

On a Reaction-Diffusion Model of COVID-19

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Abstract: Nowadays mathematical models play a major role in epidemiology since they can help in predicting the spreading and the evolution of diseases. Many of them are based on ODEs on the assumption that the populations being studied are homogenous sets of fixed points (individuals) but actually populations are far from being homogenous and people are constantly moving. In fact, thanks to science progresses, distances are no longer what they used to be in the past and a disease can travel and reach out even the most remote places on the globe in a matter of hours. HIV and Covid-19 outbreaks are perfect illustrations of how far and fast a disease can now spread. When it comes to studying the spatio-temporal spreading of a disease, instead of ODEs dynamic models the Reaction-Diffusion ones are best suited. They are inspired by the second Fick's law in physics and are getting more and more used. In this article we make a study of the spatio-temporal spreading of the COVID-19. We first present our SEIR dynamic model, we find the two equilibrium points and an expression for the basic reproduction number (\mathcal{R}_0), we use the additive compound matrices and show that only one condition is necessary to show the local stability of the two equilibrium points instead of two like it is traditionally done, and we study the conditions for the DFE (Disease Free Equilibrium point) and the EE (Endemic Equilibrium point) to be globally asymptotically stable. Then we construct a diffusive model from our previous SEIR model, we investigate on the existence of a traveling wave connecting the two equilibrium thanks to the monotone iterative method and we give an expression for the minimal wave speed. Then in the last section we use the additive compound matrices to show that the DFE remains stable when diffusion is added whereas there will be appearance of Turing instability for the EE once diffusion is added. The conclusion of our article emphasizes the importance of barrier gestures and the fact that the more people are getting tested the better governments will be able to handle and tackle the spreading of the disease.

Keywords: Reaction-Diffusion, COVID-19, Traveling Wave, Upper-solution, Lower-solution, Turing Instability

1. Introduction

The COVID-19 was declared a pandemic by the WHO on the 30th January 2020. The responsible agent is a coronavirus (SARS-Cov2) that spreads between people thanks to close contacts, usually via droplets produced by coughing, sneezing or talking. The droplets usually fall onto surfaces or to the ground rather than remaining in the air making it also possible for people to be infected by touching a contaminated surface or any contaminated object. According to the updated information available [27], incubation period ranges from 2 to 14 days and the main symptoms are fever, loss of appetite, shortness of breath, cough, fatigue, muscle aches and pain.

The majority of the infected individuals are asymptomatic and tend not to be tested though they do play a role in the spreading of the disease. The recovery time which usually ranges from 2 to 6 weeks differs from person to person and it happens that even after that period some people still complain to not be fully recovered.

Depending on the main purpose, dynamic models usually try to encapsulate as much as possible important features of the disease in the simplest way [3-6, 15-19] to provide a comprehensive view on the disease dynamic. That is why the majority of the current models on COVID-19 are very detailed in classes. For instance a quarantined class and/or

an hospitalization class are often taken into account leading to models with 5 to 8 classes [20-26]. Knowing how challenging it is to find front traveling waves for R-D models of three or more than three equations, we have chosen to build a simpler model with only four classes (the susceptibles, the asymptomatic infected individuals, the symptomatic infected individuals and the removed). other approaches are regularly

used to investigate on the existence existence of a traveling wave [5, 7] but here we use the monotone iterative method by setting up a pair of ordered super-solutions. We consider that no major action is taken to stop the spreading, therefore we have no quarantine and people are still free to move. The interactions between the four classes are given in the Figure 1

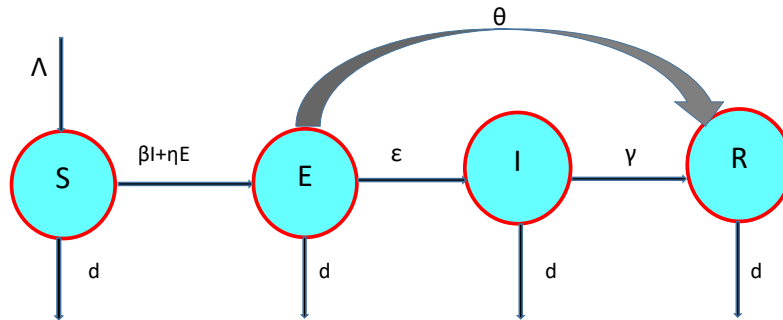


Figure 1. Dynamic graph of COVID-19 transmission.

and the assumptions we make are the following:

1. Every new-born is susceptible i.e there are only horizontal transmission;
2. An asymptomatic infected individual is an infectious person presenting no or very few symptoms;
3. A symptomatic infected individual is an infectious person presenting symptoms of COVID-19;
4. Every contact with an infectious person does not always lead to a transmission of SARS-Cov2;
5. After an infectious contact there is always an incubating period but we do not take it into account here;
6. After a susceptible has been infected by either an asymptomatic infected individual or a symptomatic infected individual, he/she will go through an asymptomatic state he can remain into until he/she is totally healed or he/she will leave that state as soon as sufficient symptoms begin to appear;
7. A symptomatic infected individual can either die of COVID-19 or get healed;
8. We do not take into account reinfection by COVID-19.
9. The entire population has a per-capita death rate

independent of COVID-19.

2. A Reaction Model of COVID-19

Consider a population with size N . We can divide it into sub-populations and denote their fractions by S, E, I , and R which respectively represent the fraction of susceptible, the fraction of asymptomatic infected individuals, the fraction of symptomatic infected individuals and the fraction of removed. Thus the sub-populations verify the identity: $S + E + I + R = 1$. Based on the assumptions made previously we can set a reaction model of COVID-19 as follows:

$$\begin{cases} S' &= \Lambda - \beta IS - \eta ES - dS \\ E' &= \beta IS + \eta ES - \epsilon E - \theta E - dE \\ I' &= \epsilon E - \mu I - \gamma I - dI \\ R' &= \gamma I + \theta E - dR. \end{cases} \quad (1)$$

The coefficients used into our model are explained in the following table:

Table 1. Coefficients meanings.

Coefficient	Meaning
Λ	The recruitment
β	The infective contact rate symptomatic infected/susceptible
η	The infective contact rate asymptomatic infected/susceptible
μ	The induced-disease death rate due to COVID-19
ϵ	The transfer rate from asymptomatic infected to symptomatic infected
d	The natural death rate out of μ
θ	The natural recovery rate of asymptomatic infected individuals
γ	The natural recovery rate of symptomatic infected individuals

The last equation in (1) does not intervene into the transmission of the disease, we can simplify our system by reducing it into three equations as follows:

$$\begin{cases} S' &= \Lambda - \beta IS - \eta ES - dS \\ E' &= \beta IS + \eta ES - \epsilon E - \theta E - dE \\ I' &= \epsilon E - \mu I - \gamma I - dI \end{cases} \quad (2)$$

To ensure the well posedness of the system we consider the proportion of the population in:

$$G = \{(S, E, I) \in \mathbb{R}_+^3 : S + E + I \leq 1\}. \quad (3)$$

2.1. Equilibrium points and \mathcal{R}_0

The equilibrium points are

$$\bar{u} = (S_0, 0, 0) = \left(\frac{\Lambda}{d}, 0, 0\right) \quad (4)$$

for the disease free equilibrium (DFE) and

$$u^* = \left(\frac{MN}{\epsilon\beta + \eta M}, \frac{\Lambda(\epsilon\beta + \eta M) - dMN}{N(\epsilon\beta + \eta M)}, \frac{\epsilon(\Lambda(\epsilon\beta + \eta M) - dMN)}{MN(\epsilon\beta + \eta M)}\right) \quad (5)$$

for the endemic equilibrium (E.E).

To find \mathcal{R}_0 we use the next generation operator [18, 21]. Our DFE is given by $\bar{u} = (\frac{\Lambda}{d}, 0, 0) = (1, 0, 0)$ due to the fact that at this equilibrium point the entire population is susceptible i.e the fraction of the healthy people is 1. So

$$J_{\bar{u}} = \begin{pmatrix} -d & -\eta\frac{\Lambda}{d} & -\beta\frac{\Lambda}{d} \\ 0 & \eta\frac{\Lambda}{d} - (\epsilon + \theta + d) & \beta\frac{\Lambda}{d} \\ 0 & \epsilon & -(d + \mu + \gamma) \end{pmatrix} \quad (6)$$

we obtain the following sub-matrices

$$F = \begin{pmatrix} \frac{\Lambda\eta}{d} & \frac{\Lambda\beta}{d} \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} (\epsilon + \theta + d) & 0 \\ -\epsilon & (\mu + \gamma + d) \end{pmatrix}. \quad (7)$$

Then

$$V^{-1} = \begin{pmatrix} \frac{1}{(\epsilon + \theta + d)} & 0 \\ \frac{\epsilon}{(\epsilon + \theta + d)(\mu + \gamma + d)} & \frac{1}{(\mu + \gamma + d)} \end{pmatrix} \text{ and } -FV^{-1} = \begin{pmatrix} -\left(\frac{\Lambda\eta}{d(\epsilon + \theta + d)} + \frac{\Lambda\beta\epsilon}{d(\epsilon + \theta + d)(\mu + \gamma + d)}\right) & -\frac{\Lambda\beta}{d(\mu + \gamma + d)} \\ 0 & 0 \end{pmatrix}. \quad (8)$$

Hence the basic reproduction number is:

$$\mathcal{R}_0 = \rho(-F.V^{-1}) = \frac{\Lambda(\eta(\mu + \gamma + d) + \beta\epsilon)}{d(\epsilon + \theta + d)(\mu + \gamma + d)} = \frac{\Lambda(\eta M + \beta\epsilon)}{dMN}. \quad (9)$$

Theorem 2.1 (Existence of equilibria). If $\mathcal{R}_0 \leq 1$, the model (2) always has a disease-free equilibrium $\bar{u} = (\frac{\Lambda}{d}, 0, 0)$. If $\mathcal{R}_0 > 1$, the model (2) has exactly one endemic equilibrium $u^* = \left(\frac{MN}{\epsilon\beta + \eta M}, \frac{\Lambda(\epsilon\beta + \eta M) - dMN}{N(\epsilon\beta + \eta M)}, \frac{\epsilon(\Lambda(\epsilon\beta + \eta M) - dMN)}{MN(\epsilon\beta + \eta M)}\right)$.

Proof. Let us suppose that $\mathcal{R}_0 \leq 1$ then we have the existence of the EE

$$u^* = (S^*, E^*, I^*) = \left(\frac{MN}{\epsilon\beta + \eta M}, \frac{\Lambda(\epsilon\beta + \eta M) - dMN}{N(\epsilon\beta + \eta M)}, \frac{\epsilon(\Lambda(\epsilon\beta + \eta M) - dMN)}{MN(\epsilon\beta + \eta M)}\right).$$

If $\mathcal{R}_0 := \frac{\Lambda(\eta M + \beta\epsilon)}{dMN} = 1$ then $\Lambda(\eta M + \beta\epsilon) = dMN$ therefore $E^* = \frac{\Lambda(\epsilon\beta + \eta M) - dMN}{N(\epsilon\beta + \eta M)} = 0$ and $I^* = \frac{\epsilon(\Lambda(\epsilon\beta + \eta M) - dMN)}{MN(\epsilon\beta + \eta M)} = 0$.

Thus $u^* = \left(\frac{MN}{\epsilon\beta + \eta M}, 0, 0\right)$. for the proportion of the population being entirely in the first component we have

$$\frac{MN}{\epsilon\beta + \eta M} = 1 \iff \frac{\epsilon\beta + \eta M}{MN} = 1 \iff \frac{\Lambda(\epsilon\beta + \eta M)}{dMN} = \frac{\Lambda}{d} \quad (\text{which keeps the proportion since } \frac{\Lambda}{d} = 1)$$

Thus $S^* = \frac{\Lambda}{d}$ and the unique equilibrium point in this situation is the disease free one.

If $\mathcal{R}_0 < 1$ then $\Lambda(\eta M + \beta\epsilon) < dMN$. Hence $E^* = \frac{\Lambda(\epsilon\beta + \eta M) - dMN}{N(\epsilon\beta + \eta M)} < 0$ and $I^* = \frac{\epsilon(\Lambda(\epsilon\beta + \eta M) - dMN)}{MN(\epsilon\beta + \eta M)} < 0$. Both

components must be positive to ensure the existence of an endemic equilibrium, therefore there is none.

Let us now suppose that $\mathcal{R}_0 > 1$, then $\Lambda(\eta M + \beta\epsilon) > dMN$, $E^* = \frac{\Lambda(\epsilon\beta + \eta M) - dMN}{N(\epsilon\beta + \eta M)} > 0$ and $I^* =$

$\frac{\epsilon(\Lambda(\epsilon\beta + \eta M) - dMN)}{MN(\epsilon\beta + \eta M)} \succ 0$. Thus there exists an endemic equilibrium u^* as defined in (5).

2.2. Stability of the Equilibria

Theorem 2.2. If

$$\mathcal{R}_0 := \frac{\Lambda(\beta\epsilon + \eta(\mu + \gamma + d))}{d(\epsilon + \theta + d)(\mu + \gamma + d)} \prec 1$$

then the DFE given in (4) is locally asymptotically stable in G .
If

$$\mathcal{R}_0 := \frac{\Lambda(\beta\epsilon + \eta(\mu + \gamma + d))}{d(\epsilon + \theta + d)(\mu + \gamma + d)} \succ 1$$

then the E.E given in (5) is locally asymptotically stable in G .

Proof: It suffices to show that the eigenvalues of the two Jacobian matrices at the two equilibria have real negative part. Next we use a property of the additive compound matrices to state:

Theorem 2.3. Let $J_{\bar{u}}$ and J_{u^*} be the Jacobian matrices at the DFE and the EE. If

$$-|J_{\bar{u}}| \succ 0$$

then the DFE given in (4) is locally asymptotically stable in G .
If

$$-|J_{u^*}| \succ 0$$

then the E.E given in (5) is locally asymptotically stable in G .

Proof: To show that \bar{u} is stable we must prove under which conditions $-|J_{\bar{u}}| \succ 0$ and $\mu(J_{\bar{u}}^{[2]}) \prec 0^*$. Where μ is a Lozinskĭ measure on $\mathcal{M}_{n \times n}$.

$$\begin{aligned} -|J_{\bar{u}}| \succ 0 &\iff d \left| \begin{array}{cc} \frac{\Lambda\eta}{d} - N & \frac{\Lambda\beta}{d} \\ \epsilon & -M \end{array} \right| \succ 0 \\ &\iff dMN - \Lambda\epsilon\beta - \eta\Lambda M \succ 0 \\ &\iff dMN \succ \Lambda(\epsilon\beta + \eta M) \\ &\iff \frac{\Lambda(\epsilon\beta + \eta M)}{dMN} \prec 1 \\ &\iff \mathcal{R}_0 \prec 1. \end{aligned}$$

The second compound matrix of $J_{\bar{u}}$ is given by:

$$J_{\bar{u}}^{[2]} = \begin{pmatrix} \frac{\Lambda\eta}{d} - (N+d) & \frac{\Lambda\beta}{d} & \frac{\Lambda\beta}{d} \\ \epsilon & -(d+M) & \frac{\Lambda\eta}{d} \\ 0 & 0 & \frac{\Lambda\eta}{d} - (M+N) \end{pmatrix}. \quad (10)$$

Let us use μ_1 as our Lozinskĭ measure with

$$\mu_1(A) = \sup_k \left(\operatorname{Re}(a_{kk}) + \sum_{i, i \neq k} |a_{ik}| \right), \quad A \in M_n(\mathbb{K}^n). \quad (11)$$

From the first column we have:

$$\begin{aligned} \frac{\Lambda\eta}{d} - (N+d) + \epsilon \prec 0 &= \frac{\Lambda\eta}{d} - 2d - \theta - \epsilon + \epsilon \prec 0 \\ &\iff \frac{\Lambda\eta}{d} - 2d - \theta \prec 0 \\ &\iff \eta \prec 2d + \theta \\ &\iff d \succ \frac{\eta - \theta}{2} \end{aligned} \quad (12)$$

We proceed the same way for the second and the third columns and find respectively:

$$d \succ \beta - M \text{ and } d \succ \frac{\beta + 2\eta - (\mu + \gamma + \epsilon + \theta)}{2}$$

Two conditions are necessary to the stability of the equilibrium \bar{u} . The first one is the sign of the Jacobian $J_{\bar{u}}$. Indeed if $(-|J_{\bar{u}}|) \succ 0$ then $\mathcal{R}_0 \prec 1$ and this condition is necessary for the local stability of \bar{u} . If $\mu(J_{\bar{u}}^{[2]}) \prec 0$, then we get a condition on the parameters and this has no much meaning and impact for our model.

Theorem 2.4. When $\eta \leq \beta$ and $\mathcal{R}_0 \leq 1$, then disease-free equilibrium \bar{u} for (2) is globally asymptotically stable.

Proof

We use an approach given by Zhisheng Shuai and P. Van Den Driessche to construct our Lyapounov function [22, 29]. Let

$$\mathcal{F} = \begin{pmatrix} \beta IS + \eta ES \\ 0 \end{pmatrix} \geq 0, \quad \mathcal{V} = \begin{pmatrix} NE \\ MI - \epsilon E \end{pmatrix} \leq 0, \quad (13)$$

F, V and V^{-1} defined like in (7) and (8).

$$\begin{aligned} V^{-1}F &= \begin{pmatrix} \frac{1}{N} & 0 \\ \frac{\epsilon}{MN} & \frac{1}{M} \end{pmatrix} \begin{pmatrix} \eta S_0 & \beta S_0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\eta S_0}{N} & \frac{\beta S_0}{N} \\ \frac{\epsilon \eta S_0}{MN} & \frac{\epsilon \beta S_0}{MN} \end{pmatrix}. \end{aligned} \quad (14)$$

If $w^T = (x \ y)$ denotes the left eigenvector of $V^{-1}F$ then we have

$$(x \ y) \begin{pmatrix} \frac{\eta S_0}{N} & \frac{\beta S_0}{N} \\ \frac{\epsilon \eta S_0}{MN} & \frac{\epsilon \beta S_0}{MN} \end{pmatrix} = (x \ y) \mathcal{R}_0 \quad (15)$$

and $w^T = (1, 1)$.

If $X = \begin{pmatrix} E \\ I \end{pmatrix}$ then we have

$$\begin{aligned} w^T V^{-1}X &= (1 \ 1) \begin{pmatrix} \frac{1}{N} & 0 \\ \frac{\epsilon}{MN} & \frac{1}{M} \end{pmatrix} \begin{pmatrix} E \\ I \end{pmatrix} \\ &= \frac{(M + \epsilon)}{MN} E + \frac{I}{M} \end{aligned}$$

*Algorithms on the calculation of $\mu(J_{\bar{u}}^{[2]})$ are given in [8-10]

and the Lyapounov function is given by:

$$L = \frac{1}{M} \left[\frac{(M + \epsilon)}{N} E + I \right] = \frac{1}{(d + \mu + \gamma)} \left[\frac{(d + \mu + \gamma + \epsilon)}{(\epsilon + \theta + d)} E + I \right]. \quad (16)$$

We have

$$L' = \frac{\partial L}{\partial E} E' + \frac{\partial L}{\partial I} I' = \frac{1}{M} \left[\frac{(M + \epsilon)}{N} E' + I' \right].$$

Therefore

$$L' = -\frac{(M + \epsilon)}{MN} (S_0 - S)(\eta E + \beta I) + \frac{1}{MN} [(M + \epsilon)(\eta S_0 E - NE + \beta S_0 I) + \epsilon NE - MNI].$$

Since $-\frac{(M + \epsilon)}{MN} (S_0 - S)(\eta E + \beta I) < 0$ the expression

$$\begin{aligned} (H) &= \frac{1}{MN} [(M + \epsilon)(\eta S_0 E - NE + \beta S_0 I) + \epsilon NE - MNI] \\ &= \frac{\eta M + \epsilon \beta}{MN} S_0 E + \frac{\eta M + \epsilon \beta}{MN} S_0 I + \frac{\epsilon(\eta - \beta)}{MN} S_0 E + \frac{M(\beta - \eta)}{MN} S_0 I - (E + I) \\ &= (\mathcal{R}_0 - 1)(E + I) + \frac{S_0(\eta - \beta)}{MN} (\epsilon E - MI) \text{ is negative.} \end{aligned}$$

From the hypothesis, $\mathcal{R}_0 \leq 1$ and from (13) $\epsilon E - MI \geq 0$, thus with $\eta \leq \beta$ we have $(H) \leq 0$ and therefore $L' \leq 0$.

Theorem 2.5. [Global stability of the EE]

The endemic equilibrium u^* for (2) is globally asymptotically stable when $\mathcal{R}_0 > 1$

Proof. Let

$$L_1 = S - S^* - S^* \ln \frac{S}{S^*}, \quad L_2 = E - E^* - E^* \ln \frac{E}{E^*} \text{ and } L_3 = I - I^* - I^* \ln \frac{I}{I^*}.$$

$$\begin{aligned} L_1' &= \frac{\partial L_1}{\partial S} S' = \frac{(S - S^*)}{S} (\beta S^* I^* + \eta S^* E^* + d S^* - \beta S I - \eta S E - d S) \\ &= \frac{(S - S^*)}{S} [-d(S - S^*) + \beta(S^* I^* - S I) + \eta(S^* E^* - S E)] \\ &= -d \frac{(S - S^*)^2}{S} + \beta \frac{(S - S^*)}{S} (S^* I^* - S I) + \eta \frac{(S - S^*)}{S} (S^* E^* - S E) \\ &\leq \beta S^* I^* \left(1 - \frac{S I}{S^* I^*} - \frac{S^*}{S} + \frac{S^* I}{S^* I^*} \right) + \eta S^* E^* \left(1 - \frac{S E}{S^* E^*} - \frac{S^*}{S} + \frac{S^* E}{S^* E^*} \right) \\ &\leq \beta S^* I^* \left(\frac{I}{I^*} - \ln \frac{I}{I^*} - \frac{S I}{S^* I^*} + \ln \frac{S I}{S^* I^*} \right) + \eta S^* E^* \left(\ln \frac{S E}{S^* E^*} - \frac{S E}{S^* E^*} + \frac{E}{E^*} - \ln \frac{E}{E^*} \right) \\ &:= a_{13} G_{13} + a_{12} G_{12} \end{aligned}$$

$$\begin{aligned} L_2' &= \frac{\partial L_2}{\partial E} E' = -N \frac{(E - E^*)^2}{E} + \beta \frac{(E - E^*)}{E} (S I - S^* I^*) + \eta \frac{(E - E^*)}{E} (S E - S^* E^*) \\ &\leq \beta S^* I^* \left(\frac{S I}{S^* I^*} - \ln \frac{S I}{S^* I^*} - \frac{E}{E^*} + \ln \frac{E}{E^*} \right) + \eta S^* E^* \left(\frac{S E}{S^* E^*} - \ln \frac{S E}{S^* E^*} - \frac{S}{S^*} + \ln \frac{S}{S^*} \right) \\ &:= a_{21} G_{21} + a_{12} G_{12} \end{aligned}$$

and similarly

$$L_3' = \frac{\partial L_3}{\partial I} I' = \frac{(I - I^*)}{I} (-\epsilon E^* + M I^* + \epsilon E - M I) \leq \epsilon E^* \left(\frac{E}{E^*} - \ln \frac{E}{E^*} - \frac{I}{I^*} + \ln \frac{I}{I^*} \right) := a_{32} G_{32}$$

$$a_{13} = \beta S^* I^*, \quad a_{12} = \eta E^* S^*, \quad a_{21} = \beta S^* I^*, \quad a_{12} = \eta E^* S^*, \quad a_{32} = \epsilon E^*.$$

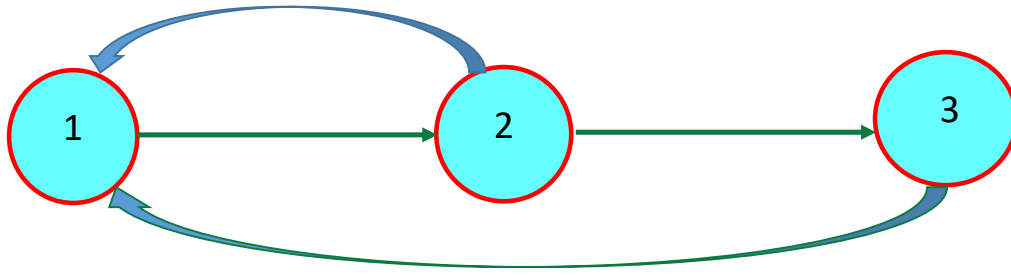


Figure 2. Interactions between the classes.

The associated weighted diagram given in Figure 2 has three vertices and two cycles. Along each cycle, $G_{21} + G_{32} + G_{13} = 0$ and $G_{21} + G_{12} = 0$. Then there exist c_i , $1 \leq i \leq 3$, such that $L = \sum_{i=1}^3 c_i L_i$ is a Lyapounov function for (2).

Let us find the relations between the c_i 's. $d^+(1) = 1 \implies c_2 a_{21} = c_1 a_{13}$, since $a_{21} = a_{13}$ then $c_2 = c_1$. $d^+(2) = 1 \implies c_3 a_{32} = c_2 a_{21} \implies c_3 = c_2 \frac{a_{21}}{a_{32}} = c_2 \frac{\beta S^* I^*}{\epsilon E^*}$. Our Lyapounov function is given by:

$$L = L_1 + L_2 + \frac{\beta S^* I^*}{\epsilon E^*} L_3 \quad (17)$$

3. A Reaction-Diffusion Model on COVID-19

Assume now that the individual in the population can move (diffuse) with the same diffusion coefficient. If the susceptibles and the asymptomatic infectious are free to move the same way, we suppose that the symptomatic infectious still have contacts with people able to diffuse. Then we can formulate our R-D model like:

$$\begin{cases} S_t = \Lambda + \Delta S - \beta IS - \eta ES - dS \\ E_t = \Delta E + \beta IS + \eta ES - (\epsilon + \theta + d)E \\ I_t = \Delta I + \epsilon E - (\mu + \gamma + d)I \end{cases} \quad (18)$$

Using wave coordinates $\xi = x + ct$ in (18) yields:

$$\begin{cases} 0 = \Lambda + S'' - cS' - (\beta I + \eta E)S - dS \\ 0 = E'' - cE' + (\beta I + \eta E)S - NE. \\ 0 = I'' - cI' + \epsilon E - MI \end{cases} \quad (19)$$

Asymptotically the system (19) satisfies the following boundary conditions:

$$\begin{pmatrix} S \\ E \\ I \end{pmatrix} (-\infty) = \begin{pmatrix} \frac{\Lambda}{d} \\ 0 \\ 0 \end{pmatrix} \quad (20)$$

$$\begin{pmatrix} S \\ E \\ I \end{pmatrix} (+\infty) = \begin{pmatrix} S^* \\ E^* \\ I^* \end{pmatrix} = \begin{pmatrix} \frac{MN}{\epsilon\beta + \eta M} \\ \frac{\Lambda(\epsilon\beta + \eta M) - dMN}{N(\epsilon\beta + \eta M)} \\ \frac{\epsilon(\Lambda(\epsilon\beta + \eta M) - dMN)}{MN(\epsilon\beta + \eta M)} \end{pmatrix} \quad (21)$$

Linearizing (19) about $(\frac{\Lambda}{d}, 0, 0) = (1, 0, 0)$ we obtain:

$$\begin{cases} 0 = \Lambda + S'' - cS' - dS \\ 0 = E'' - cE' + (\eta - N)E. \\ 0 = I'' - cI' - MI \end{cases} \quad (22)$$

The second equation in (22) provides the speed of the wave. In fact its characteristic equation is:

$$r^2 - cr + (\eta - N) = 0.$$

To ensure the existence of real solutions we must have $c \geq 2\sqrt{\eta - N}$.

Hence the minimal speed is

$$c^* = 2\sqrt{\eta - N} \quad (23)$$

and the roots to the characteristic equation are:

$$r_{1,2} = \frac{c \pm \sqrt{c^2 - 4(\eta - N)}}{2}.$$

The solutions of the first equation are:

$$q_{1,2} = \frac{c \pm \sqrt{c^2 + 4d}}{2},$$

and those of the third:

$$p_{1,2} = \frac{c \pm \sqrt{c^2 + 4M}}{2}.$$

Hence the profile of the traveling wave solution to (22) is given by:

$$\begin{pmatrix} S(\xi) \\ E(\xi) \\ I(\xi) \end{pmatrix} = \begin{pmatrix} A_{11} e^{\frac{c - \sqrt{c^2 + 4d}}{2} \xi} + A_{12} e^{\frac{c + \sqrt{c^2 + 4d}}{2} \xi} \\ A_{21} e^{\frac{c - \sqrt{c^2 - 4(\eta - N)}}{2} \xi} + A_{22} e^{\frac{c + \sqrt{c^2 - 4(\eta - N)}}{2} \xi} \\ A_{31} e^{\frac{c - \sqrt{c^2 + 4M}}{2} \xi} + A_{32} e^{\frac{c + \sqrt{c^2 + 4M}}{2} \xi} \end{pmatrix} \quad (24)$$

4. Existence of a Traveling Wave

To prove the existence of a front traveling wave solution to (22) we shall use the monotone iterative method which relies on the following principle:

Principle 4.1. [Monotone Iterative Method] Consider the general second order ODE with Dirichlet boundary conditions value given by :

$$\begin{cases} u''(t) = f(t, u(t), u'(t)), & t \in I \equiv [a, b], \\ u(a) = A, u(b) = B, \end{cases} \quad (25)$$

Hence the boundary conditions give:

$$\begin{pmatrix} \underline{S}(\xi) \\ \underline{E}(\xi) \\ \underline{I}(\xi) \end{pmatrix} (-\infty) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \leq \begin{pmatrix} \frac{\Lambda}{d} \\ 0 \\ 0 \end{pmatrix} \quad (26)$$

$$\begin{pmatrix} \underline{S}(\xi) \\ \underline{E}(\xi) \\ \underline{I}(\xi) \end{pmatrix} (+\infty) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \leq \begin{pmatrix} S^* \\ E^* \\ I^* \end{pmatrix} = \begin{pmatrix} \frac{MN}{\epsilon\beta + \eta M} \\ \frac{\Lambda(\epsilon\beta + \eta M) - dMN}{N(\epsilon\beta + \eta M)} \\ \frac{\epsilon(\Lambda(\epsilon\beta + \eta M) - dMN)}{MN(\epsilon\beta + \eta M)} \end{pmatrix} \quad (27)$$

Thus $(\underline{S}(\xi), \underline{E}(\xi), \underline{I}(\xi)) = (0, 0, 0)$ is a lower-solution.

Lemma 4.2. Suppose that $MN \geq \epsilon\beta$ and let $\overline{X}(\xi) = (\overline{S}(\xi), \overline{E}(\xi), \overline{I}(\xi))$ be a function defined by:

$$\begin{aligned} \overline{S}(\xi) &= \frac{\Lambda}{d} S^* \quad \forall \xi \in \mathbb{R}; \\ \overline{E}(\xi) &= \begin{cases} E^* e^{q_1 \xi}, & \xi \leq 0 \\ E^*, & \xi \geq 0 \end{cases} \\ \overline{I}(\xi) &= \begin{cases} \frac{N}{\beta} E^* e^{a_1 \xi}, & \xi \leq 0 \\ I^*, & \xi \geq 0 \end{cases} \end{aligned} \quad (28)$$

with $q_1 = \frac{c + \sqrt{c^2 - 4\eta}}{2}$ the greater positive root of the equation $q^2 - cq + \eta = 0$ and $a_1 = c$ the non-zero positive root of the equation $a^2 - ca = 0$. Then $\overline{X}(\xi)$ is an upper-solution to (19).

Proof

Assume that $\xi \leq 0$.

For the first equation of (19) we have:

$$\begin{aligned} & \Lambda + \left(\frac{\Lambda}{d} S^* \right)'' - c \left(\frac{\Lambda}{d} S^* \right)' \\ & - \left(\beta \frac{N}{\beta} E^* e^{a_1 \xi} + \eta E^* e^{q_1 \xi} \right) \frac{\Lambda}{d} S^* - d \frac{\Lambda}{d} S^* \\ & = \Lambda - (NE^* e^{a_1 \xi} + \eta E^* e^{q_1 \xi}) \frac{\Lambda}{d} S^* - d \frac{\Lambda}{d} S^* \\ & \leq \Lambda - (NE^* e^{a_1 \xi} + \eta E^* e^{q_1 \xi}) S^* - \Lambda S^* \\ & \leq \Lambda - (NE^* e^{a_1 \xi} + \eta E^* e^{q_1 \xi}) S^* - \Lambda \\ & \Leftrightarrow -E^* (Ne^{a_1 \xi} + \eta e^{q_1 \xi}) S^* < 0 \end{aligned} \quad (29)$$

with $f : I \times \mathbb{R}^2 \rightarrow \mathbb{R}$ a continuous function and $A, B \in \mathbb{R}$. If in $C^2(I)$ there exist $\underline{U}(t)$ a lower solution to (25) and $\overline{U}(t)$ an upper solution to (25) such that $\underline{U}(t) \leq \overline{U}(t)$ on I . Then the existence of a solution to the problem (25) lying between $\underline{U}(t)$ and $\overline{U}(t)$ is proved.

Lemma 4.1. Let $\underline{X}(\xi) = (\underline{S}(\xi), \underline{E}(\xi), \underline{I}(\xi)) = (0, 0, 0)$, then \underline{X} is a lower solution to (19).

Proof

It is obvious the last two equations of (19) vanish and for the first one we have $\Lambda \geq 0$.

For the second equation of (19) we have:

$$\begin{aligned} & q_1^2 E^* e^{q_1 \xi} - cq_1 E^* e^{q_1 \xi} \\ & + \left(\beta \frac{N}{\beta} E^* e^{a_1 \xi} + \eta E^* e^{q_1 \xi} \right) \frac{\Lambda}{d} S^* - NE^* e^{q_1 \xi} \\ & \Leftrightarrow E^* e^{q_1 \xi} (q_1^2 - cq_1 + \frac{\eta \Lambda}{d} S^*) + NE^* e^{a_1 \xi} - NE^* e^{q_1 \xi} \\ & \leq E^* e^{q_1 \xi} \overbrace{(q_1^2 - cq_1 + \eta)}^0 + NE^* e^{a_1 \xi} - NE^* e^{q_1 \xi} \\ & = NE^* (e^{a_1 \xi} - e^{q_1 \xi}) \leq 0 \text{ car } a_1 < q_1 \end{aligned} \quad (30)$$

by their definitions.

For the third equation of (19) we have:

$$\begin{aligned} & a_1^2 \frac{N}{\beta} E^* e^{a_1 \xi} - ca_1 \frac{N}{\beta} E^* e^{a_1 \xi} - \frac{MN}{\beta} E^* e^{a_1 \xi} + \epsilon E^* e^{q_1 \xi} \\ & = \frac{N}{\beta} E^* e^{a_1 \xi} \overbrace{(a_1^2 - ca_1)}^0 + \epsilon E^* e^{q_1 \xi} - \frac{MN}{\beta} E^* e^{a_1 \xi} \\ & = E^* \left(\epsilon e^{q_1 \xi} - \frac{MN}{\beta} e^{a_1 \xi} \right) \\ & E^* \left(\epsilon e^{q_1 \xi} - \frac{MN}{\beta} e^{a_1 \xi} \right) \leq 0 \Leftrightarrow \epsilon e^{q_1 \xi} - \frac{MN}{\beta} e^{a_1 \xi} \leq 0 \\ & \leq \epsilon - \frac{MN}{\beta} \leq 0 \\ & \Leftrightarrow \frac{\epsilon\beta}{MN} \leq 1 \end{aligned} \quad (31)$$

Now if $\xi \geq 0$, then:

For the first equation of (19) we have:

$$\begin{aligned} & \Lambda + \left(\frac{\Lambda}{d} S^* \right)'' - c \left(\frac{\Lambda}{d} S^* \right)' - (\beta I^* + \eta E^*) \frac{\Lambda}{d} S^* - d \frac{\Lambda}{d} S^* \\ & \leq \Lambda - (\beta I^* + \eta E^*) S^* - \Lambda \operatorname{car} \frac{\Lambda}{d} = 1 \text{ et } S^* \leq 1 \\ & = -(\beta I^* + \eta E^*) S^* < 0 \end{aligned} \quad (32)$$

For the second equation of (19) we have:

$$\begin{aligned} & (\beta I^* + \eta E^*) \frac{\Lambda}{d} S^* - N E^* \\ & \leq (\beta I^* + \eta E^*) S^* - N E^* = 0. \end{aligned} \quad (33)$$

For the third equation of (19) we have:

$$M I^* - \epsilon E^* = 0 \quad (34)$$

(4.8) and (4.9) are obtained due to the fact that (S^*, E^*, I^*) is an equilibrium to (2).

For the boundary conditions we have:

$$\begin{pmatrix} \bar{S} \\ \bar{E} \\ \bar{I} \end{pmatrix} (-\infty) = \begin{pmatrix} \frac{\Lambda}{d} \\ 0 \\ 0 \end{pmatrix} \geq \begin{pmatrix} \frac{\Lambda}{d} \\ 0 \\ 0 \end{pmatrix} \quad (35)$$

$$\begin{pmatrix} \bar{S} \\ \bar{E} \\ \bar{I} \end{pmatrix} (+\infty) = \begin{pmatrix} \frac{\Lambda}{d} S^* \\ E^* \\ I^* \end{pmatrix} = \begin{pmatrix} \frac{MN}{\epsilon\beta + \eta M} \\ \frac{\Lambda(\epsilon\beta + \eta M) - dMN}{N(\epsilon\beta + \eta M)} \\ \frac{\epsilon(\Lambda(\epsilon\beta + \eta M) - dMN)}{MN(\epsilon\beta + \eta M)} \end{pmatrix} \quad (36)$$

Hence, for both values $\xi > 0$ and $\xi \leq 0$, $\bar{X}(\xi)$ is an upper-solution to (19).

Theorem 4.1. If $\mathcal{R}_0 > 1$ then there exists a traveling wave solution to (19) with a minimal speed $c^* = 2\sqrt{\eta - \bar{N}}$. If $\mathcal{R}_0 < 1$ then there does not exist any traveling wave solution to (19).

Proof

Assume

$$\mathcal{R}_0 > 1 \text{ then } u^* = \left(\frac{MN}{\epsilon\beta + \eta M}, \frac{\Lambda(\epsilon\beta + \eta M) - dMN}{N(\epsilon\beta + \eta M)}, \frac{\epsilon(\Lambda(\epsilon\beta + \eta M) - dMN)}{MN(\epsilon\beta + \eta M)} \right) > (0, 0, 0).$$

Hence the function $\bar{X}(\xi)$ in (28) is well defined. By their definitions it obvious that:

$$(\underline{S}(\xi), \underline{E}(\xi), \underline{I}(\xi)) \leq (\bar{S}(\xi), \bar{E}(\xi), \bar{I}(\xi)).$$

Then by principle on monotone iterative method we are ensured of the existence of a traveling wave $(S(\xi), E(\xi), I(\xi))$ solution to (18) that verifies

$$(\underline{S}(\xi), \underline{E}(\xi), \underline{I}(\xi)) \leq (S(\xi), E(\xi), I(\xi)) \leq (\bar{S}(\xi), \bar{E}(\xi), \bar{I}(\xi)).$$

5. Turing Instability

When diffusion is added to a dynamic model it can radically change the nature of the equilibrium points and generate

diffusion-driven (Turing) instabilities [12-14]. In this section we also use the additive compound matrices to investigate whether there will be appearance of Turing instability.

Theorem 5.1. Suppose $0 < \frac{\epsilon\beta}{N-\eta} < M$, the DFE will be locally asymptotically stable for all diffusion matrix $D > 0$.

Proof

The principal minor matrices to $J_{\bar{u}}$ are

$$J_1 = \begin{bmatrix} \eta - N & \beta \\ \epsilon & -M \end{bmatrix}, \quad J_2 = \begin{bmatrix} -d & -\beta \\ 0 & -M \end{bmatrix} \text{ and } J_3 = \begin{bmatrix} -d & -\eta \\ 0 & N - \eta \end{bmatrix}$$

$$\begin{aligned} |J_1| > 0 & \Leftrightarrow M(N - \eta) - \epsilon\beta > 0 \\ & \Leftrightarrow M(N - \eta) > \epsilon\beta \\ & \Leftrightarrow \frac{\epsilon\beta}{N - \eta} < M \quad (a) \end{aligned}$$

$$|J_2| = dM > 0$$

$$\begin{aligned} |J_3| > 0 & \Leftrightarrow d(N - \eta) > 0 \\ & \Leftrightarrow N - \eta > 0 \\ & \Leftrightarrow N > \eta \quad (b) \end{aligned}$$

From (a) and (b) we know that $J_{\bar{u}}$ satisfy the minor conditions if $0 < \frac{\epsilon\beta}{N-\eta} < M$ i.e under that condition the DFE will remain locally asymptotically stable even if diffusion is introduced in the Reaction model (2).

Theorem 5.2. There will always be a Turing instability at the EE defined in (5) for all diffusion matrix $D > 0$.

Proof

The principal minor matrices of J_{u^*} are

$$K_1 = \begin{bmatrix} -\frac{\Lambda(\epsilon\beta+\eta M)}{MN} & -\frac{\eta MN}{\epsilon\beta+\eta M} \\ \frac{\Lambda(\epsilon\beta+\eta M)-dMN}{MN} & \frac{N\epsilon\beta}{\epsilon\beta+\eta M} \end{bmatrix}, K_2 = \begin{bmatrix} -\frac{\Lambda(\epsilon\beta+\eta M)}{MN} & -\frac{\beta MN}{\epsilon\beta+\eta M} \\ 0 & -M \end{bmatrix} \text{ and } K_3 = \begin{bmatrix} -\frac{N\epsilon\beta}{\epsilon\beta+\eta M} & \frac{\beta MN}{\epsilon\beta+\eta M} \\ \epsilon & -M \end{bmatrix}$$

$$\begin{aligned} |K_1| > 0 &\Leftrightarrow \frac{\eta MN(\Lambda(\epsilon\beta+\eta M)-dMN)}{MN(\epsilon\beta+\eta M)} - \frac{\epsilon\beta\Lambda N(\epsilon\beta+\eta M)}{MN(\epsilon\beta+\eta M)} > 0 \\ &\Leftrightarrow \frac{\Lambda(\epsilon\beta+\eta M)(\eta MN - \epsilon\beta\Lambda N) - d\eta(MN)^2}{MN(\epsilon\beta+\eta M)} > 0 \\ &\Leftrightarrow \Lambda N(\epsilon\beta+\eta M)(\eta M - \Lambda\epsilon\beta) > d\eta(MN)^2 \\ &\Leftrightarrow \mathcal{R}_0 > \frac{\eta M}{\eta M - \Lambda\epsilon\beta} \end{aligned}$$

$$|K_2| = \frac{M\Lambda(\epsilon\beta+\eta M)}{MN} = \frac{\Lambda(\epsilon\beta+\eta M)}{N} > 0$$

$$|K_3| = \frac{\epsilon\beta MN}{\epsilon\beta+\eta M} - \frac{\epsilon\beta MN}{\epsilon\beta+\eta M} = 0.$$

$|K_3| = 0$ implies that the minor conditions will never be satisfied on J_{u^*} . Then if diffusion is introduced into the Reaction model (2) we will have a Turing instability for u^* .

6. Conclusion

The model we built and studied can provide capital information on the dynamic of the pandemic: The expression given in (23) enable us to calculate the minimum speed rate for the appearance of a wave spreading the disease, the stability analysis of the equilibrium points have shown different situations that can occur.

If the DFE is stable then the dynamic model remains stable under the condition $\mathcal{R}_0 < 1$ even if a diffusion term is introduced and there will not be any appearance of a traveling wave solution but some conditions must be fulfilled:

- The contact rate given in β must be reduced. Quarantine and barrier gestures remain the best means so far;
- In the absence of an effective vaccine, it is important to reinforce immunity within populations by maintaining symptomatic individuals alive long enough for them to acquire immunity;
- From both the transfer rate ϵ and the contact rate η we understand that more people should be tested especially the asymptomatic infectious keeping in mind that they

seem to be the most dangerous in the spreading of the pandemic since they show no symptoms of the disease.

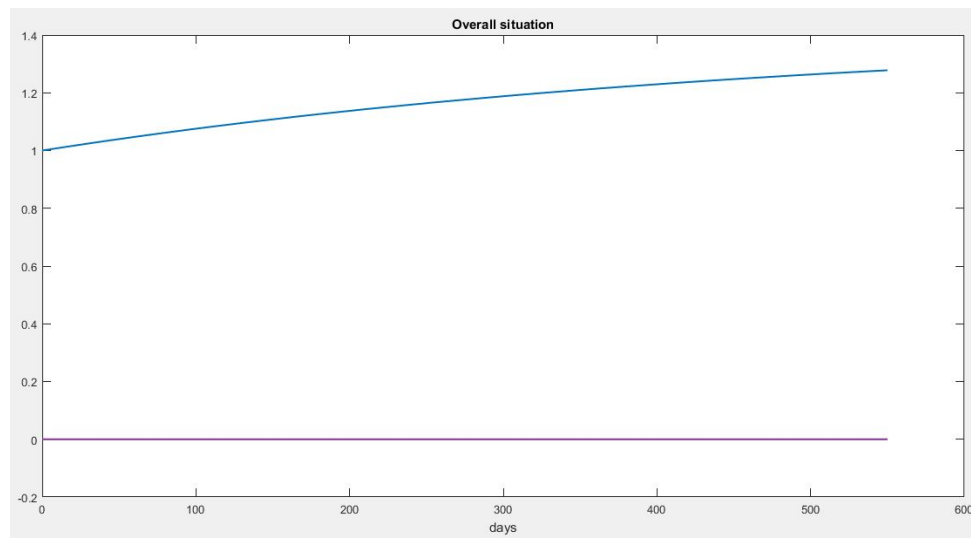
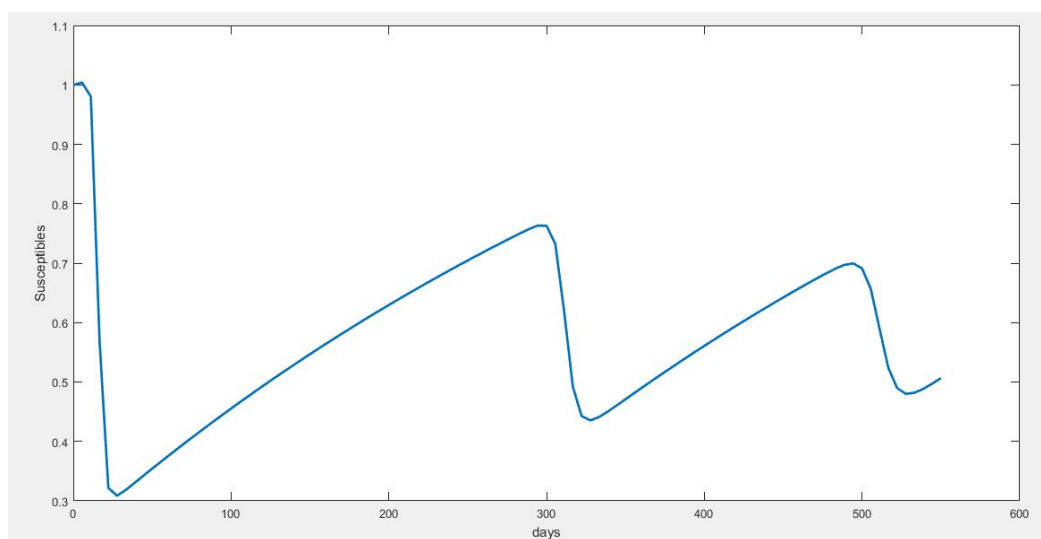
When $\mathcal{R}_0 > 1$ we have the existence of an endemic equilibrium point the E.E and when diffusion is introduced there is appearance of a Turing instability. We have fluctuations in the number of infectious individuals even if $\mathcal{R}_0 > 1$ i.e the EE will somehow lose its asymptotic behavior.

7. Simulations

We suppose an asymptomatic infected is likely to be in contact with more people than a symptomatic one because he shows no signs for people to be suspicious. The two infection rates are obtained by $\beta = s_1 \cdot p$ and $\eta = s_2 \cdot p$ where p is the percentage of contamination for an infected individual in one day, s_1 is the number of contacts that a symptomatic infected person can meet in one day and s_2 is the number of contacts that an asymptomatic infected person can meet in one day. The initial condition is given by $X_0 = (\frac{N-100}{N}, \frac{50}{N}, \frac{30}{N}, \frac{20}{N})$ and from the literature (see [20]-[26]) we can give the following table containing the estimated values of the parameters.

Table 2. Coefficients Values

Coefficient	Estimated Values
Λ	0.0028
N	12.10^6 (The estimated population size of Kinshasa)
p	0.03
μ	0.03
ϵ	0.3
d	0.002
θ	0.9
γ	0.8

**Figure 3.** When $\beta = 0.3$ ($s_1 = 10$) and $\eta = 0.45$ ($s_2 = 15$), $\mathcal{R}_0 = 0.6579$ and the trajectories are going to the DFE in fact $S=1$, $E=I=R=0$.**Figure 4.** If $s_1 = 40$ and $s_2 = 55$ ($\beta = 1.2$ and $\eta = 1.65$), $\mathcal{R}_0 = 2.4546$. We can see 3 different waves in the susceptible population and in the first weeks after the beginning of the infection the number of susceptibles dwindles quickly.

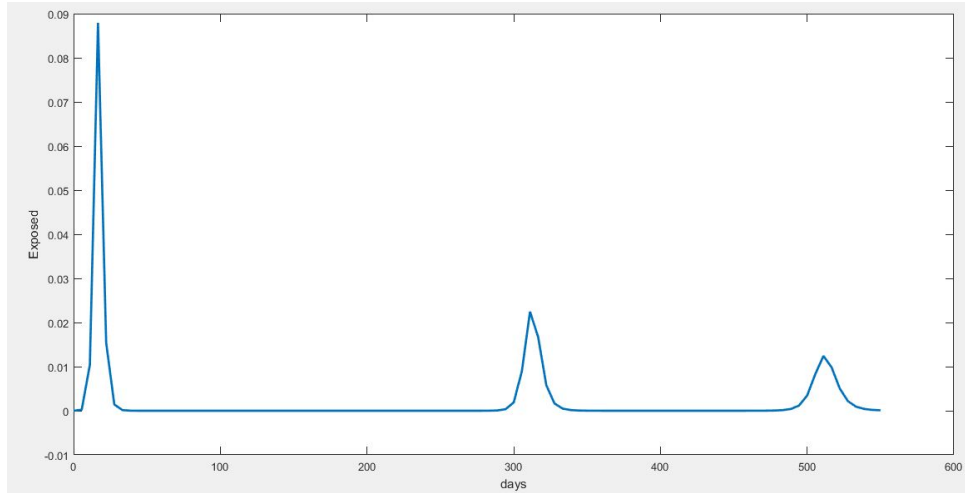


Figure 5. $\mathcal{R}_0 = 2.4546$, we have 3 different waves of infections where the number of exposed increases for a certain length of time and reach a pic before a decrease and so on.

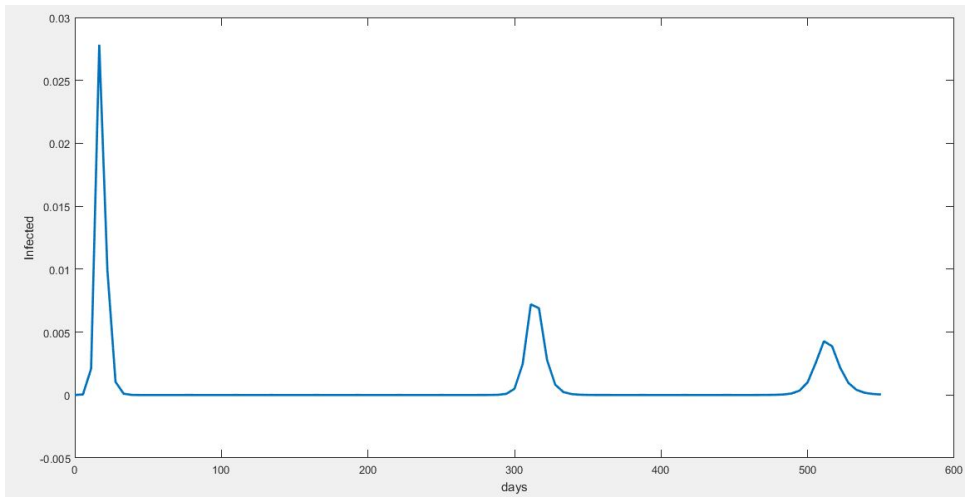


Figure 6. $\mathcal{R}_0 = 2.4546$, we also have 3 different waves of infections and the number of symptomatic infected is always smaller then the number of asyptomatic infected.

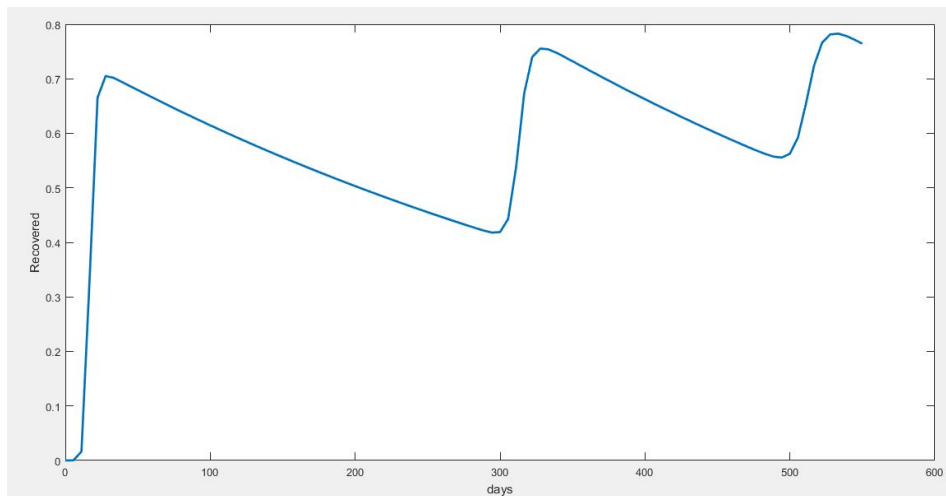


Figure 7. $\mathcal{R}_0 = 2.4546$, we also have 3 different waves where the number of recovered increases and decreases.

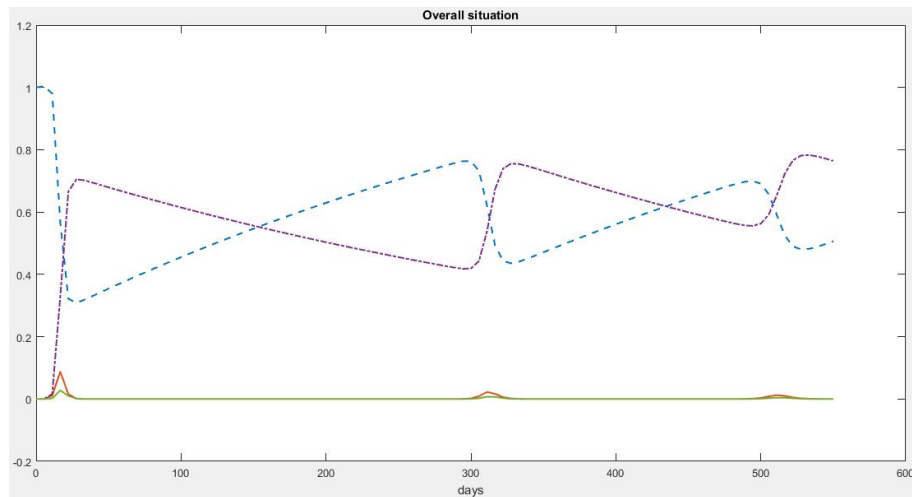


Figure 8. Here we can clearly see the interactions between the four classes when $\mathcal{R}_0 = 2.4546$.

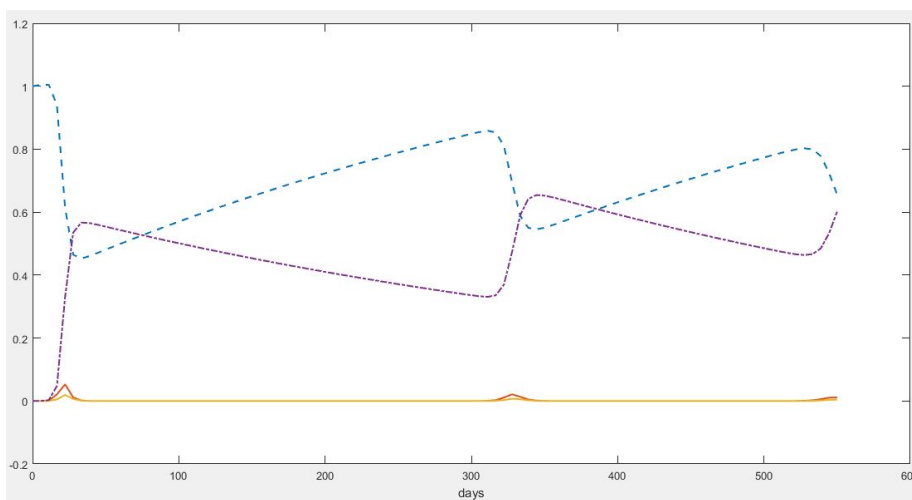


Figure 9. If $s_1 = 10$ and $s_2 = 55$ ($\beta = 0.3$ and $\eta = 1.65$), $\mathcal{R}_0 = 1.0403$. We can see that even if the number of contacts for symptomatic infected has been strongly reduced, we still have lots of infections due to the asymptomatic infected contact number which is still high. We can also see it slows down the infection waves.

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