) = log\left[\frac{k\beta}{\lambda^k\alpha^\beta}x^{\beta-1}\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{-1} log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{k-1}\right]$$

$$\lambda_X(t) = log\left(\frac{k\beta}{\lambda^k\alpha^\beta}\right) + (\beta-1)logx + log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{-1} + log\left[log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{k-1}\right]$$

$$log\left(\frac{k\beta}{\lambda^k\alpha^\beta}\right) = \beta_0$$

$$\lambda_X(t) = \beta_0 + (\beta-1)logx - log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right) + log\left[log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{k-1}\right] \quad (11)$$

Reversed Hazard Function (RH) of the WLLED is:

$$RH(x) = \frac{\frac{k\beta}{\lambda^k \alpha^\beta} x^{\beta-1}\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{-1} log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{k-1} e^{-(1/\lambda)^k log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^k}}{1-e^{-(1/\lambda)^k log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^k}} \quad (12)$$

Cumulative Hazard Function

The cumulative hazard function of the WLLE distribution is derived from this definition:

$$H_X(x) = -log_e S_X(x)$$

where S(x) is the survival function of WLLE distribution. Hence,

$$H_X(x) = -log_e\left[e^{-(1/\lambda)^k log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^k}\right] \quad (13)$$

Simplifying (13) further, we have:

$$H_X(x) = (1/\lambda)^k log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^k \quad (14)$$

The plots for the cumulative hazard function is displayed in Figure 6 while its plot against the survival rate is displayed in Figure 7.

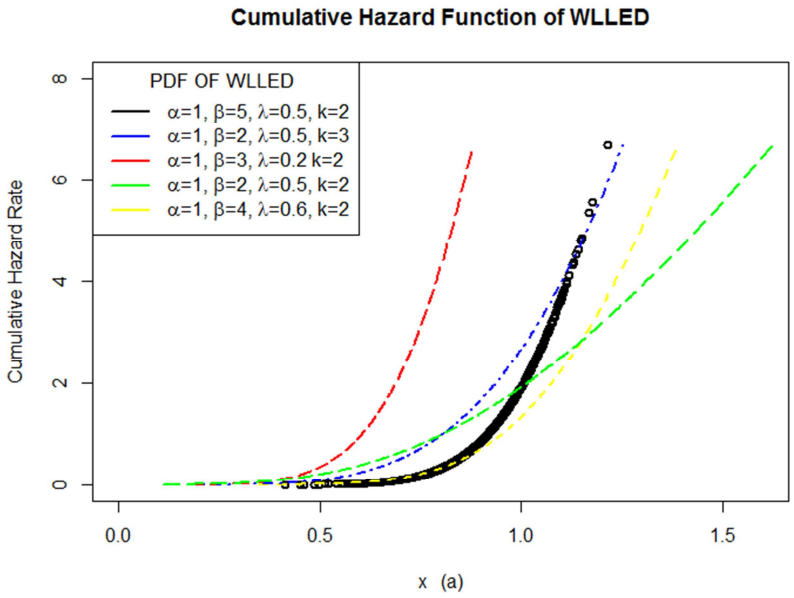

Figure 6. *The Cumulative hazard rate of the WLLED.*

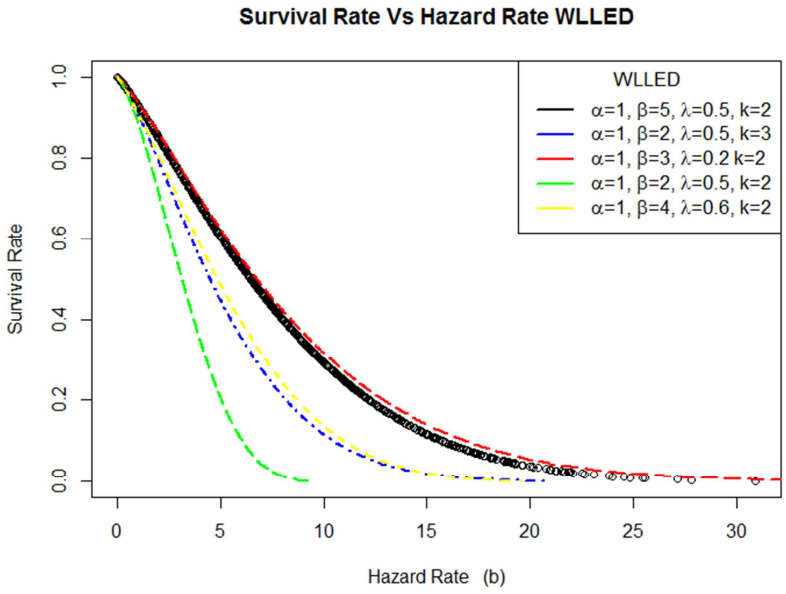

Figure 7. *Survival rate Versus Cumulative hazard rate of the WLLED.*

Shapes of the density and hazard rate function.

The shapes of the density and hazard rate can be described numerically. The critical points of the new proposed density are the roots of the equation:

$$\left(\frac{x}{\alpha}\right)^{\beta} k\beta\, log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{k-1}$$

$$\left\{\left(\frac{x}{\alpha}\right)^{\beta}(k-1)\beta\, log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}+(\beta-1)\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)-\frac{\left(\frac{x}{\alpha}\right)^{\beta}k\beta\, log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{k-1}}{\lambda^{k}}-\left(\frac{x}{\alpha}\right)^{\beta}\right\}=0$$

$$\left(\frac{x}{\alpha}\right)^{\beta}(k-1)\beta\, log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}+(\beta-1)\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)-\frac{\left(\frac{x}{\alpha}\right)^{\beta}k\beta\, log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{k-1}}{\lambda^{k}}-\left(\frac{x}{\alpha}\right)^{\beta}=0$$

$$\left(\frac{x}{\alpha}\right)^{\beta}\beta\, log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}\left((k-1)-\frac{k}{\lambda^{k}}log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{k}\right)+(\beta-1)\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)-\left(\frac{x}{\alpha}\right)^{\beta}=0 \quad (15)$$

The critical points of the WLLE hazard are obtained from the equation:

$$\left(\frac{x}{\alpha}\right)^{2\beta} k\beta log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{k-1}$$

$$\left\{\beta log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}(k-1)-\beta\right\}+\left(\frac{x}{\alpha}\right)^{\beta}k(\beta-1)\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)\left\{\beta log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{k-1}\right\}=0$$

$$\left(\frac{x}{\alpha}\right)^{2\beta} k\beta log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{k-1}$$

$$\left\{\beta log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}(k-1)-\beta\right\}+\left(\frac{x}{\alpha}\right)^{\beta}k(\beta-1)\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)\left\{\beta log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{k-1}\right\}=0$$

$$\left(\frac{x}{\alpha}\right)^{2\beta} k\beta log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{k-1}\left\{\beta(k-1)log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}-\beta+\left(\frac{x}{\alpha}\right)^{-\beta}(\beta-1)\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)\right\}=0$$

$$\beta(k-1)log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1}-\beta+\left(\frac{x}{\alpha}\right)^{-\beta}(\beta-1)\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)=0 \quad (16)$$

By using any numerical software, the equations (15) and (16) can be examined to determine the local maximum and minimum and inflexion points of WLLED.

Quantile Function for the WLLE Distribution

This is used for the generation of some moments of random variables such as skewness and kurtosis. Furthermore, can be used to obtain the median and generation of random numbers. It is derived by taking the inverse of the inverse of a given probability distribution. The quantile function of WLLE distribution is given as:

$$X_Q = \alpha\left\{e^{\lambda(-\log(1-p))^{1/k}}-1\right\}^{1/\beta} \quad (17)$$

Where p ~ U (0,1)

Asymptotic behavior of WLLE distribution

The asymptotic behavior of the WLLE distribution when $x \to 0$ and when $x \to \infty$ is determined as follows:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{k\beta}{\lambda^{k}\alpha^{\beta}}x^{\beta-1}\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1} log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{k-1} e^{-(1/\lambda)^{k}log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{k}}=0 \quad (18)$$

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty}\frac{k\beta}{\lambda^{k}\alpha^{\beta}}x^{\beta-1}\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{-1} log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{k-1} e^{-(1/\lambda)^{k}log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{k}}=0 \quad (19)$$

Since $\lim_{x\to 0} x^{\beta-1}=0$, then equation equals zero.

Similarly, since $\lim_{x\to\infty} e^{-(1/\lambda)^{k}log\left(1+\left(\frac{x}{\alpha}\right)^{\beta}\right)^{k}}=0$ as x approaches zero and infinity, the density of the proposed distribution also tends towards zero.

3. Mixture Representation and Nth Moment of WLLE Distribution

In this section, a useful representation for the WLLE distribution is provided. Using the pdf (8) of the WLLED, we have:

Theorem 1: let X be a random variable of WLLE distribution, the linear representation is given as:

$$f_X(x) = \frac{k^2\beta}{\lambda^k\alpha^\beta}(k-1)\omega_{j,m,r}x^{\beta(mj+i+1)-1} \tag{20}$$

Where $i \geq 0, j > 0, m > 0, r \geq 0$ and $\alpha, \lambda > 0$.

$$f_X(x) = \frac{k\beta}{\lambda^k\alpha^\beta}x^{\beta-1}\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{-1}\underbrace{log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{k-1}}_{A}\underbrace{e^{-(1/\lambda)^k log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^k}}_{B}$$

$$A = log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{k-1} = (k-1)log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)$$

$$B = e^{-(1/\lambda)^k log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^k}$$

Using series expansion, A and B can be rewritten as:

$$A = (k-1)\sum_{m=1}^{\infty}(-1)^{m+1}\frac{\left(\frac{x}{\alpha}\right)^{\beta m}}{m}$$

$$B = \sum_{r=0}^{\infty}(-1)^r\frac{(1/\lambda)^{rk}log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{rk}}{r!}$$

And the Taylor series expansion of $\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{-1}$ is given as:

$$\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{-1} = \sum_{i}^{\infty}(-1)^i\binom{1}{i}\left(\frac{x}{\alpha}\right)^{i\beta}$$

After substituting A and B, then,

$$f_X(x) = \frac{k^2\beta}{\lambda^k\alpha^\beta}(k-1)\sum_{J=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+j+r+i+2}\binom{1}{i}(1/\lambda)^{rk}r\frac{\left(\frac{x}{\alpha}\right)^{\beta(mj)}}{j\times m\times r!}x^{\beta-1}$$

$$f_X(x) = \frac{k^2\beta}{\lambda^k\alpha^\beta}(k-1)\sum_{i,j,m=0,r>0}^{\infty}(-1)^{m+j+r+i+2}(1/\lambda)^{rk}r\binom{1}{i}\frac{x^{\beta(mj+1+i)-1}}{\alpha^{\beta(mj+i)}\times j\times m\times r!}$$

$$f_X(x) = \frac{k^2\beta}{\lambda^k\alpha^\beta}(k-1)\sum_{j.m=0,r>0}^{\infty}\binom{1}{i}\frac{(-1)^{m+j+r+i+2}(1/\lambda)^{rk}r}{\alpha^{\beta(mj+i)}\times j\times m\times r!}x^{\beta(mj+1+i)-1}$$

Let:

$$\omega_{i,j,m,r} = \frac{k^2\beta}{\lambda^k\alpha^\beta}(k-1)\sum_{j.m,r=0}^{\infty}\binom{1}{i}\frac{(-1)^{m+j+r+i+2}(1/\lambda)^{rk}}{\alpha^{\beta(mj+i)}\times j\times m\times r!}r \tag{21}$$

Where $i \geq 0, j > 0, m > 0, r \geq 0$.
Then,

$$f_X(x) = \omega_{j,m,r}x^{\beta(mj+i+1)-1} \tag{22}$$

Hence the PDF of Weibull Log logistic {Exponential}Distribution is a mixture of PDF and Exponentiated x with power parameter $\beta(mj+i+1)$:

$$F_X(x) = \frac{1}{\beta(mj+i+1)}\omega_{j,m,r}x^{\beta(mj+i+1)}$$

The n[th] Moment of WLLE Distribution:

If X_1, X_2,... X_n be independent and identically distributed (IID) random variables that follow the WLLED$(\alpha, \beta, \lambda, k)$ model, let μ_1' and μ_2' be the first two sample moments.

$$\mu_n' = \int_0^\infty x^n f_X(x) dx$$

$$\mu_n' = \int_0^\infty x^n \omega_{j,m,r} x^{\beta(mj+i+1)-1} dx$$

$$\mu_n' = \omega_{j,m,r} \int_0^\infty x^{n+\beta(mj+i+1)-1} dx$$

Given that $\int_0^\infty x^{n+\beta(mj+1)-1} dx = I_\infty B(n + \beta(mj + i + 1), 1)$

Theorem 2: If X ~ WLLED $(\alpha, \beta, \lambda, k)$, then the nth moment of X is given by:

$$\mu_n' = \omega_{j,m,r} I_\infty B(n + \beta(mj + i + 1), 1)$$

$$\mu_n' = \frac{\omega_{j,m,r}}{n+\beta(mj+i+1)} \quad (23)$$

And first, second moments and variance are given below:

$$\mu_1' = \frac{\omega_{j,m,r}}{1+\beta(mj+i+1)}, \mu_2' = \frac{\omega_{j,m,r}}{2+\beta(mj+i+1)}$$

$$variance = \frac{\omega_{j,m,r}}{2 + \beta(mj + i + 1)} - \left(\frac{\omega_{j,m,r}}{1 + \beta(mj + i + 1)}\right)^2$$

$$variance = \omega_{j,m,r} \left\{ \frac{1}{2 + \beta(mj + i + 1)} - \frac{\omega_{j,m,r}}{\left(1 + \beta(mj + i + 1)\right)^2} \right\}$$

Standard deviation of X is given by:

$$\sigma = \sqrt{\omega_{j,m,r} \left\{ \frac{1}{2 + \beta(mj + i + 1)} - \frac{\omega_{j,m,r}}{\left(1 + \beta(mj + i + 1)\right)^2} \right\}}$$

Skewness and Kurtosis

In this research, the coefficient of skewness and kurtosis are calculated using quantile function which is defined by [17] and [18] respectively:

$$S = \frac{Q(6/8) - 2Q(4/8) + Q(2/8)}{Q(6/8) - Q(2/8)} \quad (24)$$

and

$$K = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)} \quad (25)$$

Theorem 3: The skewness and kurtosis of WLLE distribution do not depend on parameter α.

Proof. Recall the quantile function of WLLE distribution given in (17) as:

$$X_Q(p) = \alpha \left\{ e^{\lambda(-\log(1-p))^{1/k}} - 1 \right\}^{1/\beta}$$

Put (17) in (24) to have

$$S = \frac{\alpha\left\{e^{\lambda(-\log(2/8))^{1/k}} - 1\right\}^{1/B} - 2\alpha\left\{e^{\lambda(-\log(4/8))^{1/k}} - 1\right\}^{1/\beta} + \alpha\left\{e^{\lambda(-\log(6/8))^{1/k}} - 1\right\}^{1/\beta}}{\alpha\left\{e^{\lambda(-\log(2/8))^{1/k}} - 1\right\}^{1/\beta} - \alpha\left\{e^{\lambda(-\log(6/8))^{1/k}} - 1\right\}^{1/\beta}}$$

Dividing the numerator and denominator by α

$$S = \frac{\left\{e^{\lambda(-\log(2/8))^{1/k}} - 1\right\}^{1/B} - \left\{e^{\lambda(-\log(4/8))^{1/k}} - 1\right\}^{1/\beta} + \left\{e^{\lambda(-\log(6/8))^{1/k}} - 1\right\}^{1/\beta}}{\left\{e^{\lambda(-\log(2/8))^{1/k}} - 1\right\}^{1/\beta} - \left\{e^{\lambda(-\log(6/8))^{1/k}} - 1\right\}^{1/\beta}}$$

The same procedure is also used to prove that the kurtosis does not contain parameter α. Put (17) in (25) to have

$$K = \frac{\alpha\left\{e^{\lambda(-\log(1/8))^{1/k}} - 1\right\}^{1/\beta} - \alpha\left\{e^{\lambda(-\log(3/8))^{1/k}} - 1\right\}^{1/\beta} + \alpha\left\{e^{\lambda(-\log(5/8))^{1/k}} - 1\right\}^{1/\beta}}{\alpha\left\{e^{\lambda(-\log(2/8))^{1/k}} - 1\right\}^{1/\beta} - \alpha\left\{e^{\lambda(-\log(6/8))^{1/k}} - 1\right\}^{1/\beta}}$$

Dividing through by α:

$$K = \frac{\left\{e^{\lambda(-\log(1/8))^{1/k}} - 1\right\}^{1/\beta} - \left\{e^{\lambda(-\log(3/8))^{1/k}} - 1\right\}^{1/\beta} + \left\{e^{\lambda(-\log(5/8))^{1/k}} - 1\right\}^{1/\beta}}{\left\{e^{\lambda(-\log(2/8))^{1/k}} - 1\right\}^{1/\beta} - \left\{e^{\lambda(-\log(6/8))^{1/k}} - 1\right\}^{1/\beta}}$$

Table 1. *The Galtons skewness and Moors kurtosis for some values of $(\alpha, \beta, \lambda, k)$ of the new WLLE distribution.*

α	β	λ	k	Skewness	Kurtosis	α	β	λ	k	Skewness	Kurtosis
0.1	0.1	0.1	10	0.2277	1.37	0.1	2	0.1	20	-0.1300	1.32
0.1	0.1	0.1	20	0.0472	1.24	0.1	2	0.5	2	0.0014	1.27
0.1	0.1	0.1	5	0.5305	2.22	0.1	2	0.5	5	-0.0810	1.28
0.1	0.1	0.5	20	0.0880	1.25	0.1	2	0.5	10	-0.1101	1.30
0.1	0.1	0.5	5	0.6284	2.89	0.1	2	0.5	20	-0.1250	1.32

α	β	λ	k	Skewness	Kurtosis	α	β	λ	k	Skewness	Kurtosis
0.1	0.1	2	10	0.5817	2.59	0.1	2	2	2	0.1448	1.39
0.1	0.1	2	20	0.2671	1.44	0.1	2	2	5	-0.0164	1.26
0.1	0.1	5	10	0.9111	11.37	0.1	2	2	10	-0.0766	1.28
0.1	0.1	5	20	0.6237	2.85	0.1	2	2	20	-0.1080	1.30
0.1	0.1	10	10	0.9957	134.90	0.1	2	5	2	0.3723	1.94
0.1	0.1	10	20	0.9102	10.58	0.1	2	5	5	0.0917	1.29
0.1	0.1	20	20	0.9958	120.71	0.1	2	5	10	-0.0203	1.26
0.1	0.5	0.1	0.5	0.8551	7.95	0.1	2	5	20	-0.0794	1.28
0.1	0.5	0.1	10	-0.0640	1.27	0.1	2	10	5	0.2500	1.48
0.1	0.5	0.1	20	-0.1021	1.29	0.1	2	10	10	0.0653	1.26
0.1	0.5	0.5	10	-0.0429	1.26	0.1	2	10	20	-0.0355	1.26
0.1	0.5	0.5	20	-0.0913	1.29	0.1	2	20	10	0.2297	1.41
0.1	0.5	2	10	0.0432	1.25	0.1	2	20	20	0.0513	1.25
0.1	0.5	2	20	-0.0472	1.26	0.1	5	0.1	0.5	0.0225	1.29
0.1	0.5	5	5	0.5216	2.40	0.1	5	0.1	2	-0.0962	1.29
0.1	0.5	5	10	0.2310	1.41	0.1	5	0.1	5	-0.1222	1.31
0.1	0.5	5	20	0.0520	1.25	0.1	5	0.1	10	-0.1311	1.32
0.1	2	0.1	0.5	0.2351	1.47	0.1	5	0.1	20	-0.1356	1.33
0.1	2	0.1	2	-0.0406	1.26	0.1	5	0.5	0.5	0.1025	1.55
0.1	2	0.1	5	-0.0999	1.29	0.1	5	0.5	2	-0.0637	1.29
0.1	5	2	0.5	0.3598	3.33	0.1	5	0.5	5	-0.1076	1.30
0.1	5	2	2	0.0377	1.30	0.1	5	0.5	10	-0.1235	1.32
0.1	2	0.1	10	-0.1199	1.31	0.1	5	0.5	20	-0.1317	1.32

Table 1 shows that WLLE distribution has tendency to exhibit both positive skewness and negative skewness, when $\alpha, \beta, \lambda, k$ are greater than zero but less than one the WLLE distribution exhibits positive skewness hence otherwise.

Entropy

In information theory, entropy is an important concept in information theory and can be defined as measure of the randomness or uncertainty associated with a random variable. However, the Shannon entropy for a random variable X with pdf $f_X(x)$ is defined as $E\{-log(f_X(x))\}$.

Theorem 4 If X is a random variable that has WHLED$(x; k, \theta, \gamma)$ then the Shannon's entropy is given as:

$$E\{-log(f_X(x))\} = \{-log(\omega_{j,m,r}x^{\beta(mj+i+1)-1})\} = E\{-log(\omega_{i,j,m,r})\} + E\{-log(x^{\beta(mj+i+1)-1})\}$$

$$= E\left\{-log\left[\frac{k^2\beta}{\lambda^k\alpha^\beta}(k-1)\sum_{i,j.m,r=0}^{\infty}\binom{1}{i}\frac{(-1)^{m+j+r+i+2}(1/\lambda)^{rk}}{\alpha^{\beta(mj+i)}\times j\times m\times r!}r\,k\right]\right\} + E\{-log(x^{\beta(mj+i+1)-1})\}$$

Order Statistics

Order statistics is an important concept in probability theory. Let a random sample $X_1, X_2, \ldots X_n$, from the distribution function F(x) and corresponding pdf f(x), therefore the pdf of ith order statistic is given as

Proposition 5 If X is a random variable that has WLED$(x; k, \theta, \gamma)$ and let $f(x_p)$ denote the pdf of ith order statistic which is given as:

$$f(X_p) = \frac{n!}{(p-1)!(n-p)!}\sum_{q=0}^{\infty}(-1)^q\binom{n-1}{q}\left(\frac{1}{\beta(mj+i+1)}\right)^{p+q-1}(\omega_{i,j,m,r})^{p+q}\frac{[x^{\beta(mj+i+1)}]^{p+q-1}}{x} \quad (26)$$

where $k, \theta, \gamma \geq 0$ are parameters of WHLE distribution

Proof: Given

$$f(X_p) = \frac{n!}{(i-1)!(n-i)!}f(x)F(x)^{i-1}[1-F(x)]^{n-1} \quad (27)$$

$$f(X_p) = \frac{n!}{(p-1)!(n-p)!}\sum_{q=0}^{\infty}(-1)^q\binom{n-1}{q}f(x)F(x)^{p+q-1}$$

From equations (20) and (21)

$$f(X_p) = \frac{n!}{(p-1)!(n-p)!}\sum_{q=0}^{\infty}(-1)^q\binom{n-1}{q}[\omega_{i,j,m,r}x^{\beta(mj+i+1)-1}]\left[\frac{1}{\beta(mj+i+1)}\omega_{i,j,m,r}x^{\beta(mj+i+1)}\right]^{p+q-1} \quad (28)$$

By factorizing equation (28), we have

$$f(X_p) = \frac{n!}{(p-1)!(n-p)!}\sum_{q=0}^{\infty}(-1)^q\binom{n-1}{q}\left(\frac{1}{\beta(mj+i+1)}\right)^{p+q-1}\left[\omega_{i,j,m,r}x^{\beta(mj+i+1)-1}\right]\left[\omega_{i,j,m,r}x^{\beta(mj+i+1)}\right]^{p+q-1}$$

Then, it becomes

$$f(X_p) = \frac{n!}{(p-1)!(n-p)!}\sum_{q=0}^{\infty}(-1)^q\binom{n-1}{q}\left(\frac{1}{\beta(mj+i+1)}\right)^{p+q-1}\left(\omega_{i,j,m,r}\right)^{p+q}\frac{\left[x^{\beta(mj+i+1)}\right]^{p+q-1}}{x}$$

4. Parameter Estimation

To estimate the parameters of the WLLE distribution, the method of Maximum Likelihood Estimation (MLE) was used as follows; Let $X=x_1, x_2,..., x_n$ be a random sample of n independently and identically distributed random variables each having the WLLE distribution, the likelihood function is given as follows:

$$f_X(x) = \frac{k\beta}{\lambda^k\alpha^\beta}x^{\beta-1}\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)^{-1}\left(log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)\right)^{k-1}e^{-(1/\lambda)^k\left(log\left(1+\left(\frac{x}{\alpha}\right)^\beta\right)\right)^k}$$

$$\mathrm{L}(x_i;\alpha,\beta,\lambda,k) = \prod_{i=1}^{n}\frac{k\beta}{\lambda^k\alpha^\beta}x_i^{\beta-1}\left(1+\left(\frac{x_i}{\alpha}\right)^\beta\right)^{-1}\left(log\left(1+\left(\frac{x_i}{\alpha}\right)^\beta\right)\right)^{k-1}e^{-(1/\lambda)^k\left(log\left(1+\left(\frac{x_i}{\alpha}\right)^\beta\right)\right)^k} \quad (29)$$

And the log-likelihood function is given as:

$$logL(x_i;\alpha,\beta,\lambda,k) = nlog(k\beta) - nlog\left(\lambda^k\alpha^\beta\right) + (\beta-1)\sum_{i=1}^{n}log(x_i) + \sum_{i=1}^{n}log\left(1+\left(\frac{x_i}{\alpha}\right)^\beta\right)^{-1}$$

$$+(k-1)\sum_{i=1}^{n}log\left(\log(1+\left(\frac{x_i}{\alpha}\right)^\beta\right) - \left(\frac{1}{\lambda^k}\right)\sum_{i=1}^{n}\left(log\left(1+\left(\frac{x_i}{\alpha}\right)^\beta\right)\right)^k \quad (30)$$

The log-likelihood function is differentiated with respect to the unknown parameters, equated to zero and solved simultaneously. The solutions are not in closed form, so the estimates of the parameters (α,β,λ,k) must be obtained using iterative methods such as Newton-Raphson procedure.

5. Application and Discussion

In this section, the WLLE distribution is compared with other existing distributions using a real life application. In each case, the parameters are estimated by maximum likelihood method using the R3.4.4 version. First, we describe the data sets and give the MLEs (and the corresponding standard errors in parentheses) of the parameters and the values of the Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC) and Bayesian Information Criterion (BIC) statistics.

Data set: Breast Cancer data

The data represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 (Lee 1992). It was also reported and used by [18]. The observations are:

Table 2. *Breast Cancer data set.*

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0

Table 3. *Data summary for Breast Cancer.*

Statistic	Values
N	100
Minimum	0.39
1st Quarter	1.84
Median	2.7
Mean	2.621
3rd Quarter	3.22
Variance	1.027964
St Deviation	1.013885
Maximum	5.56
Skewness	0.3681541
kurtosis	3.104939

Table 4 shows the performance of the WLLE distribution with the McDonald Weibull (McW), ZBLL, Beta Log-Logistic (BLL) and Kumaraswamy Log-Logistic (KwLL) distributions. The following goodness of fit statistics (Anderson-Darling A* and Cramér-Von Mises W*) and model selection criteria: AIC (Akaike information criterion), CAIC (consistent Akaike information criterion) and BIC (Bayesian information criterion) are used.

$$\text{AIC}=-2(l)+2p,\ \text{CAIC}=-2(l)+\frac{2pn}{n-p-1}\ \text{and BIC}=-2(l)+p\log n$$

where (l) denotes the log-likelihood function evaluated at the maximum likelihood estimates, p is the number of parameters, and n is the sample size.

Table 4. *Parameter Estimates, Standard Errors (in parentheses), AIC, CAIC, BIC, A* and W*.*

Distributions	Estimates				A*	W*	AIC	CAIC	BIC
WLLE (α, β, λ, k)	29.21	0.024	0.699	76.905	0.256	0.040	1166.152	1166.573	1174.539
	(3.29)	(0.002)	(0.002)	(1.118)					
KLL (a, b, γ, a)	33.70	23.048	0.336	0.044	1.511	0.232	1189.937	1190.282	1201.120
	(4.81)	(13.979)	(0.043)	(0.020)					
BLL (a, b, α, β)	0.36	0.732	53.251	3.368	0.494	0.066	1171.861	1172.206	1183.045
	(0.23)	(0.482)	(9.731)	(1.716)					
ZBLL (a, α, β)	0.35	77.856	3.098		0.454	0.053	1167.063	1167.268	1175.450
	(0.10)	(12.562)	(0.579)						
ELL (a, α, β)	0.32	70.715	3.401		0.455	0.053	1167.341	1167.546	1175.728
	(0.09)	(10.458)	(0.658)						

6. Conclusion

A new class of distributions called the T-Log logistic {Exponential} family has been proposed in this study. Some structural properties of the Weibull Log-logistic {Exponential} distribution including linear representation of its density function have been established. Explicit expressions for the moments, model, mean deviations and quantile function have been obtained. The maximum likelihood method is employed to estimate the WLLE distribution parameters. The proposed distribution (WLLE) was fitted to a real data set using five goodness-of-fit statistics. The WLLE distribution however provides consistently better fits than the other four competing models. The WLLE distribution is therefore a competitive model and it is hoped it will attract wider applications in several fields of study such as reliability engineering, insurance, hydrology and economics.

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