

Solution of Nonlinear Fractional Differential Equations Using Adomain Decomposition Method

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Abstract: In this paper, Adomian decomposition method (ADM) will apply to solve nonlinear fractional differential equations (FDEs) of Caputo sense. These type of equations is very important in engineering applications such as electrical networks, fluid flow, control theory and fractals theory. ADM give analytical solution in form of series solution so the convergence of the series solution and the error analysis will discuss. In addition, existence and uniqueness of the solution will prove. Some numerical examples will solve to test the validity of the method and the given theorems. A comparison of ADM solution with exact and numerical methods are given.

Keywords: Fractional Differential Equation, Adomian Method, Existence, Uniqueness, Error Analysis

1. Introduction

Fractional Differential equations (FDEs) have many applications in engineering and science [1-6], including electrical networks, fluid flow, control theory, fractals theory, electromagnetic theory, viscoelasticity, potential theory, chemistry, biology, optical and neural network systems [7-13]. In this paper, Adomian decomposition method (ADM) [14-19] will use to solve nonlinear FDEs of Caputo sense. This method has many advantages; it is efficiently works with different types of linear and nonlinear equations in deterministic or stochastic fields and gives an analytic solution for all these types of equations without linearization or discretization [20-23]. The paper organize as follows: In section two ADM will apply to the problem under consideration. In section three uniqueness, convergence and error analysis will discuss. Finally, six numerical examples presented by using MATHEMATICA package.

2. Formulation of the Problem

Consider the nonlinear FDE,

$${}_0\mathcal{D}_t^{\sigma_n} y(t) + a(t)f(y(t)) = x(t), \quad (1)$$

Subject to the initial conditions,

$$y^{(j-1)}(0) = c_j, j = 1, 2, \dots, n. \quad (2)$$

Where

$${}_0\mathcal{D}_t^{\sigma_n} = {}_0\mathcal{D}_t^{\alpha_n} {}_0\mathcal{D}_t^{\alpha_{n-1}} \dots {}_0\mathcal{D}_t^{\alpha_1},$$

$$\sigma_n = \sum_{k=1}^n \alpha_k, 0 \leq \alpha_k \leq 1, k = 1, 2, \dots, n.$$

The fractional derivative is of sequential Caputo sense. In the applications, the Caputo sense are preferred to use because the initial conditions of $y(t)$ and its derivatives will be of integer orders and has a physical meaning. Now performing subsequently the fractional integration of order $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$, this reduces the problem (1)-(2) to the fractional integral equation (FIE):

$$y(t) = \sum_{j=1}^n \frac{c_j}{\Gamma(\sigma_j)} t^{\sigma_j-1} + \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n-1} x(\tau) d\tau$$

$$- \frac{1}{\Gamma(\sigma_n)} \int_0^t a(\tau) (t-\tau)^{\sigma_n-1} f(y(\tau)) d\tau \quad (3)$$

Assume that $x(t)$ bounded $\forall t \in J = [0, T], T \in R^+$, $|a(\tau)| \leq M \forall 0 \leq \tau \leq t \leq T, M$ is a finite constant and $f(y)$ is Lipschitz continuous with Lipschitz constant L such as,

$$|f(y) - f(z)| \leq L|y - z| \quad (4)$$

Which has Adomian polynomials representation,

$$f(y) = \sum_{n=0}^{\infty} A_n(y_0, y_1, \dots, y_n) \tag{5}$$

Where

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} [f(\sum_{i=0}^{\infty} \lambda^i y_i)]_{\lambda=0} \tag{6}$$

Substitute from equation (5) into equation (3) we get,

$$y(t) = \sum_{j=1}^n \frac{c_j}{\Gamma(\sigma_j)} t^{\sigma_j-1} + \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n-1} x(\tau) d\tau - \frac{1}{\Gamma(\sigma_n)} \int_0^t a(\tau)(t-\tau)^{\sigma_n-1} \sum_{n=0}^{\infty} A_n d\tau \tag{7}$$

Let $y(t) = \sum_{n=0}^{\infty} y_n(t)$ in (7) and applying ADM, we get the following recursive relations,

$$y_0(t) = \sum_{j=1}^n \frac{c_j}{\Gamma(\sigma_j)} t^{\sigma_j-1} + \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n-1} x(\tau) d\tau, \tag{8}$$

$$y_i(t) = -\frac{1}{\Gamma(\sigma_n)} \int_0^t a(\tau)(t-\tau)^{\sigma_n-1} A_{i-1} d\tau, i \geq 1. \tag{9}$$

Finally, the solution is,

$$y(t) = \sum_{i=0}^{\infty} y_i(t) \tag{10}$$

2.1. Existence and Uniqueness

Theorem 1 If $0 < \alpha < 1$ where $\alpha = \frac{LMT^{\sigma_n}}{\Gamma(\sigma_n+1)}$, then the series (10) is the solution of the problem (1)-(2) and this solution is unique.

Proof For existence,

$$\begin{aligned} y(t) &= \sum_{i=0}^{\infty} y_i(t) \\ &= y_0(t) + \sum_{i=1}^{\infty} y_i(t) \\ &= y_0(t) - \frac{1}{\Gamma(\sigma_n)} \sum_{i=1}^{\infty} \int_0^t a(\tau)(t-\tau)^{\sigma_n-1} A_{i-1} d\tau \\ &= y_0(t) - \frac{1}{\Gamma(\sigma_n)} \int_0^t a(\tau)(t-\tau)^{\sigma_n-1} \sum_{i=1}^{\infty} A_{i-1} d\tau \\ &= y_0(t) - \frac{1}{\Gamma(\sigma_n)} \int_0^t a(\tau)(t-\tau)^{\sigma_n-1} \sum_{i=0}^{\infty} A_i d\tau \\ &= \sum_{j=1}^n \frac{c_j}{\Gamma(\sigma_j)} t^{\sigma_j-1} + \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n-1} x(\tau) d\tau \\ &\quad - \frac{1}{\Gamma(\sigma_n)} \int_0^t a(\tau)(t-\tau)^{\sigma_n-1} f(y(\tau)) d\tau \end{aligned}$$

Then the Adomian's series solution satisfy equation (3) which is the equivalent FIE to the problem (1)-(2).

For uniqueness of the solution: Assume that y and z are two different solutions to the problem (1)-(2) and hence,

$$\begin{aligned} |y-z| &= \left| \frac{1}{\Gamma(\sigma_n)} \int_0^t a(\tau)(t-\tau)^{\sigma_n-1} [f(y) - f(z)] d\tau \right| \\ &\leq \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n-1} |a(\tau)| |f(y) - f(z)| d\tau \\ &\leq \frac{LM}{\Gamma(\sigma_n)} |y-z| \int_0^t (t-\tau)^{\sigma_n-1} d\tau \end{aligned}$$

$$\leq \frac{LMT^{\sigma_n}}{\Gamma(\sigma_n+1)} |y-z|$$

Let $\frac{LMT^{\sigma_n}}{\Gamma(\sigma_n+1)} = \alpha$ where, $0 < \alpha < 1$ then,

$$\begin{aligned} |y-z| &\leq \alpha |y-z| \\ (1-\alpha) |y-z| &\leq 0 \end{aligned}$$

However, $(1-\alpha) |y-z| \geq 0$ and since, $(1-\alpha) \neq 0$ then, $|y-z| = 0$ this imply that, $y = z$ and this completes the proof.

2.2. Proof of Convergence

Theorem 2: The series solution (10) of the problem (1)-(2) using ADM converges if $|y_1| < \infty$ and $0 < \alpha < 1$, $\alpha = \frac{LMT^{\sigma_n}}{\Gamma(\sigma_n+1)}$.

Proof Define the Banach space $(C[J], \|\cdot\|)$, the space of all continuous functions on J with the norm $\|y(t)\| = \max_{t \in J} |y(t)|$. Define the sequence $\{S_n\}$ such that, $S_n = \sum_{i=0}^n y_i(t)$ the sequence of partial sums from the series solution $\sum_{i=0}^{\infty} y_i(t)$ since,

$$f(y) = f\left(\sum_{i=0}^{\infty} y_i(t)\right) = \sum_{i=0}^{\infty} A_i(y_0, y_1, \dots, y_i)$$

So,

$$\begin{aligned} f(y_0) &= f(S_0) = A_0, \\ f(y_0 + y_1) &= f(S_1) = A_0 + A_1, \\ f(y_0 + y_1 + y_2) &= f(S_2) = A_0 + A_1 + A_2, \\ &\vdots \\ f(S_n) &= \sum_{i=0}^n A_i(y_0, y_1, \dots, y_i). \end{aligned}$$

Let, S_n and S_m be two arbitrary partial sums with, $n \geq m$. Now, we are going to prove that $\{S_n\}$ is a Cauchy sequence in this Banach space.

$$\begin{aligned} \|S_n - S_m\| &= \max_{t \in J} |S_n - S_m| = \max_{t \in J} \left| \sum_{i=m+1}^n y_i(t) \right| \\ &= \max_{t \in J} \left| \sum_{i=m+1}^n -\frac{1}{\Gamma(\sigma_n)} \int_0^t a(\tau)(t-\tau)^{\sigma_n-1} A_{i-1} d\tau \right| \\ &= \max_{t \in J} \left| \frac{1}{\Gamma(\sigma_n)} \int_0^t a(\tau)(t-\tau)^{\sigma_n-1} \sum_{i=m}^{n-1} A_i d\tau \right| \\ &= \max_{t \in J} \left| \frac{1}{\Gamma(\sigma_n)} \int_0^t a(\tau)(t-\tau)^{\sigma_n-1} [f(S_{n-1}) - f(S_{m-1})] d\tau \right| \\ &\leq \frac{1}{\Gamma(\sigma_n)} \max_{t \in J} \int_0^t (t-\tau)^{\sigma_n-1} |a(\tau)| |f(S_{n-1}) - f(S_{m-1})| d\tau \end{aligned}$$

$$\begin{aligned} &\leq \frac{LM}{\Gamma(\sigma_n)} \max_{t \in J} |S_{n-1} - S_{m-1}| \int_0^t (t - \tau)^{\sigma_n - 1} d\tau \\ &\leq \frac{LMT^{\sigma_n}}{\Gamma(\sigma_n + 1)} \|S_{n-1} - S_{m-1}\| \\ &\leq \alpha \|S_{n-1} - S_{m-1}\| \end{aligned}$$

Let $n = m + 1$ then,

$$\begin{aligned} \|S_{m+1} - S_m\| &\leq \alpha \|S_m - S_{m-1}\| \leq \alpha^2 \|S_{m-1} - S_{m-2}\| \\ &\leq \dots \leq \alpha^m \|S_1 - S_0\| \end{aligned}$$

From the triangle inequality we have,

$$\begin{aligned} \|S_n - S_m\| &\leq \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| \\ &\quad + \dots + \|S_n - S_{n-1}\| \\ &\leq [\alpha^m + \alpha^{m+1} + \dots + \alpha^{n-1}] \|S_1 - S_0\| \\ &\leq \alpha^m [1 + \alpha + \dots + \alpha^{n-m-1}] \|S_1 - S_0\| \\ &\leq \alpha^m \left[\frac{1 - \alpha^{n-m}}{1 - \alpha} \right] \|y_1(t)\| \end{aligned}$$

Since, $0 < \alpha < 1$ and $n \geq m$ then $(1 - \alpha^{n-m}) \leq 1$. Consequently,

$$\|S_n - S_m\| \leq \frac{\alpha^m}{1 - \alpha} \|y_1(t)\| \leq \frac{\alpha^m}{1 - \alpha} \max_{t \in J} |y_1(t)|$$

However, $|y_1(t)| \leq \infty$ and as $m \rightarrow \infty$ then, $\|S_n - S_m\| \rightarrow 0$ and hence, $\{S_n\}$ is a Cauchy sequence in this Banach space so, the series $\sum_{n=0}^{\infty} y_n(t)$ converges and the proof is complete.

2.3. Error Analysis

For ADM, we can estimate the maximum absolute truncated error of the Adomian's series solution in the following theorem.

Theorem 3: The maximum absolute truncation error of the series solution (10) to the problem (1)-(2) estimated to be,

$$\max_{t \in J} \left| y(t) - \sum_{i=0}^m y_i(t) \right| \leq \frac{\alpha^m}{1 - \alpha} \max_{t \in J} |y_1(t)|$$

Proof. From Theorem 2 we have

$$\|S_n - S_m\| \leq \frac{\alpha^m}{1 - \alpha} \max_{t \in J} |y_1(t)|$$

But, $S_n = \sum_{i=0}^n y_i(t)$ as $n \rightarrow \infty$ then, $S_n \rightarrow y(t)$ so,

$$\|y(t) - S_m\| \leq \frac{\alpha^m}{1 - \alpha} \max_{t \in J} |y_1(t)|$$

Therefore, the maximum absolute truncation error in the interval J is,

$$y_n = h^\mu + h^\mu y_{n-1}^2 - \sum_{j=1}^n (-1)^j \frac{\Gamma(\mu+1)}{\Gamma(j+1)\Gamma(\mu-j+1)} y_{n-j}, \quad m - 1 < \mu \leq m, t \in [0, T], n = m, m + 1, \dots \quad (17)$$

$$\max_{t \in J} \left| y(t) - \sum_{i=0}^m y_i(t) \right| \leq \frac{\alpha^m}{1 - \alpha} \max_{t \in J} |y_1(t)|$$

Hence, the proof is completed.

3. Numerical Examples

Example 1. Consider the initial value problem [20],

$$D^\mu y = y^2 + 1, m - 1 < \mu \leq m, 0 < t < 1, \quad (11)$$

$$y^{(k)}(0) = 0, k = 0, 1, \dots, m - 1. \quad (12)$$

Operating with J^μ on both sides of equation (11) and using the initial conditions (12) we obtain,

$$y(t) = J^\mu[1] + J^\mu[y^2], \quad (13)$$

Using ADM and replace the nonlinear term $f(y) = y^2$ by its corresponding Adomian polynomials we have,

$$y_0 = J^\mu[1], \quad (14)$$

$$y_n = J^\mu[A_{n-1}], n \geq 1. \quad (15)$$

From the two relations (14) and (15), the six terms approximation are,

$$\phi_6 = \sum_{k=0}^5 C_k t^{(2k+1)\mu}, \quad (16)$$

Where, the coefficients given by,

$$\begin{aligned} C_0 &= \frac{1}{\Gamma(\mu + 1)}, \quad C_1 = \frac{\Gamma(2\mu + 1)}{\Gamma(3\mu + 1)} C_0^2, \\ C_2 &= \frac{\Gamma(4\mu + 1)}{\Gamma(5\mu + 1)} (2C_0C_1), \quad C_3 = \frac{\Gamma(6\mu + 1)}{\Gamma(7\mu + 1)} (2C_0C_2 + C_1^2), \\ C_4 &= \frac{\Gamma(8\mu + 1)}{\Gamma(9\mu + 1)} (2C_0C_3 + 2C_1C_2), \\ C_5 &= \frac{\Gamma(10\mu + 1)}{\Gamma(11\mu + 1)} (2C_0C_4 + 2C_1C_3 + C_2^2). \end{aligned}$$

The solution of the problem (14)-(15) by using the numerical method is:

$$h^{-\mu} \sum_{j=0}^n w_j^{(\mu)} y_{n-j} - y_n^2 = 1,$$

Where, $t_n = nh, y_n = y(t_n), w_j^{(\mu)} = (-1)^j \binom{\mu}{j}, (n, j = 0, 1, 2, \dots)$ therefore, we get

$$h^{-\mu} \sum_{j=0}^n (-1)^j \frac{\Gamma(\mu + 1)}{\Gamma(j + 1)\Gamma(\mu + 1)} y_{n-j} - y_n^2 = 1,$$

Then,

Figures 1-6 illustrate the comparison between ADM solution ($n = 5$) and the numerical solution ($h = 0.01$). For $\mu = 0.5$; the numerical method gives unbounded solution when $t \in [0,1]$, see Figure 1, while, ADM gives a bounded solution in the same interval, see Figure 2.

Notices:

1. All computations and figures made using MATHEMATICA software for all the given examples.
2. In all figures, the solid curve represents ADM solution, while the other curve for the other method.

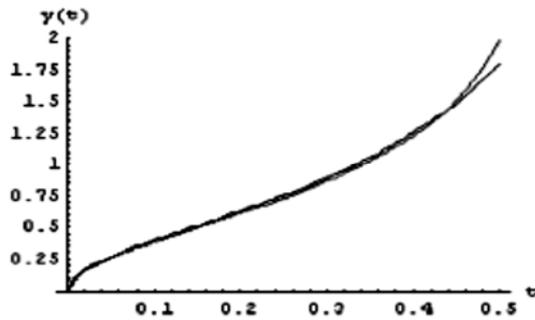


Figure 1. ADM and Numerical Sol. [$\mu = 0.5$].

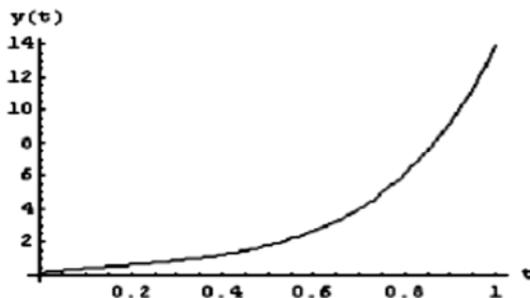


Figure 2. ADM Sol. [$\mu = 0.5$].

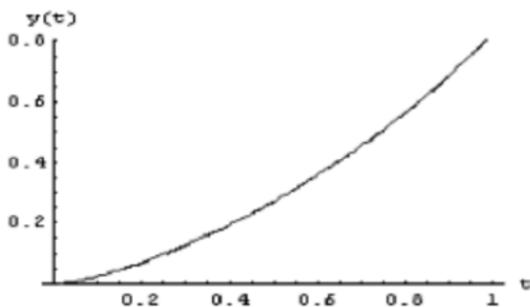


Figure 3. ADM and Numerical Sol. [$\mu = 1.5$].

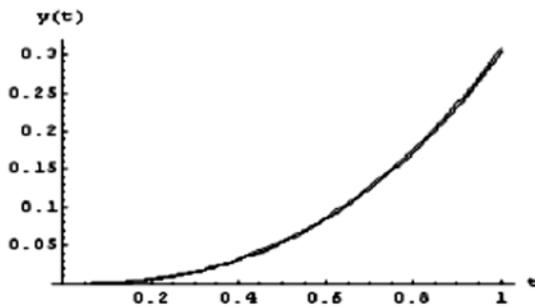


Figure 4. ADM and Numerical Sol. [$\mu = 2.5$].

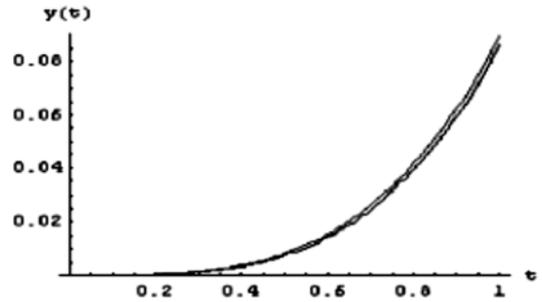


Figure 5. ADM and Numerical Sol. [$\mu = 3.5$].

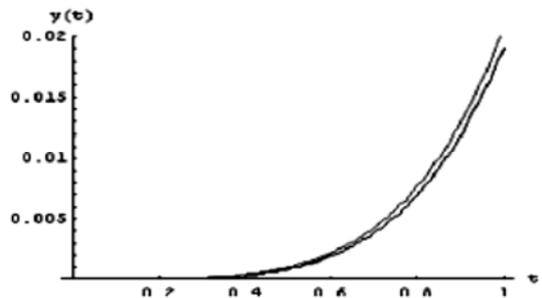


Figure 6. ADM and Numerical Sol. [$\mu = 4.5$].

Table 1 shows the relative error between exact and ADM solution of $\mu = 1$. The value $\mu = 1$ (ODE) is the only case for which we know the exact solution ($y = \tan t$) and our approximate solution is in good agreement with the exact values.

Table 1. Relative Error.

t	Exact solution	ADM solution	Relative error
0.1	0.100334672	0.100334672	$3.3195594351 \times 10^{-15}$
0.2	0.2027100355	0.2027100355	$1.4756006633 \times 10^{-11}$
0.3	0.3093362496	0.309336249	$1.921474304 \times 10^{-9}$
0.4	0.42279321874	0.42279319296	$6.097058134 \times 10^{-8}$
0.5	0.54630248984	0.54630200191	$8.931495255 \times 10^{-7}$
0.6	0.68413680834	0.68413131533	$8.0291143934 \times 10^{-6}$
0.7	0.84228838046	0.84224495232	0.0000515597
0.8	1.02963855705	1.02937191571	0.000258966
0.9	1.26015821755	1.25879891374	0.001078677
1.0	1.55740772466	1.5513676447	0.003878291

Example 2. Consider the following nonlinear FDE with nonhomogeneous initial conditions,

$$D^\mu y = \frac{9}{4}\sqrt{y} + y, t \geq 0, 1 < \mu \leq 2, \tag{18}$$

$$(0) = 1, y'(0) = 2.$$

This problem was solved by Nabil Shawagfeh in [20] by using ADM but the given solution was incorrect. Here, we give the correct solution.

Operating with J^μ on both sides of (18), we get

$$y = 1 + 2t + \frac{9}{4}J^\mu(\sqrt{y}) + J^\mu(y). \tag{19}$$

Using ADM and Adomian polynomials to the equation (19) and since the computation of A_n depends heavily on y_0 we

will use a slight modification [24]. This will ease the computations considerably. Thus,

$$y_0 = 1, \tag{20}$$

$$y_1 = 2t + \frac{9}{4}J^\mu(A_0) + J^\mu(y_0), \tag{21}$$

$$y_n = \frac{9}{4}J^\mu(A_{n-1}) + J^\mu(y_{n-1}), \quad n \geq 2. \tag{22}$$

Using relations (20)-(22), the first three-terms of the series solution for $\mu = 1.25$ are,

$$y(t) = 1 + 2t + 2.86848t^{1.25} + 1.66715t^{2.25} + 2.0781t^{2.5} + \dots, \tag{23}$$

And for $\mu = 1.5$,

$$y(t) = 1 + 2t + 2.44482t^{1.5} + 1.27883t^{2.5} + 1.15104t^3 + \dots, \tag{24}$$

And for $\mu = 1.75$,

$$y(t) = 1 + 2t + 2.02069t^{1.75} + 0.960889t^{2.75} + 0.593742t^{3.5} + \dots, \tag{25}$$

And for $\mu = 2$,

$$y(t) = 1 + 2t + \frac{13t^2}{8} + \frac{17}{768}t^3(32 + 13t) + \dots \tag{26}$$

The plots for some values of $\mu \in (1,2]$ given in figures 7-10 ($n = 5$).

In Shawaghfeh's study [20], the mistake was ignoring the third term in equation (21). A comparison between ADM solution and exact solution ($\mu = 2$) is given in Figure 10 and its relative error is given in table 2.

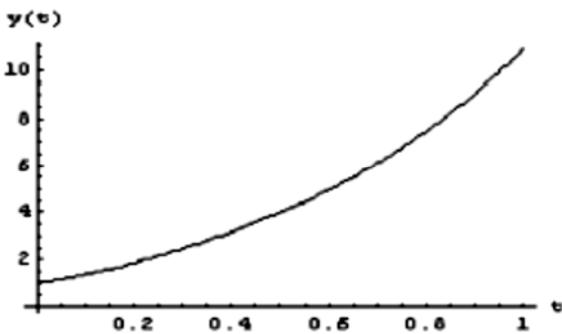


Figure 7. ADM and Exact Sol. [$\mu = 1.25$].

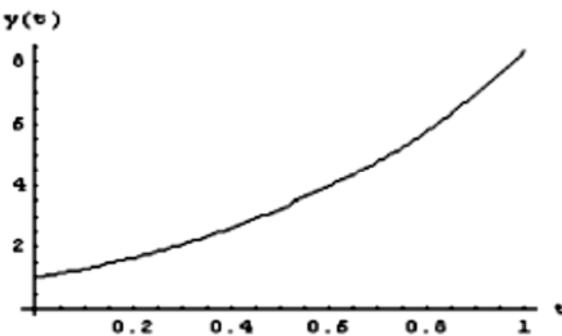


Figure 8. ADM Sol. [$\mu = 1.5$].

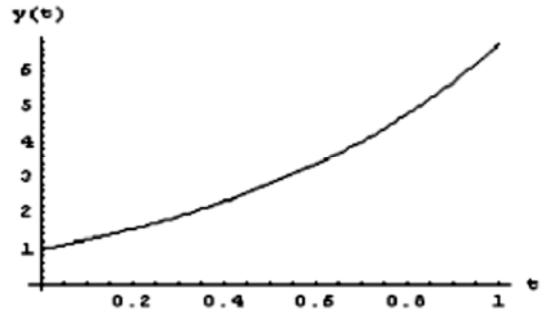


Figure 9. ADM Sol. [$\mu = 1.75$].

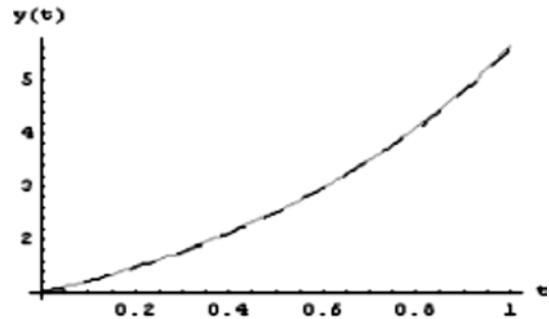


Figure 10. ADM and Exact Sol. [$\mu = 2$].

Table 2. Relative Error.

t	Exact solution	ADM solution	Relative error
0.1	1.216978142472	1.216978192217	4.08761×10^{-8}
0.2	1.470990374716	1.470993749521	2.29424×10^{-6}
0.3	1.767049231411	1.767089933573	0.0000230339715
0.4	2.110740852297	2.11098272054	0.0001145892651
0.5	2.508287449206	2.509262171435	0.0003886006884
0.6	2.966616537584	2.969687894858	0.0010353064631
0.7	3.493437642763	3.501601616183	0.0023369455119
0.8	4.097327267145	4.116482251175	0.0046749948883
0.9	4.787822988191	4.82867126594	0.0085317017462
1.0	5.575527649637	5.656298489041	0.0144866718416

Example 3. Consider the following nonlinear FDE,

$$Dy + D^{1/2}y - 2y^2 = 0, y(0) = c. \tag{27}$$

Where c is a constant.

This problem was solved by Saha Ray in [25] by using ADM but the given solution was incorrect. Here, we give the correct solution.

Operating with J^1 on both sides of equation (27), we get

$$y = c - J^1(D^{1/2}y) + 2J^1(y^2). \tag{28}$$

Then we obtain,

$$y = c + \frac{ct^{1/2}}{\Gamma(3/2)} - J^{1/2}(y) + 2J^1(y^2). \tag{29}$$

In [25], the mistake was ignoring the second term in equation (29). Using ADM and the reliable modification [24] to equation (29). Thus,

$$y_0 = c, \tag{30}$$

$$y_1 = \frac{ct^{1/2}}{\Gamma(3/2)} - J^{1/2}(y_0) + 2J^1(A_0) = 2c^2t, \tag{31}$$

$$y_n = -J^{\frac{1}{2}}(y_{n-1}) + 2J^1(A_{n-1}), n \geq 2. \quad (32)$$

From relation (32), we have

$$y_2 = -\frac{8c^2t^{3/2}}{3\sqrt{\pi}} + 4c^2t^2,$$

$$y_3 = c^2t^2 - \frac{128c^3t^{5/2}}{15\sqrt{\pi}} + 8c^4t^3, \quad (33)$$

$$y_4 = \frac{4}{105}c^2t^{5/2} \left(105c\sqrt{t}(1 + 4c^2t) - \frac{4(7+152c^2t)}{\sqrt{\pi}} \right),$$

$$\vdots$$

$$y(t) = 1 + (2t) + \left(-\frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} + 4t^2 \right) + \left(t^2 - \frac{128t^{5/2}}{15\sqrt{\pi}} + 8t^3 \right) + \dots \quad (36)$$

Consequently, the practical solution may take as n-term approximation to y as follow,

$$\phi_n = \sum_{i=0}^{n-1} y_i. \quad (37)$$

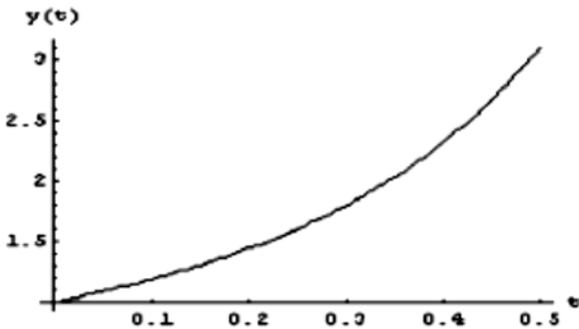


Figure 11. ADM Sol. [n=5].

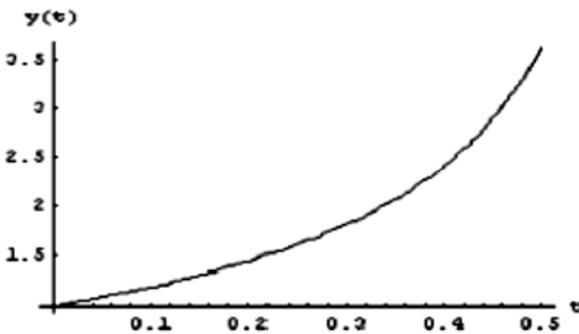


Figure 12. ADM Sol. [n=10].

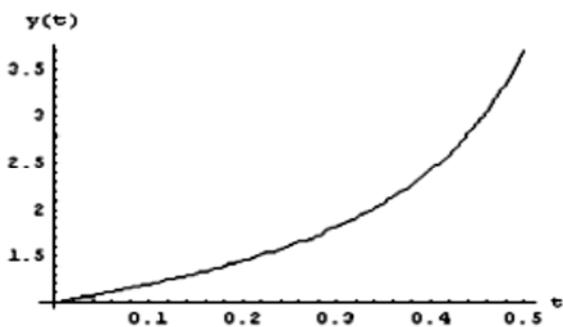


Figure 13. ADM Sol. [n=15].

Therefore, the solution of equation (27) is:

$$y(t) = c + (2c^2t) + \left(-\frac{8c^2t^{\frac{3}{2}}}{3\sqrt{\pi}} + 4c^2t^2 \right) + \left(c^2t^2 - \frac{128c^3t^{5/2}}{15\sqrt{\pi}} + 8c^4t^3 \right) + \dots \quad (34)$$

Now, consider the special case when $c = 1$, equation (27) will be:

$$Dy + D^{1/2}y - 2y^2 = 0, y(0) = 1. \quad (35)$$

From equation (34), the solution of equation (35) will be,

Figures 11-13 show ADM solution of problem (27) at different values of n . Table 3 shows the absolute error between the truncated series at different values of n . We see from this table that, as the number of terms n increases, the error will be decrease.

Table 3. Absolute Error.

t	$ \phi_5 - \phi_3 $	$ \phi_{10} - \phi_5 $	$ \phi_{15} - \phi_{10} $	$ \phi_{20} - \phi_{15} $
0.0	0	0	0	0
0.1	0.00234004	0.000076351	1.99985×10^{-8}	4.93579×10^{-12}
0.2	0.0179593	0.0010773	8.40649×10^{-7}	2.46319×10^{-9}
0.3	0.0833554	0.0130684	0.0000717527	7.04296×10^{-7}
0.4	0.260574	0.0974659	0.00419549	0.000197369
0.5	0.637506	0.507904	0.0890769	0.0161493

Example 4. Consider the nonlinear FDE,

$$D^{1/2}y - y^2 = \Gamma\left(\frac{3}{2}\right) - t, y(0) = 0, \quad (38)$$

Which has the exact solution $t^{1/2}$.

Applying $J^{1/2}$ to both sides of equation (38), we obtain

$$y = t^{1/2} - \frac{t^{3/2}}{\Gamma(5/2)} + J^{1/2}(y^2). \quad (39)$$

Using ADM to the equation (39), we get

$$y_0 = t^{1/2} - \frac{t^{3/2}}{\Gamma(5/2)}, \quad (40)$$

$$y_n = J^{\frac{1}{2}}(A_{n-1}), n \geq 1. \quad (41)$$

From the relations (40) and (41), we have

$$y_0 = \sqrt{t} - \frac{4t^{\frac{3}{2}}}{3\sqrt{\pi}},$$

$$y_1 = \frac{4t^{\frac{3}{2}}(105\pi - 224\sqrt{\pi}t + 128t^2)}{315\pi^{\frac{3}{2}}}, \quad (42)$$

$$y_2 = \frac{128t^{\frac{5}{2}}(72765\pi^{\frac{3}{2}} - 216216\pi t + 22580\sqrt{\pi}t^2 - 81920t^3)}{3274425\pi^{\frac{5}{2}}},$$

$$\vdots$$

From equation (42), the solution of equation (38) will be,

$$y(t) = \left(\sqrt{t} - \frac{4t^{3/2}}{3\sqrt{\pi}}\right) + \left(\frac{4t^{3/2}(105\pi - 224\sqrt{\pi}t + 128t^2)}{315\pi^{3/2}}\right) + \left(\frac{128t^{5/2}(72765\pi^{3/2} - 216216\pi t + 22580\sqrt{\pi}t^2 - 81920t^3)}{3274425\pi^{5/2}}\right) + \dots \quad (43)$$

Consequently, the practical solution may take as,

$$\phi_n = \sum_{i=0}^{n-1} y_i. \quad (44)$$

Figures 14-15. It is clear that, as the number of terms n increases, the solution will be more accurate.

Example 5. Consider the nonlinear FDE,

$$D^{3/2}y = \frac{1}{2}y^2 + t^2, \quad 0 < t \leq 1, \quad (45)$$

$$y(t) = \left(\frac{32t^{7/2}}{105\sqrt{\pi}}\right) + \left(\frac{1048576t^{17/2}}{1206079875\sqrt{\pi}}\right) + \left(\frac{140737488355328t^{27/2}}{57788429686596984375\sqrt{\pi}}\right) + \dots \quad (48)$$

The comparison between ADM and exact solution given in figures 16-19 show ADM solution of problem (45) at different values of m .

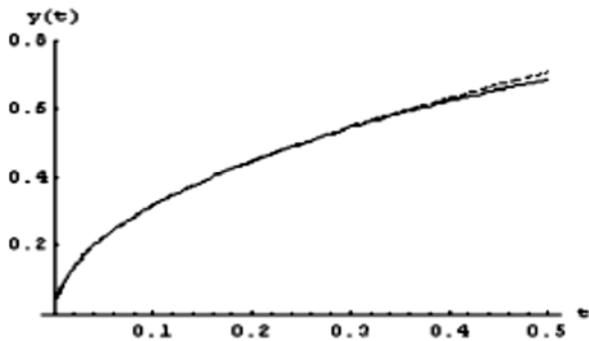


Figure 14. ADM and Exact Solutions. [n=5].

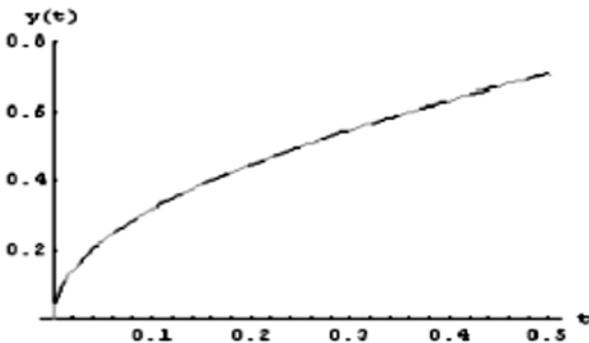


Figure 15. ADM and Exact Solutions. [n=10].

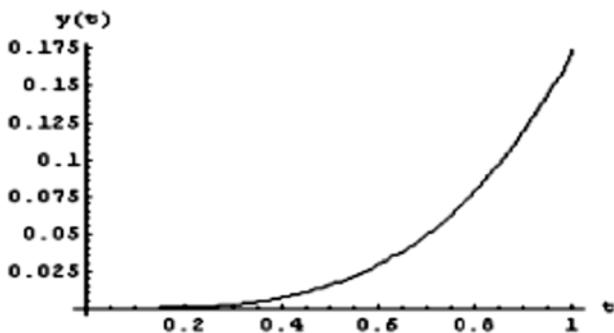


Figure 16. ADM Sol. [m=5].

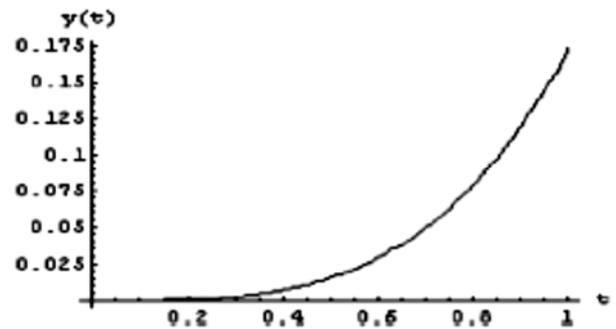


Figure 17. ADM Sol. [m=10].

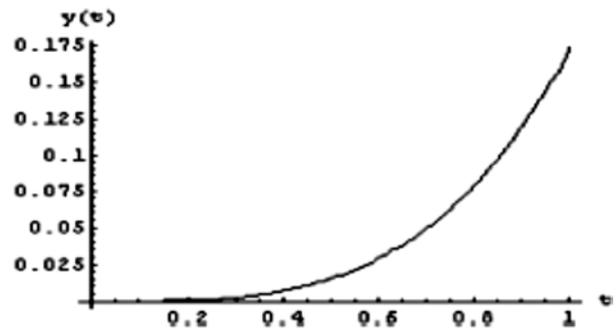


Figure 18. ADM Sol. [m=15].

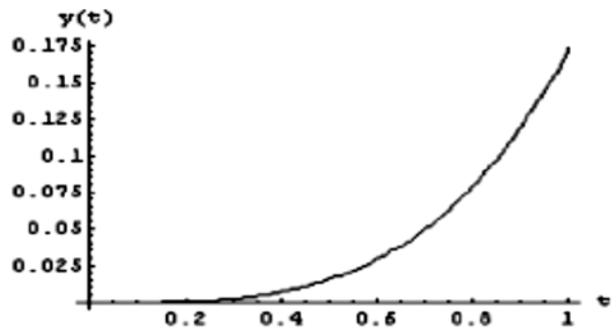


Figure 19. ADM Sol. [m=20].

Now, we will use Theorem 3 to evaluate the maximum absolute truncated error of the series solution (48). So, we evaluate the following values,

- 1) Lipschitz constant (L):

$$|f(y) - f(z)| = |y^2 - z^2| \leq |y + z||y - z| \leq 2|y - z|$$

$$\Rightarrow L = 2.$$

2) $M |a(\tau)| \leq \frac{1}{2} \Rightarrow M = \frac{1}{2}.$

3) $\alpha: \alpha = \frac{LMT^\mu}{\Gamma(\mu+1)} = \frac{1}{\Gamma(5/2)}.$

4) $\max_{t \in J} |y_1(t)| = \frac{1048576}{1206079875\sqrt{\pi}}$

5) The maximum error:

$$\max_{t \in J} \left| y(t) - \sum_{i=0}^m y_i(t) \right| \leq \frac{\alpha^m}{1 - \alpha} \max_{t \in J} |y_1(t)|$$

1) For $m = 5$:

$$\max_{t \in J} \left| y(t) - \sum_{i=0}^5 y_i(t) \right| \leq 0.00047693.$$

2) For $m = 10$:

$$\max_{t \in J} \left| y(t) - \sum_{i=0}^{10} y_i(t) \right| \leq 0.000114889.$$

3) For $m = 15$:

$$y(t) = \left(\frac{16t^{7/2}}{105\sqrt{\pi}} \right) + \left(\frac{274877906944t^{33/2}}{2009196669692953125\pi^{3/2}} \right) + \left(\frac{633825300114114700748351602688t^{59/2}}{5816608736316564766348394981381260458984375\pi^{5/2}} \right) + \dots \quad (52)$$

Figures 20-23 show ADM solution of problem (49) at different values of m .

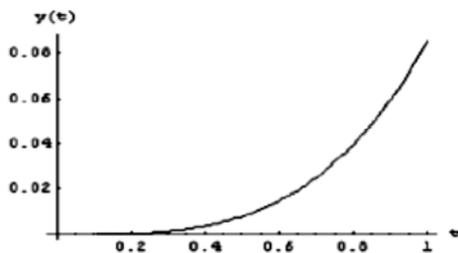


Figure 20. ADM Sol. [m=5].

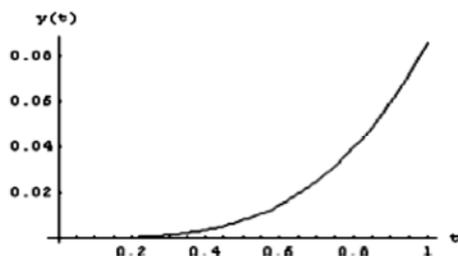


Figure 21. ADM Sol. [m=10].

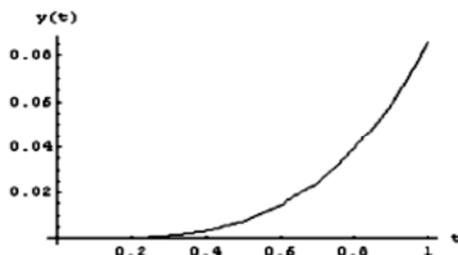


Figure 22. ADM Sol. [m=15].

$$\max_{t \in J} \left| y(t) - \sum_{i=0}^{15} y_i(t) \right| \leq 0.0000276756.$$

4) For $m = 20$:

$$\max_{t \in J} \left| y(t) - \sum_{i=0}^{20} y_i(t) \right| \leq 6.66678 \times 10^{-6}.$$

Example 6. Consider the nonlinear FDE,

$$D^{5/2}y = \frac{1}{4}y^4 + t, \quad 0 < t \leq 1, \quad (49)$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0.$$

Using ADM to the equation (49), we get

$$y_0 = J^{5/2}(t), \quad (50)$$

$$y_n = \frac{1}{4}J^{5/2}(A_{n-1}), \quad n \geq 1. \quad (51)$$

From the relations (50) and (51), the first three-terms of the series solution are,

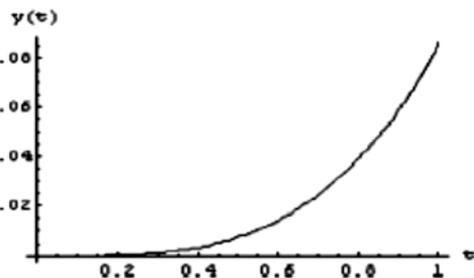


Figure 23. ADM Sol. [m=20].

Now, we will evaluate the maximum absolute truncated error of the series solution (52). Therefore, we evaluate the following values,

1) $L: |f(y) - f(z)| = |y^4 - z^4| \leq |y^2 + z^2||y + z||y - z| \leq 4|y - z| \Rightarrow L = 4.$

2) $M: |a(\tau)| \leq \frac{1}{4} \Rightarrow M = \frac{1}{4}.$

3) $\alpha: \alpha = \frac{LMT^\mu}{\Gamma(\mu+1)} = \frac{1}{\Gamma(7/2)}.$

4) $\max_{t \in J} |y_1(t)| = \frac{274877906944}{2009196669692953125\pi^{3/2}}.$

5) The maximum error:

a) For $m = 5$:

$$\max_{t \in J} \left| y(t) - \sum_{i=0}^5 y_i(t) \right| \leq 8.66908 \times 10^{-11}.$$

b) For $m = 10$:

$$\max_{t \in J} \left| y(t) - \sum_{i=0}^{10} y_i(t) \right| \leq 2.13841 \times 10^{-13}.$$

c) For $m = 15$:

$$\max_{t \in J} \left| y(t) - \sum_{i=0}^{15} y_i(t) \right| \leq 5.27486 \times 10^{-16}.$$

d) For $m = 20$:

$$\max_{t \in J} \left| y(t) - \sum_{i=0}^{20} y_i(t) \right| \leq 1.30116 \times 10^{-18}.$$

4. Conclusion

In this paper, an interesting method (ADM) used to solve fractional differential equations. This method gives analytical solution and when we comparing ADM solution with the exact solution and Numerical solution method, we see that it gives a good approximate solution and it enclosed to the results obtained from using Theorem 3.

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