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Investigating the Impact of Variable Dividends and Tail Dependence in a Compound Poisson Risk Model

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Abstract: This paper extends the compound Poisson risk model with a variable threshold dividend payment strategy and dependence between claims and inter-claim times, modeled via the Spearman copula. The objective is to establish the ultimate ruin probability in this framework. Following an introduction that motivates the study and highlights limitations of traditional risk models, the paper reviews relevant literature on risk models, dividend strategies, and copulas. Subsequently, it describes the extended model, including the dividend strategy and dependence structure. The Gerber-Shiu transform and Laplace transform of the ruin probability are then derived. Finally, the ultimate ruin probability is determined within the proposed model. Concluding remarks discuss the implications of the findings and suggest directions for future research. By considering a more realistic and comprehensive approach to financial risk modeling in insurance, this paper aims to contribute to the field of insurance risk management and provide industry professionals with improved tools for risk assessment and management.

Keywords: Gerber-Shiu Function, Copula, Integro-Differential Equation, Ruin Probability

1. Introduction

Traditionally, financial risk models in insurance assumed independence between the underlying random variables [1-3]. However, in certain pratical contexts, this assumption is inadequate and too restrictive. For instance, in flood insurance, the occurrence of multiple floods in a sort period can lead to significant damages and claim amounts due to the accumulation of water. In earthquake insurance, it is the the opposite, as in a high-risk area, the longer the time between two earthquakes, the more significant the second earthquake due to the accumulation of energy.

To remedy this insufficiency, many works integrate dependence between certain random variables, particularly claims amounts and inter-claim times, into the risk model, using Farlie-Gumbel- Morgenstern copula (see for example [4-11]). Although this copula is commonly used in the literature, it has certain limitations and cannot model tail dependencies (see [12-17]).

This work extends the compound Poisson risk model by incorporating a dependence between claim amounts and inter-claim times via the Spearman copula, addressing the limitations of the Farlie-Gumbel-Morgenstern copula.

Acknowledging the practical context of insurance companies, this work incorporates a dividend payment strategy for shareholders into the risk model. Notably, the first introduction of an insurance risk model with dividend distribution dates back to De Finetti in 1957 (see [27]). Many authors have been interested in its study (see for example [19-27].

This work analyzes a dividend payment strategy with a variable threshold barrier defined by equation $b_t = at +$

b_0 where $0 < u \leq b_0;\ 0 \leq a < c.$ The risk model with affine threshold dividend payment strategy was proposed to compensate the non-optimality of the risk model with constant threshold dividend payment strategy (see [20]). In this model, when the surplus process reaches the variable threshold barrier b_t, the premium are paid to shareholders as a constant rate $c - a$. Denoting by $U_b(t)$ the surplus process in the presence of the threshold dividend barrier b_t (with $U_b(0) = u$), the model follows the following dynamic:

$$dU_b(t) = \begin{cases} cdt - dS(t)\ si\ U_b(t) < b_t \\ adt - dS(t)\ si\ U_b(t) = b_t \end{cases} \quad (1)$$

where:

$U_b(t)$ is the surplus process in the presence of the dividend barrier of threshold b_t (with $U_b(0) = u$ the initial surplus and $0 < u \leq b$).

c is a constant rate of premium collected by the insurer per unit of time, $0 \leq a < c$.

$S(t) = \sum_{i=1}^{N(t)} X_i$ is the aggregated loss process with a compound Poisson distribution where:

$\{N(t), t \geq 0\}$ is the total number of recorded claims up to time t, following a Poisson process with intensity $\lambda > 0$. (Note that $S(t) = 0$ if $N(t) = 0$).

$\{X_i, i \geq 1\}$ is a sequence of random variables representing the individual claim amount with a common density function f and cumulative distribution function F assumed to follow an exponential distribution with parameter β.

The inter-claim times $\{V_i, i \geq 1\}$ form a sequence of random variables following an exponential distribution with parameter λ, with probability density function $k(t) = \lambda e^{-\lambda t}$ and cumulative distribution $K(t) = 1 - e^{-\lambda t}$.

This paper investigates the ultimate ruin probability within the risk model defined by equation (1). To achieve this goal, the work is structured as follows: section 2 establishes the fundamental concepts and mathematical framework associated with the model. Section 3 introduces the crucial tail dependence structure, capturing the relationship between extreme events. Section 4 derives the integro-differential equation satisfied by the Gerber-Shiu function, a key tool for analyzing ruin probabilities. Building upon this, section 5 determines both the Gerber-Shiu transform and the Laplace transform of the ultimate ruin probability, providing essential analytical insights. Finally, section 6 culminates in the derivation of the ultimate ruin probability itself, representing the central result of this investigation.

2. Preliminaries

2.1. Instant of Ruin

Let T_b be the instant of ruin of the insurance company. T_b is defined by:

$$T_b = inf\{t \geq 0, U(t) < 0\} \quad (2)$$

The situation where the ruin probability is perpetually zero leads to the conventional notation denote $T_b = \infty$. Consequently, in this setting,

$$U(t) \geq 0\ \forall\ t \geq 0.$$

2.2. Expected Discounted Penalty Function of Gerber-Shiu

The expected discounted penalty function of Gerber-Shiu, first appeared in the work of Gerber and Shiu in 1998 (see [1]). Nowadays, this function is of significant interest in research.

Its analysis remains a central question both in insurance and finance, as it is a valuable tool not only for studying the probability of ruin but also calculating retirement and reinsurance premiums, option pricing, and more.

It is defined by

$$\phi(u) = E[e^{-\delta T_b} w\left(U_{(T_b^-)}, |U_b(T_b)|\right) I(\tau < \infty) | U(0) = u] \quad (3)$$

where:

T_b is the instant of ruin defined by equation (2).

${T_b}^-$ is the instant just before ruin.

δ is a interest force.

The penalty function $w(x, y)$ is a positive function of the surplus just before ruin, $U_b({T_b}^-)$ and the deficit at ruin $|U_b(T_b)|, \forall\ x, y \geq 0$.

1_A is a indicator function, wich equal 1 if event 1_A occurs and 0 otherwise.

3. Model of Dependence Based on Spearman Copula

Copulas introduced by Abe Sklar in 1998, are an innovative and relevant tool for introducing dependence between multiples random variables. Given the marginal distribution functions of several random variables, copulas allow us to establish their joint distribution function. Nowadays, they are fundamental in modeling multivariate distributions in finance, insurance and hydrology. Key references on copulas theory include Joe [11] and Nelsen [18]. In this article, the structure of dependence is ensured by the Spearman copulas. Il is defined for all $(u, v) \in [0,1]^2$ and $\alpha \in [0,1]$ as follow:

$$C_\alpha(u, v) = (1 - \alpha) C_I(u, v) + \alpha C_M(u, v) \quad (4)$$

Where: $C_I(u, v) = uv$; $C_M(u, v) = min(u, v)$ and α is a dependency parameter.

The Spearman copulas allows for the introduction of positive dependence as well as tail dependencies in many situations. It also includes independence when $\alpha = 0$. Using formula (4), the random vectors of claim amounts and inter-claim times (X, V) has the joint distribution function given by:

$$F_{X,V}(x, t) = C_\alpha(F_X(x), F_V(t)) = (1 - \alpha) F_I(x, t) + \alpha F_M(x, t) \quad (5)$$

Where F_X and F_V are the marginal distributions of the random variables X and V, respectively.

In the risk model defined by equation (1), the Gerber Shiu function $\phi_b(u)$ takes the following form (see [12-16]):

$$\phi_b(u) = (1-\alpha)\left(I_{b,1}(u) + I_{b,2}(u)\right) + \alpha\left(I_{b,3}(u) + I_{b,4}(u)\right) \quad (6)$$

where:

$$I_{b,1}(u) = \int_0^\infty \int_0^{u+ct} e^{-\delta t}\phi_b(u+ct-x)dF_I(x,t);$$

$$I_{b,2}(u) = \int_0^\infty \int_{u+ct}^\infty e^{-\delta t}w(u+ct, x-u-ct)dF_I(x,t);$$

$$I_{b,3}(u) = \int_0^\infty \int_0^{u+ct} e^{-\delta t}\phi_b(u+ct-x)dF_M(x,t);$$

$$I_{b,4}(u) = \int_0^\infty \int_{u+ct}^\infty e^{-\delta t}w(u+ct, x-u-ct)dF_M(x,t)$$

4. Integro-Differential Equation Satisfied by the Gerber-Shiu Function $\phi_b(u)$

To obtain integro-differential equation satisfied by Gerber-Shiu function $\phi_b(u)$ in the risk model defined by equation (1), the following approach was adopted:

1) The first claim occurs at time t befor the surplus process reaches the barrier b_t (ie $t < \frac{b_0-u}{c-a}$). Its amount x satisfies $x < u+ct$.
2) The first claim occurs at time t befor the surplus process reaches the barrier b_t (ie $t < \frac{b_0-u}{c-a}$). Its amount x satisfies $x > u+ct$.
3) The first claim occurs at time t after the surplus process reaches the barrier b_t (ie $t > \frac{b_0-u}{c-a}$). Its amount x satisfies $x < at+b_0$.
4) The first claim occurs at time t after the surplus process reaches the barrier b_t (ie $t > \frac{b_0-u}{c-a}$). Its amount x satisfies $x > at+b_0$.

By conditioning on the time and the amount of the first claim and considering the different scenarios mentioned above, the integrals $I_{b,1}(u)$ and $I_{b,2}(u)$ in relation (6) become:

$$I_{b,1}(u) = \int_0^{\frac{b_0-u}{c-a}} \int_0^{u+ct} e^{-\delta t}\phi_b(u+ct-x)dF_I(x,t) + \int_{\frac{b_0-u}{c-a}}^\infty \int_0^{at+b_0} e^{-\delta t}\phi_b(at+b_0-x)dF_I(x,t).$$

$$I_{b,2}(u) = \int_0^{\frac{b_0-u}{c-a}} \int_{u+ct}^\infty e^{-\delta t}w(u+ct, x-u-ct)dF_I(x,t) + \int_{\frac{b_0-u}{c-a}}^\infty \int_{at+b_0}^\infty e^{-\delta t}w(at+b_0, x-at-b_0)dF_I(x,t).$$

By defining $I_b(u) = I_{b,1}(u) + I_{b,2}(u)$, it can be shown that:

$$I_b(u) = \int_0^{\frac{b_0-u}{c-a}} \int_0^{u+ct} e^{-\delta t}\phi_b(u+ct-x)dF_I(x,t) + \int_{\frac{b_0-u}{c-a}}^\infty \int_0^{at+b_0} e^{-\delta t}\phi_b(at+b_0-x)dF_I(x,t) + \int_0^{\frac{b_0-u}{c-a}} \int_{u+ct}^\infty e^{-\delta t}w(u+ct, x-u-ct)dF_I(x,t) + \int_{\frac{b_0-u}{c-a}}^\infty \int_{at+b_0}^\infty e^{-\delta t}w(at+b_0, x-at-b_0)dF_I(x,t)$$

$$I_b(u) = \lambda\int_0^{\frac{b_0-u}{c-a}} \int_0^{u+ct} e^{-(\delta+\lambda)t}\phi_b(u+ct-x)f_X(x)dxdt + \lambda\int_{\frac{b_0-u}{c-a}}^\infty \int_0^{at+b_0} e^{-(\delta+\lambda)t}\phi_b(at+b_0-x)f_X(x)dxdt + \lambda\int_0^{\frac{b_0-u}{c-a}} \int_{u+ct}^\infty e^{-(\delta+\lambda)t}w(u+ct, x-u-ct)f_X(x)dxdt + \lambda\int_{\frac{b_0-u}{c-a}}^\infty \int_{at+b_0}^\infty e^{-(\delta+\lambda)t}w(at+b_0, x-at-b_0)f_X(x)dxdt \quad (7)$$

To streamline the notation of equation (7), the following definitions are introduced:

$$\omega(u) = \int_u^\infty w(u, x-u)f(x)dx;\ \sigma_b(u) = \int_0^u \phi_b(u-x)f(x)dx + \omega(u) \quad (8)$$

Equation (7) becomes:

$$I_b(u) = \lambda\int_0^{\frac{b_0-u}{c-a}} e^{-(\delta+\lambda)t}\sigma_b(u+ct)dt + \lambda\int_{\frac{b_0-u}{c-a}}^\infty e^{-(\delta+\lambda)t}\sigma_b(at+b_0)dt \quad (9)$$

The relation (9) can be put in the form:

$$I_b(u) = \lambda\int_0^\infty e^{-(\delta+\lambda)t}\sigma_b\left((u+ct)\wedge(at+b_0)\right)dt \quad (10)$$

where: $u \wedge v = \min(u,v)$

Let us now determine the integrales $I_{b,3}(u)$ and $I_{b,4}(u)$ in relation (6).

The support of copula C_M is $D = \{(u,v) \in [0,1]^2 : u = v\}$.

On the domain $[0,1]^2 \setminus D$, $\frac{\partial^2 C_M}{\partial u \partial v} = 0$ and on D, C_M follows an uniform distribution.

Since the structure of dependence between the claim amounts and inter-claim times is described by the copula C_M, these are monotonous and there almost surely exists an increasing function l, such that $X = l(V)$ (see Nelsen [18, P.27]. The following result is established (see [12-15]):

$$l(t) = \frac{\lambda}{\beta}t \quad (11)$$

The joint distribution $F_{X,V}(x,t)$ of the random vector (X,V) is singular, and its support is the domain $D' = \{(x,t): F_X(x) = F_V(t)\} = \{(x,t): x = l(t)\}$.

Its distribution is $G(t) = F_M(l(t),t) = 1 - e^{-\lambda t}$ on the domaine $D' = \left\{(x,t): x = \frac{\lambda}{\beta}t\right\}$.

The integral $I_{b,3}(u)$ becomes:

$$I_{b,3}(u) = \int_0^{\frac{b_0-u}{c-a}} \int_0^{u+ct} e^{-\delta t}\phi_b(u+ct-x)\, dF_M(x;t) + \int_{\frac{b_0-u}{c-a}}^{\infty} \int_0^{b} e^{-\delta t}\phi_b(at+b_0-x)\, dF_M(x;t)$$

$$= \int_K e^{-\delta t}\phi_b(u+ct-x)dG(t) + \int_J e^{-\delta t}\phi_b(at+b_0-x)dG(t) \quad (12)$$

where:

$$K = \left\{t \in \mathbb{R}^+ : 0 \le t \le \frac{b_0-u}{c-a}\ and\ 0 \le x = \frac{\lambda}{\beta}t \le u+ct\right\}$$

$$= \left\{t \in \mathbb{R}^+ : 0 \le t \le \frac{b_0-u}{c-a}\ and\ \left(\frac{\lambda}{\beta} - c\right)t \le u\right\}$$

$$= \left\{t \in \mathbb{R}^+ : 0 \le t \le \frac{b_0-u}{c-a}\ and\ t \in \mathbb{R}^+\right\}$$

Because $c > \frac{\lambda}{\beta}\ and\ u > 0$ (solvency condition: $\mathbb{E}[cV - X] > 0$) and $u \ge 0$.

Therefore:

$$K = \left[0;\ \frac{b_0-u}{c-a}\right] \quad (13)$$

$$J = \left\{t \in \mathbb{R}^+ : t \ge \frac{b_0-u}{c-a}\ and\ x = \frac{\lambda}{\beta}t \le at + b_0\right\}$$

$$= \left\{t :\in \mathbb{R}^+ : t \ge \frac{b_0-u}{c-a}\ and \left(\frac{\lambda}{\beta} - a\right)t \le b_0\right\}$$

$$\frac{\lambda}{\beta} - a < 0; u > 0\ and\ t \ge 0\ \Longrightarrow \left\{t \in \mathbb{R}^+ : \left(\frac{\lambda}{\beta} - a\right)t \le b_0\right\} = \mathbb{R}_+$$

Therefore:

$$J = \left[\frac{b_0-u}{c-a}; +\infty\right[\quad (14)$$

Using relations (13) and (14), the integral $I_{b,3}(u)$ can be written as:

$$I_{b,3}(u) = \int_0^{\frac{b_0-u}{c-a}} e^{-\delta t}\phi_b(u+ct-x)\, dG(t) + \int_{\frac{b_0-u}{c-a}}^{\infty} e^{-\delta t}\phi_b(at+b_0-x)\, dG(t)$$

$$= \lambda \int_0^{\frac{b_0-u}{c-a}} e^{-(\delta+\lambda)t}\phi_b\left(u+ct-\frac{\lambda}{\beta}t\right) dt + \lambda \int_{\frac{b_0-u}{c-a}}^{\infty} e^{-(\delta+\lambda)t}\phi_b\left(at+b_0-\frac{\lambda}{\beta}t\right) dt \quad (15)$$

The analogous case yields:

$$I_{b,4}(u) = \int_0^{\frac{b_0-u}{c-a}} \int_{u+ct}^{\infty} e^{-\delta t} w(u+ct, x-u-ct)\, F_M(x;t) + \int_{\frac{b_0-u}{c-a}}^{\infty} \int_{at+b_0}^{\infty} e^{-\delta t} w(at+b_0, x-at-b_0)\, F_M(x;t)$$

$$= \int_{K'} e^{-\delta t} e^{-\delta t} w(u+ct, x-u-ct)dG(t) + \int_{J'} e^{-\delta t} w(at+b_0, x-at-b_0)dG(t) \quad (16)$$

where:

$$K' = \left\{t \in \mathbb{R}^+ : t \le \frac{b_0-u}{c-a}\ and\ 0 \le x = \frac{\lambda}{\beta}t \ge u+ct\right\}$$

$$= \left\{t \in \mathbb{R}^+ : t \le \frac{b_0-u}{c-a}\ and\ \left(\frac{\lambda}{\beta} - c\right)t \ge u\right\}$$

But: $\left\{t \in \mathbb{R}^+\ and\ \left(\frac{\lambda}{\beta} - c\right)t \ge u\right\} = \emptyset$

Therefore:

$$K' = \emptyset \quad (17)$$

$$J' = \left\{t \in \mathbb{R}^+ : t \ge \frac{b_0-u}{c-a}\ and\ x = \frac{\lambda}{\beta}t \ge at + b_0\right\}$$

$$= \left\{t :\in \mathbb{R}^+ : t \ge \frac{b_0-u}{c-a}\ and\ \left(\frac{\lambda}{\beta} - a\right)t \ge b_0\right\}$$

The formula $\left\{t \in \mathbb{R}^+ \; and \; \left(\frac{\lambda}{\beta} - a\right) t \geq b_0\right\} = \emptyset$ is obtained.
Therefore:

$$J' = \emptyset \tag{18}$$

Substituting relations (17) and (18) into equation (16) yields:

$$I_{b,4}(u) = 0 \tag{19}$$

Assume $I_b^\star(u) = I_{b,3}(u) + I_{b,4}(u)$. Applying relations (15) and (19) yields:

$$I_b^\star(u) = \lambda \int_0^{\frac{b_0-u}{c-a}} e^{-(\delta+\lambda)t} \phi_b\left(u + ct - \frac{\lambda}{\beta} t\right) dt + \lambda \int_{\frac{b_0-u}{c-a}}^{\infty} e^{-(\delta+\lambda)t} \phi_b\left(at + b_0 - \frac{\lambda}{\beta} t\right) dt \tag{20}$$

The relation (20) can be put in the form:

$$I_b^\star(u) = \lambda \int_0^\infty e^{-(\delta+\lambda)t} \phi_b\left(\left(u + ct - \frac{\lambda}{\beta} t\right) \wedge \left(at + b_0 - \frac{\lambda}{\beta} t\right)\right) dt \tag{21}$$

From the relations (6), (10) and (21), the Gerber-Shiu function $\phi_b(u)$ can be put into the form:

$$\phi_b(u) = \lambda(1-\alpha) \int_0^\infty e^{-(\delta+\lambda)t} \sigma_b\big((u+ct) \wedge (at+b_0)\big) dt \; + \alpha\lambda \int_0^\infty e^{-(\delta+\lambda)t} \phi_b\left(\left(u + ct - \frac{\lambda}{\beta} t\right) \wedge \left(at + b_0 - \frac{\lambda}{\beta} t\right)\right) dt \tag{22}$$

By setting $s = u + ct; \; s = u + ct - \frac{\lambda}{\beta} t$ in the relation (22), it can be shown that:

$$\phi_b(u) = \frac{\lambda}{c}(1-\alpha) \int_u^\infty e^{-\left(\frac{\delta+\lambda}{c}\right)(s-u)} \sigma_b\left(s \wedge \left(a\frac{s-u}{c} + b_0\right)\right) ds$$

$$+ \frac{\alpha\beta\lambda}{\beta c - \lambda} \int_u^\infty e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)} \phi_b\left(s \wedge \left(\beta\left(\frac{a - \frac{\lambda}{\beta}}{\beta c - \lambda}\right)(s-u)\right) + b_0\right) ds \tag{23}$$

Theorem 4.1: *The Gerber-Shiu function $\phi_b(u)$ satisfies the following integro-differential equation::*

$$\left(\mathcal{D} - \frac{\beta(\delta+\lambda)}{\beta c - \lambda}\ell\right)\left(\mathcal{D} - \frac{\delta+\lambda}{c}\ell\right)\phi_b(u) = \left(\frac{\beta\lambda(\delta+\lambda)(1-\alpha)}{c(\beta c-\lambda)}\ell - \frac{\lambda}{c}(1-\alpha)\mathcal{D}\right)\sigma_b(u) + \left(\frac{\alpha\beta\lambda(\delta+\lambda)}{c(\beta c-\lambda)}\ell - \frac{\alpha\beta\lambda}{\beta c-\lambda}\mathcal{D}\right)\phi_b(u) \tag{24}$$

where: $\mathcal{D}$ and ℓ are the differentiation and identity operators respectively.
Proof of theorem 4.1:
Let us derive the function $\phi_b(u)$ in the relation (23) with respect to u.

$$\phi'_b(u) = \frac{\lambda}{c}(1-\alpha)\left(\frac{\delta+\lambda}{c}\right) \int_u^\infty e^{-\left(\frac{\delta+\lambda}{c}\right)(s-u)} \sigma_b\left(s \wedge \left(a\frac{s-u}{c} + b_0\right)\right) ds$$

$$+ \frac{\alpha\beta^2\lambda(\delta+\lambda)}{(\beta c-\lambda)^2} \int_u^\infty e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)} \phi_b\left(s \wedge \left(\beta\left(\frac{a-\frac{\lambda}{\beta}}{\beta c-\lambda}\right)(s-u)\right) + b_0\right) ds - \frac{\lambda}{c}(1-\alpha)\sigma_b(u) - \frac{\alpha\beta\lambda}{\beta c-\lambda}\phi_b(u) \tag{25}$$

Using the differentiation and identity operators $\mathcal{D}$ and ℓ, let us calculate $g(u) = \left(\mathcal{D} - \frac{\delta+\lambda}{c}\ell\right)\phi_b(u)$.

$$g(u) = -\frac{\lambda}{c}(1-\alpha)\sigma_b(u) - \frac{\alpha\beta\lambda}{\beta c-\lambda}\phi_b(u) + \frac{\alpha\beta\lambda}{\beta c-\lambda}\left(\frac{\beta}{\beta c-\lambda} - \frac{1}{c}\right) \times \int_u^\infty e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)} \phi_b\left(s \wedge \left(\beta\left(\frac{a-\frac{\lambda}{\beta}}{\beta c-\lambda}\right)(s-u)\right) + b_0\right) ds \tag{26}$$

Let us derive the function $g(u)$ in the relation (26) with respect to u.

$$g'(u) = -\frac{\lambda}{c}(1-\alpha)\sigma'_b(u) - \frac{\alpha\beta\lambda}{\beta c-\lambda}\phi'_b(u) + \frac{\alpha\beta^2\lambda}{\beta c-\lambda}\left(\frac{\beta}{\beta c-\lambda} - \frac{1}{c}\right)\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)$$

$$\times \int_u^\infty e^{-\beta\left(\frac{\delta+\lambda}{\beta c-\lambda}\right)(s-u)} \phi_b\left(s \wedge \left(\beta\left(\frac{a-\frac{\lambda}{\beta}}{\beta c-\lambda}\right)(s-u)\right)+b_0\right)ds - \frac{\alpha\beta\lambda(\delta+\lambda)}{\beta c-\lambda}\left(\frac{\beta}{\beta c-\lambda}-\frac{1}{c}\right)\phi_b(u) \tag{27}$$

Using the differentiation and identity operators $\mathcal{D}$ and ℓ let us calculate $h(u)=\left(\mathcal{D}-\frac{\beta(\delta+\lambda)}{\beta c-\lambda}\ell\right)g(u)$.

$$h(u)=\frac{\beta\lambda(\delta+\lambda)}{c(\beta c-\lambda)}(1-\alpha)\sigma_b(u)-\frac{\lambda}{c}(1-\alpha)\sigma'_b(u)-\frac{\alpha\beta\lambda}{\beta c-\lambda}\phi'_b(u)+\frac{\alpha\beta\lambda(\delta+\lambda)}{c(\beta c-\lambda)}\phi_b(u) \tag{28}$$

From the relations (26) and (28), the equation (24) can be deduced.

5. Laplace Transforms of Gerber-Shiu and Ultimate Ruin Probability

Lemma 5.1: *The Laplace transform of Gerber Shiu $\phi_b(u)$ is given by:*

$$\hat{\phi}_b(s)=\frac{L_{1,b}(s)+L_{2,b}(s)}{D_{1,b}(s)+D_{2,b}(s)} \tag{29}$$

where:

$$L_{1,b}(s)=\left(\frac{\beta\lambda(1-\alpha)(\delta+\lambda)}{c(\beta c-\lambda)}-\frac{\lambda(1-\alpha)}{c}s\right)\widehat{\omega}(s)+\frac{\lambda(1-\alpha)}{c}\omega(0)$$

$$L_{2,b}(s)=\left(s+\frac{\alpha\beta\lambda c-(\delta+\lambda)(2\beta c-\lambda)}{c(\beta c-\lambda)}\right)\phi_b(0)+\phi'_b(0)$$

$$D_{1,b}(s)=s^2+\frac{\alpha\beta\lambda c-(\delta+\lambda)(2\beta c-\lambda)}{c(\beta c-\lambda)}s+\frac{\beta(\delta+\lambda)^2-\alpha\beta\lambda(\delta+\lambda)}{c(\beta c-\lambda)}$$

$$D_{2,b}(s)=\frac{\beta\lambda(1-\alpha)(\beta c-\lambda)s}{c(\beta+s)(\beta c-\lambda)}-\frac{\beta^2\lambda(1-\alpha)(\delta+\lambda)}{c(\beta+s)(\beta c-\lambda)}$$

Proof of the lemma 5.1:
Let's put:

$$\gamma_1(u)=\left(\mathcal{D}-\frac{\beta(\delta+\lambda)}{\beta c-\lambda}\ell\right)\left(\mathcal{D}-\frac{\delta+\lambda}{c}\ell\right)\phi_b(u)$$

$$\gamma_2(u)=\left(\frac{\beta\lambda(1-\alpha)(\delta+\lambda)}{c(\beta c-\lambda)}\ell-\frac{\lambda(1-\alpha)}{c}\mathcal{D}\right)\sigma_b(u)+\left(\frac{\alpha\beta\lambda(\delta+\lambda)}{c(\beta c-\lambda)}\ell-\frac{\alpha\beta\lambda}{\beta c-\lambda}\mathcal{D}\right)\phi_b(u)$$

By taking the Laplace transform of both sides of the equation (7), it can be shown that:

$$\int_0^\infty e^{-su}\gamma_1(u)du=\left(s^2-\frac{(\delta+\lambda)(2\beta c-\lambda)}{c(\beta c-\lambda)}s+\frac{\beta(\delta+\lambda)^2}{c(\beta c-\lambda)}\right)\hat{\phi}_b(s)+\left(\frac{(\delta+\lambda)(2\beta c-\lambda)}{c(\beta c-\lambda)}-s\right)\phi_b(0)-\phi'_b(0) \tag{30}$$

$$\begin{aligned}\int_0^\infty e^{-su}\gamma_2(u)du=&\frac{\alpha\beta\lambda(\delta+\lambda)}{c(\beta c-\lambda)}\hat{\phi}_b(s)-s\frac{\alpha\beta\lambda}{\beta c-\lambda}\hat{\phi}_b(s)+\frac{\alpha\beta\lambda}{\beta c-\lambda}\phi_b(0)+\left(\frac{\beta\lambda(1-\alpha)(\delta+\lambda)}{c(\beta c-\lambda)}-\frac{\lambda(1-\alpha)}{c}s\right)\left(\frac{\beta}{\beta+s}\right)\hat{\phi}_b(s)\\&+\left(\frac{\beta\lambda(1-\alpha)(\delta+\lambda)}{c(\beta c-\lambda)}-\frac{\lambda(1-\alpha)}{c}s\right)\widehat{\omega}(s)+\frac{\lambda(1-\alpha)}{c}\omega(0)\end{aligned} \tag{31}$$

From the relations (30) and (31), the relation (29) can be deduced.
Theorem 5.1: *The Laplace transform of the ultimate ruin probability is given by:*

$$\hat{\psi}_b(s)=\frac{L_{3,b}(s)+L_{4,b}(s)}{D_{3,b}(s)+D_{4,b}(s)} \tag{32}$$

where:

$$L_{3,b}(s)=\frac{\beta\lambda^2(1-\alpha)}{c(\beta c-\lambda)(\beta+s)}-\frac{\lambda(1-\alpha)}{c(\beta+s)}s+\frac{\lambda(1-\alpha)}{c}\omega(0)$$

$$L_{4,b}(s)=\left(s+\frac{\alpha\beta\lambda c-\lambda(2\beta c-\lambda)}{c(\beta c-\lambda)}\right)\phi_b(0)+\phi'_b(0)$$

$$D_{3,b}(s)=s^2+\frac{\alpha\beta\lambda c-\lambda(2\beta c-\lambda)}{c(\beta c-\lambda)}s+\frac{\beta\lambda^2(1-\alpha)}{c(\beta c-\lambda)}$$

$$D_{4,b}(s) = \frac{\beta\lambda(1-\alpha)(\beta c-\lambda)s}{c(\beta+s)(\beta c-\lambda)} - \frac{(\beta\lambda)^2(1-\alpha)}{c(\beta+s)(\beta c-\lambda)}$$

Proof of the theorem 5.1:

By setting $w(x,y) = 1$ and $\delta = 0$ in the equation (29), $\widehat{\omega}(s) = \frac{1}{\beta+s}$ can be obtained and the relation (32) can be deduced.

6. Ultimate Ruin Probability

Lemme 6.1: *The Laplace transform of ultimate ruin probability can be put in the form:*

$$\widehat{\psi}_b(s) = \frac{C_1 s^2 + C_2 s + C_3}{sD(s)} \tag{33}$$

where:

$$C_1 = c(\beta c - \lambda)\phi_b(0)$$

$$C_2 = \lambda(1-\alpha)(\beta c-\lambda)(\omega(0)-1) + (\alpha\beta\lambda c - \beta\lambda c + (\beta c-\lambda)^2)\phi_b(0) + c(\beta c-\lambda)\phi'_b(0)$$

$$C_3 = \beta\lambda(1-\alpha)(\beta c-\lambda)\omega(0) + \left(\alpha\beta^2\lambda c - \beta\lambda(2\beta c-\lambda)\right)\phi_b(0) + \beta c(\beta c-\lambda)\phi'_b(0)$$

$$D(s) = cs^2(\beta c-\lambda) + s(\alpha\beta\lambda c - \beta\lambda c + (\beta c-\lambda)^2) - \beta\lambda(\beta c-\lambda)$$

Proof of the lemma 6.1: By multiplying the numerator and the denominator of the relation (10) by $c(\beta+s)(\beta c-\lambda)$, the desired result is obtained after simplification.

Theorem 6.1: The *ultimate ruin probability in the risk model defined by relation (1) has the explicit expression given by*:

$$\psi_b(u) = \frac{\lambda(1-\alpha)(\beta c-\lambda)}{cR_2+\beta c} e^{R_1 u} \tag{34}$$

where:

$$R_1 = \frac{\beta c\lambda - \alpha\beta\lambda c - (\beta c-\lambda)^2 - \sqrt{(\alpha\beta\lambda c - \beta\lambda c + (\beta c-\lambda)^2)^2 + 4\beta\lambda c(\beta c-\lambda)^2}}{2c(\beta c-\lambda)} < 0 \tag{35}$$

$$R_2 = \frac{\beta c\lambda - \alpha\beta\lambda c - (\beta c-\lambda)^2 + \sqrt{(\alpha\beta\lambda c - \beta\lambda c + (\beta c-\lambda)^2)^2 + 4\beta\lambda c(\beta c-\lambda)^2}}{2c(\beta c-\lambda)} > 0 \tag{36}$$

Proof of the the theorem 6.1:

The polynomial $D(s)$ in the relation (33) is clearly a polynomial of degree 2 in s of discriminant $\Delta = (\alpha\beta\lambda c - \beta\lambda c + (\beta c-\lambda)^2)^2 + 4\beta\lambda c(\beta c-\lambda)^2 > 0$ and poles R_1 and R_2 given by relations (35) and (36).

The denominator of relation (33) is clearly a polynomial of degree 3 in s while its numerator is a polynomial of degree 2. By decomposition into simple elements, the Laplace transform of the ultimate ruin probability in the relation (33) can therefore be put in the form:

$$\widehat{\psi}_b(s) = \frac{c(\zeta_1+\zeta_2+\zeta_3)s^2 + c(-\zeta_1 R_1 - \zeta_1 R_2 - \zeta_2 R_2 - \zeta_3 R_1)s + \zeta_1 c R_1 R_2}{cs(s-R_1)(s-R_2)} \tag{37}$$

where: $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{R}$; R_1 and R_2 are given by the relations (35) and (36).

The analysis of relations (33) and (36) leads to:

$$\zeta_1 = \frac{C_3}{cR_1R_2} \tag{38}$$

$$\zeta_3 = \frac{c\zeta_1 R_1 + C_1 R_2 + C_2}{c(R_2-R_1)} \tag{39}$$

$$\zeta_2 = \frac{C_1}{c} - \zeta_1 - \zeta_3 \tag{40}$$

where: C_1, C_2 and C_3 are given by the relation (33).

By the inverse Laplace transform, the ultimate ruin probabilty can therefore be put in the form:

$$\psi_b(u) = \zeta_1 + \zeta_2 e^{R_1 u} + \zeta_3 e^{R_2 u} \tag{41}$$

where: ζ_1, ζ_2 and ζ_3 are respectively given by the relations (38), (39) and (40).

Given that $\lim_{u\to+\infty}\psi_b(u) = 0$ (natural condition), it can be deduced that:

$$\zeta_3 = 0 \tag{42}$$

$$\zeta_1 = 0 \tag{43}$$

$$\zeta_2 = (\beta c-\lambda)\phi_b(0) \tag{44}$$

$$\phi_b(0) = \frac{\lambda(1-\alpha)}{\beta c + cR_2} \tag{45}$$

By injecting the relations (42); (43); (44) and (45) in the relation (41), the desired result can be obtained.

Graphic illustration

Example 6.1. By fixing the parameters $c = 1$; $\lambda = 0.5$; $\beta = 0.7$; $b_0 = 10$ and using MATLAB, curves illustrating the probability of failure for various values of the dependent parameter α. are presented.

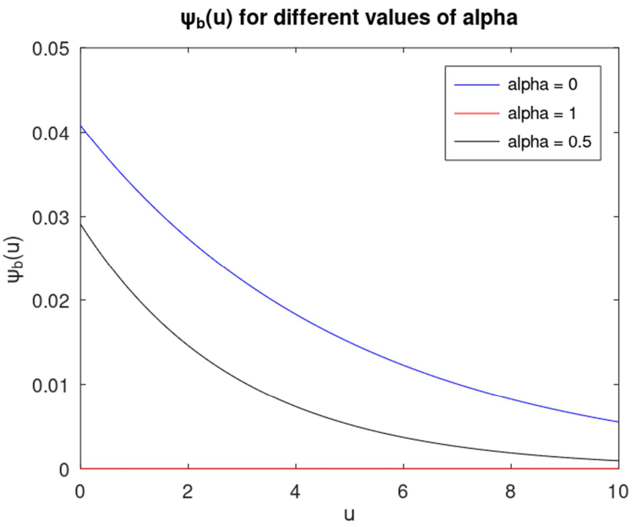

Figure 1. *Curve of* $\psi_b(u)$ *as a function of* u *and dependent parameter* α.

Remark: Analysis reveals that the ultimate ruin probability ψ_b is a decreasing function of u and the dependent parameter α.

7. Conclusion

This work focuses on the ultimate ruin probability in a compound Poisson risk model with variable threshold dividends and Spearman copula-dependent claims.

The remainder of this work will be devoted to the applications of the results obtained in insurance companies.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] H. U. Gerber and E. S. W. Shiu, On the time value of ruin, North American Actuarial Journal (1998), 48-78.

[2] H. U. Gerber, An extension of the renewal equation and its application in the collective theory of risk, Skandinavissk Actuarietidskrift (1970) 205-210.

[3] Grandell, J. (1991) Aspects of risk theory. Springer Series in statistics: Probability and its applications. Springer-Verlag, New York.

[4] S. Asmussen, Stationary distributions for fluid flow models with or without Brownian noise, Communications in Statististics-stochastics Model 11(1995) 21-49.

[5] Cosette, H., Marceau, E., and Marri, F., 2010, analysis of ruin measure for the classical compound Poisson risk model with dependence. Scand. Actuar. J. 3, 221-245.

[6] Hürlimann, W., 2004. a. Multivariate Frechet copulas and conditional value- at- risque. Ind. J. Math. Sci. 7, 345-364.

[7] M. Boudreault, «Modeling and pricing earthquake risk,», scor Canada Actuarial Price, 2003.

[8] M. Boudreault, H. Cosette, D. Landriault and E. Marceau, «On a risk model with dependence between interclaim arrivals and claim sizes,» Scandinavian Actuarial Journal, vol. 5, pp. 301-323, 2006.

[9] H. Albrecher, O. J. Boxma, A ruin model with dependance between claim sizes ans claim intervals, Insurance: Mathematics and Economics 35(2004) 245-254.

[10] H. Albrecher, J. Teugels, Exponential behavior in the presence of dependance in risk theory, Journal and Applied Probability 43 (1) (2006) 265-285.

[11] H. Joe, Multivariate Models and dependence concepts, Chapman and Hall/CRC, 1997.

[12] S. Heilpern, 2014,. ruin measures for a compound Poisson risk model with dependence based on the Spearman copula and the exponential claim sizes. Insurance: Mathematics and economic 59 251-257.

[13] Delwendé Abdoul-Kabir Kafando, Victorien Konané, Frédéric Béré and Pierre Clovis Nitiéma, Extension of the Sparre Andersen via the Spearman copula, Advances and Applications in statistics 86(1) (2023), 79-100.

[14] Kiswendsida Mahamoudou OUEDRAOGO, Francois Xavier OUEDRAOGO, Delwendé Abdoul-Kabir KAFANDO and Pierre Clovis NITIEMA. On compound risk model with partial premium payment strategy to shareholders and dependence between claim amount and inter-claim times through the Spearman copula, Advances and applications in statistics 89(2) (2023), 175-188.

[15] Kiswendsida Mahamoudou OUEDRAOGO, Delwendé Abdoul-Kabir KAFANDO, Francois Xavier OUEDRAOGO, Lassané SAWADOGO and Pierre Clovis NITIEMA, an integro-differential equation in compound Poisson risk model with variable threshold dividend payment strategy to shareholders and tail dependence between claims amounts and inter-claim time, Advances in differential equations and control processes 30(4) 2023, 413-429.

[16] Kiswendsida Mahamoudou OUEDRAOGO, Delwendé Abdoul-Kabir KAFANDO, Lassané SAWADOGO, Francois Xavier OUEDRAOGO and Pierre Clovis NITIEMA, Laplace transform for the compound risk model with a strategy of partial payment of premium to shareholders and dependence between claims amounts and inter-claim time using the Spearman copula, Far east journal of theoretical statistics 68(1) (2024), 23-39.

[17] Delwendé Abdoul-Kabir KAFANDO, Frédéric BERE, Victorien KONANE and Pierre Clovis NITIEMA, Extension of the Compound Poisson model via the Spearman copula, Far East journal of theoretical Statistics 67(2) (2023), 147-184.

[18] Nelsen, R. B., 2006,. An introduction to copula, second edition: Springer Series in statistic, Springer-Verlag, New York.

[19] Cossette, H.; Marceau, E., F. On a compound poisson risk model with dependence and in a presence of a constant dividend barrier. Appl. Stoch. Models Bus. Ind. 2014, 30, 82-98.

[20] Albrecher, H.; Hartinger, J. On the non-optimality of horizontal barrier strategies in the Sparre Andersen model. HERMES International Journal of computer Mathematics and its applications 7(2006), 1-14.

[21] A. K. Nikoloulopoulos and D. Karlis, «Fiiting copulas to bivariate earthquake data: the seismic gap hypothesis revisited», Environmetrics, vol. 19, no. 3, PP. 251-269, 2008.

[22] D Landriault, «Constant dividend barrier in a risk model with interclaim-dependent claim sizes,» Insurance: Mathematics and economics, vol. 42, no. 1, pp. 31-38, 2008.

[23] K. C. Yue, G. Wang, and W. K. Li, «The Gerber Shiu expected discounted penalty function for risk process with interest and a constant dividend barrier,» Insurance: Mathematics and economics, vol. 40, no. 1, pp. 104-112, 2007.

[24] X. S. Lin, G. E. Wilmot, and S. Drekic, « The classical risk model with a constant dividend barrier: analysis of the Gerber Shiu discounted penalty function, » Insurance: Mathematics and economics, vol. 33, no. 3, pp. 551-556, 2003.

[25] H. U. Gerber. On the probability of ruin in the presence of a linear dividend barrier. Scandinavian Actuarial Journal, pages 105-115, 1981.

[26] D. Dickson (2004). Dividends. In: Encyclopedia of actuarial science (Eds. J. L. Teugels and B Sundt) Wiley, to appear.

[27] B De Finetti (1957). Su un' imposttazione alternativa dell teoria colletiva del rischio. Transactions of the XVth International Congress of Actuaries 2, 433-443.