
Topological Structure of Fuzzy PU-new Ideal in PU-algebra

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Abstract: In this manuscript, we consider the fuzzification of the notion PU-new ideal of PU-algebra and define the fuzzy topological terms such that fuzzy topology, τ -open fuzzy set, fuzzy neighborhood, fuzzy interior, sequence of fuzzy sets, fuzzy neighborhood system, fuzzy continuity of a function with respect to PU-new ideal of PU-algebra. We explore the new theorems and related properties of above mention notions with respect to PU-new ideal on PU-algebras. Such that for (Z, τ) to be a TSFP on Z and the set \tilde{F} is a fuzzy in Z and $N_{\tilde{F}}$ be a fuzzy neighborhood system of \tilde{F} then the finite intersection of elements of ' $N_{\tilde{F}}$ ' is also an element of ' $N_{\tilde{F}}$ ' also any fuzzy set of Z which contains an element of " $N_{\tilde{F}}$ " is also an element of " $N_{\tilde{F}}$ ". Furthermore we prove the conditions with respect to fuzzy neighborhood, convergence of a sequence of fuzzy sets, fuzzy interior set of a fuzzy set under which a fuzzy set \tilde{F} is τ -open. We show that how the function Ψ from (Z_1, τ) to (Z_2, ω) is fuzzy continuous. We prove that if Ψ is a fuzzy continuous function then for every fuzzy set \tilde{F} in Z_1 , inverse of each neighborhood of $\Psi(\tilde{F})$ is a neighborhood of a fuzzy set \tilde{F} .

Keywords: PU-algebra, Fuzzy PU-new Ideal, Fuzzy Topology, Topology of Fuzzy PU-new Ideal, Fuzzy Continuity

1. Introduction

The concept of topological space is one of the important concepts that deal with sets theory and this notion is discussed in different fields [1-13]. Imai and Iseki [14] introduced two classes of abstract algebras, which are the BCK – algebra and the BCI – algebra [15]. Next, a wide class of abstract algebras (BCH –algebras) was introduced by Hu [16] in 1983 and Li [17] in 1985. Further, they showed that the classes of BCK – algebras were proper subclasses of the Khalil 98 classes of BCI – algebras and the classes of BCI – algebras were proper subclasses of the classes of BCH-algebras. Next, some new classes of algebras were given [18-22]. Also, many applications on set theory of groups and topological spaces are shown [23-28]. A Fuzzy set, which is a class of objects with a continuum of grades of membership, was a concept proposed by Zadeh [29] in 1965. After that many concepts were introduced on Fuzzy sets like fuzzy topology by Chang [30] in 1968 and in recent years, some interesting studies and applications on fuzzy sets have been discussed by Shuker [31-35] and others. Jun [36] combined the structure of Fuzzy topological spaces with that

of a Fuzzy BCK-algebra to formulate the theory of Fuzzy topological BCK-algebras. In 1999, the concept of d – algebra, which is another generalization of BCK-algebras, is introduced by Neggers and Kim [37]. Also, the notion of d –ideal in d –algebra is discussed by Jun, Neggers and Kim [38]. After that, they introduced the notions of fuzzy d –sub algebra, fuzzy d –ideal, fuzzy B – algebras, fuzzy BCI – algebras and the relations among them are shown [39-41]. Keawrahan and Leerawat [42] determined another class of abstract algebras known as SU-Algebra. In 2018, Muralikrishna and Vijayan [43] combined SU-algebras and fuzzy topology. During 2015 Mostafa [44] introduced a new structured algebra known as PU-algebra while fuzzy PU-new ideal of PU-algebra have been discussed by Mostafa [45].

In this manuscript, we consider the fuzzification of the notion PU-new ideal of PU-algebra and define the fuzzy topological terms such that fuzzy topology, τ -open fuzzy set, fuzzy neighborhood, fuzzy interior, sequence of fuzzy sets, fuzzy neighborhood system, fuzzy continuity of a function with respect to PU-new ideal of PU-algebra. We explore the new theorems and related properties of above mention notions with respect to PU-new ideal on PU-algebras. Furthermore we prove that under what conditions a fuzzy set

\hat{F} is τ -open. We show that how the function Ψ from (Z_1, τ) to (Z_2, ω) is fuzzy continuous. we prove that if Ψ is a fuzzy continuous function then for every fuzzy set \hat{F} in Z_1 , inverse of each neighborhood of $\Psi(\hat{F})$ is a neighborhood of \hat{F} .

2. Preliminaries

In this section we take into account some of the basic definitions which play an essential role in the proof of theorems.

Definition (1): Let $Z \neq \{\}$ then the mapping $T: Z \rightarrow [0, 1]$ is called a fuzzy set of Z .

Definition (2): Suppose that T_1 and T_2 are any two fuzzy sets of Z then $T_1 \cup T_2$ is also a fuzzy set of Z which is defined as

$$(T_1 \cup T_2)(z) = \max \{ (T_1)(z), (T_2)(z) \}, \text{ for any } z \in Z.$$

Definition (3): Suppose that T_1 and T_2 are any two fuzzy sets of Z then $T_1 \cap T_2$ is also a fuzzy set of Z which is defined as

$$(T_1 \cap T_2)(z) = \min \{ (T_1)(z), (T_2)(z) \}, \text{ for any } z \in Z.$$

Definition (4): Suppose that T_1 and T_2 are any two fuzzy sets of a set Z and if $T_1 \subset T_2$ then $(T_1)(z) \leq (T_2)(z), \forall z \in Z$.

Definition (5): Suppose that T is any fuzzy set of a set Z then the complement T' of T is defined by $T'(z) = 1 - T(z), \forall z \in Z$.

Definition (6): Suppose that τ is a family of fuzzy sets in a set Z then τ is said to be a fuzzy topology if it satisfies the following conditions

(τ_i): $\bar{0}, \bar{1} \in \tau$ where $\bar{0} = T(z) = 0, \forall z \in Z$ and $\bar{1} = T(z) = 1, \forall z \in Z$.

(τ_{ii}): If $T_1, T_2 \in \tau$ then $T_1 \cap T_2 \in \tau$.

(τ_{iii}): If $T_i \in \tau$ for all $i \in I$ then $\cup_{i \in I} T_i \in \tau$, where I represents the indexing set.

Definition (7): Let Z is a non-vacuous set and τ be a family of fuzzy sets in Z then the pair (Z, τ) is termed as a fuzzy topological space while the elements of τ are termed as τ -open fuzzy sets in Z .

Definition (8): Suppose that (Z, τ) is a fuzzy topological space and let the fuzzy set T_1 be in Z . Then a fuzzy set $T_2 \in \tau$ is defined to be a neighborhood of T_1 if there exists a τ -open fuzzy set δ such that $T_1 \subset \delta \subset T_2$ i.e. $(T_1)(z) \leq \delta(z) \leq (T_2)(z), \forall z \in Z$.

Definition (9): Let T_1 and T_2 are any two fuzzy sets in (Z, τ) and let $T_1 \supset T_2$ then T_2 is said to be an interior of T_1 if T_1 is a neighborhood T_2 .

Definition (10): If T_1 is any fuzzy set in (Z, τ) then all of its fuzzy interior sets union is also its interior which is represented by T_1° .

Definition (11): A PU-algebra $(Z, *, 0)$ is a class of the type $(2, 0)$ algebras satisfying (PU-i) and (PU-ii) conditions for all $a, b, c \in Z$, where

(PU-i) $0 * a = 0$

(PU-ii) $(a * c) * (b * c) = b * a$

Example (1): Suppose that $Z = \{0, a, b, c\}$ in which binary

operation $*$ is defined

Table 1. Tabular arrangement of the values of the set Z satisfying the axioms of PU-algebra.

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Shows $(Z = \{0, a, b, c\}, *)$ is a PU-Algebra.

Definition (12): Let $(Z, *, 0)$ is a PU-algebra and let \hat{F} be a fuzzy subset in Z then \hat{F} is called a fuzzy PU-Subalgebra of Z if $\hat{F}(z_1 * z_2) \geq \min \{ \hat{F}(z_1), \hat{F}(z_2) \}, \forall z_1, z_2 \in Z$.

Definition (13): Let $(Z, *, 0)$ is a PU-algebra and let \hat{F} be a fuzzy subset in Z then \hat{F} is called a fuzzy PU new-ideal of Z if $\hat{F}((z_1 * (z_2 * z_3)) * z_3) \geq \min \{ \hat{F}(z_1), \hat{F}(z_2) \}, \forall z_1, z_2, z_3 \in Z$.

3. Topological Structure of Fuzzy PU-new Ideal (TSFP)

In this portion we describe the topological structure of fuzzy PU-new ideal (TSFP) on PU-Algebra $(Z, *, 0)$. For simplicity we use the notations $\hat{F}_1, \hat{F}_2, \hat{F}_3$ etc. for representing the elements of " τ ".

Definition (14): Suppose that $(Z, *, 0)$ is a PU-Algebra then the pair (Z, τ) is said to be a topological structure of a fuzzy PU-new ideal (TSFP) on a PU-algebra $(Z, *, 0)$ if the family τ of fuzzy PU-new ideals in Z holds the following axioms.

(τ_1): $\bar{0}, \bar{1} \in \tau$

(τ_2): If any $\hat{F}_1, \hat{F}_2 \in \tau \implies \hat{F}_1 \cap \hat{F}_2 \in \tau$.

(τ_3): If $\hat{F}_i \in \tau$ for all $i \in I \implies \cup_{i \in I} \hat{F}_i \in \tau$ where I is an indexing set.

Definition (15): If (Z, τ) is a topological structure of a fuzzy PU-new ideal (TSFP) than any member of τ is called τ -open fuzzy set in the PU-Algebra Z .

Definition (16): Let us consider the PU-Algebra which has been defined in [example 2.12] therefore for that algebra we define the fuzzy PU-new ideals $\hat{F}_i, i = 1, 2, 3, 4, 5, 6$ on Z as follows

$$\hat{F}_1(z) = \begin{cases} 0.7; 0 \\ 0.4; a \\ 0.4; b \\ 0.5; c \end{cases} \quad \hat{F}_2(z) = \begin{cases} 0.6; 0 \\ 0.3; a \\ 0.3; b \\ 0.4; c \end{cases}$$

$$\hat{F}_3(z) = \begin{cases} 0.8; 0 \\ 0.5; a \\ 0.5; b \\ 0.6; c \end{cases} \quad \hat{F}_4(z) = \begin{cases} 0.5; 0 \\ 0.2; a \\ 0.2; b \\ 0.3; c \end{cases}$$

$$\hat{F}_5(z) = \begin{cases} 0.9; 0 \\ 0.5; a \\ 0.5; b \\ 0.7; c \end{cases} \quad \hat{F}_6(z) = \begin{cases} 0.7; 0 \\ 0.4; a \\ 0.4; b \\ 0.6; c \end{cases}$$

Then $(Z, \tau = \{ \bar{0}, \bar{1}, \hat{F}_1, \hat{F}_2, \hat{F}_3, \hat{F}_4, \hat{F}_5, \hat{F}_6 \})$ is a TSFP on a PU-algebra

Definition(17): Let (Z, τ) is a topological structure of a fuzzy PU-new ideal (TSFP) on a PU-Algebra Z then the

fuzzy set \hat{F}_1 in (Z, τ) is defined to be a fuzzy neighborhood of \hat{F}_2 if there exists a τ -open set θ so that

$$\hat{F}_2 \subset \theta \subset \hat{F}_1 \text{ or } \hat{F}_2(z) \leq \theta(z) \leq \hat{F}_1(z), \forall z \in Z$$

Definition (18): Consider the PU-Algebra which has been defined in [example 2.12] and the TSFP (Z, τ) which has been defined in example 3.3 from these examples we have noted that \hat{F}_5 is a fuzzy neighborhood of a fuzzy set \hat{F}_2 such that

$$\hat{F}_2(z) \leq \hat{F}_3(z) \leq \hat{F}_5(z)$$

Definition (19): Let \hat{F}_1 and \hat{F}_2 are any two fuzzy sets in a TSFP (Z, τ) on a PU-algebra Z and suppose that $\hat{F}_1 \supset \hat{F}_2$ then \hat{F}_2 is called a fuzzy interior of $\hat{F}_1 \iff \hat{F}_1$ is a fuzzy neighborhood of \hat{F}_2 .

Remarks (1): If \hat{F}_1 is any fuzzy set in (Z, τ) then all of its fuzzy interior sets union is also its interior which is designated by \hat{F}_1° .

Example (2): Consider the PU-Algebra which has been defined in example 2.12 and the TSFP (Z, τ) which has been defined in example 3.3. From these examples we have noted that \hat{F}_5 is a fuzzy neighborhood of fuzzy sets $\hat{F}_1, \hat{F}_2, \hat{F}_4, \hat{F}_6$ and

$$\hat{F}_2(z) \leq \hat{F}_1(z) \leq \hat{F}_5(z), \forall z \in Z$$

$$\hat{F}_1(z) \leq \hat{F}_3(z) \leq \hat{F}_5(z), \forall z \in Z$$

$$\hat{F}_4(z) \leq \hat{F}_1(z) \leq \hat{F}_5(z), \forall z \in Z$$

$$\hat{F}_6(z) \leq \hat{F}_3(z) \leq \hat{F}_5(z), \forall z \in Z$$

Which shows that $\hat{F}_1, \hat{F}_2, \hat{F}_4, \hat{F}_6$ are the fuzzy interiors of \hat{F}_5 and

$\hat{F}^\circ = \cup \{ \hat{F}_1, \hat{F}_2, \hat{F}_4, \hat{F}_6 \} = \max \{ \hat{F}_1(z), \hat{F}_2(z), \hat{F}_4(z), \hat{F}_6(z) \} = \hat{F}_6(z)$, which implies that \hat{F}° is also an interior of \hat{F}_5 .

Definition (20): Let (Z, τ) is a TSFP on Z and let a set $\{ \hat{F}_n : n \geq 1 \wedge n \in \mathbb{N} \}$ is a sequence of fuzzy sets in a PU-algebra Z such a set of sequences is said to be contained in \hat{F} Where \hat{F} is a fuzzy set \iff there exists $i \in \mathbb{N}$ so that $n \geq i \implies \hat{F}_n \subset \hat{F}$.

Definition (21): Let (Z, τ) is a TSFP on Z then a sequence $\{ \hat{F}_n : n \geq 1 \wedge n \in \mathbb{N} \}$ of fuzzy PU-new ideals in Z converges to a fuzzy PU-new ideal \hat{F} in $Z \iff$ every fuzzy neighborhood of \hat{F} contains such a sequence.

Definition (22): Consider the PU-Algebra which has been defined in [example 2.12]. For that algebra we determine the fuzzy PU-new ideals $\hat{F}_i, i = 1, 2, 3, 4, 5, 6, 7, 8$ on Z as follows.

$$\hat{F}_1(z) = \begin{cases} 0.7; 0 \\ 0.4; a \\ 0.4; b \\ 0.5; c \end{cases} \hat{F}_2(z) = \begin{cases} 0.6; 0 \\ 0.3; a \\ 0.3; b \\ 0.4; c \end{cases}$$

$$\hat{F}_3(z) = \begin{cases} 0.8; 0 \\ 0.5; a \\ 0.5; b \\ 0.6; c \end{cases} \hat{F}_4(z) = \begin{cases} 0.5; 0 \\ 0.2; a \\ 0.2; b \\ 0.3; c \end{cases}$$

$$\hat{F}_5(z) = \begin{cases} 0.9; 0 \\ 0.5; a \\ 0.5; b \\ 0.7; c \end{cases} \hat{F}_6(z) = \begin{cases} 0.7; 0 \\ 0.4; a \\ 0.4; b \\ 0.6; c \end{cases}$$

$$\hat{F}_7(z) = \begin{cases} 0.4; 0 \\ 0; a \\ 0; b \\ 0.1; c \end{cases} \hat{F}_8(z) = \begin{cases} 0.8; 0 \\ 0.2; a \\ 0.2; b \\ 0.3; c \end{cases}$$

Then $(Z, \tau = \{ \bar{0}, \bar{1}, \hat{F}_1, \hat{F}_2, \hat{F}_3, \hat{F}_4, \hat{F}_5, \hat{F}_6, \hat{F}_7, \hat{F}_8 \})$ is a TSFP on a PU-algebra Z and take

$$\hat{F} = \hat{F}_4(z) = \begin{cases} 0.5; 0 \\ 0.2; a \\ 0.2; b \\ 0.3; c \end{cases}$$

Consider the sequence $\hat{F}_n = \{ \hat{F}_1, \hat{F}_2, \hat{F}_6, \hat{F}_7 \}$ while $\hat{F}_4(z) \leq \hat{F}_8(z) \leq \hat{F}_3(z)$ and $\hat{F}_4(z) \leq \hat{F}_8(z) \leq \hat{F}_5(z)$ thus $\hat{F}_3(z)$ and $\hat{F}_5(z)$ are the neighborhoods of \hat{F} and \hat{F}_n is contained in \hat{F}_3 and \hat{F}_5 . Ultimately the sequence \hat{F}_n converges to \hat{F} .

Theorem (1): Let (Z, τ) is a TSFP on Z . A fuzzy set \hat{F} is τ -open iff for every fuzzy set G that \hat{F} contains, \hat{F} is a fuzzy neighborhood of G .

Proof: Let the fuzzy set \hat{F} is τ -open. Let G be any fuzzy set contained in \hat{F} so that $G \subset \hat{F}$. Since \hat{F} is open therefore we have $G \subset \hat{F} \subset \hat{F}$ which implies that \hat{F} is a fuzzy neighborhood of G .

Conversely, for every fuzzy set G that \hat{F} contains, \hat{F} is a fuzzy neighborhood of G . it is obvious that $\hat{F} \subset \hat{F}$ by our supposition \hat{F} is a fuzzy neighborhood of \hat{F} therefore there exists τ -open set θ so that $\hat{F} \subset \theta \subset \hat{F}$. Which implies $\hat{F} = \theta \implies \hat{F}$ is τ -open.

Definition (23): Let (Z, τ) is a TSFP on a PU-Algebra. Let the set \hat{F} be a fuzzy in Z then the set that contains all fuzzy neighborhoods of \hat{F} is called fuzzy neighborhood system of \hat{F} which we denote by " $N_{\hat{F}}$ ".

Theorem (2): Let (Z, τ) is a TSFP on Z and the set \hat{F} is supposed to a fuzzy in Z and $N_{\hat{F}}$ be a fuzzy neighborhood system of \hat{F} then

- (i) The finite intersection of elements of ' $N_{\hat{F}}$ ' is also an element of ' $N_{\hat{F}}$ '.
- (ii) Any fuzzy set of Z which contains an element of " $N_{\hat{F}}$ " is also an element of " $N_{\hat{F}}$ ".

Proof (i): Let (Z, τ) be a TSFP on Z . Let the set \hat{F} is a fuzzy in Z and $N_{\hat{F}}$ be a fuzzy neighborhood system of \hat{F} . Let $G, H \in N_{\hat{F}}$ then G and H are the neighborhoods of \hat{F} therefore there exist open fuzzy sets G_0 and H_0 of G and H respectively so that

$$\hat{F} \subset G_0 \subset G \tag{1}$$

$$\hat{F} \subset H_0 \subset H \tag{2}$$

from (1) and (2) we have

$$\hat{F} \subset G_0 \cap H_0 \subset G \cap H \tag{3}$$

Equation (3) reveals that $G \cap H$ is a fuzzy neighborhood of $\hat{F} \implies$ intersection of two elements of $N_{\hat{F}}$ is also an element of

$N_{\dot{F}}$ and it conclusively results that the finite intersection of elements of $N_{\dot{F}}$ is also an element of $N_{\dot{F}}$.

Proof (ii): Let G is a fuzzy set containing an element of $N_{\dot{F}}$ and let that element is 'J'. This means that G contains a neighborhood J of \dot{F} that is $J \subset G$ where $J \in N_{\dot{F}}$. Since J is a fuzzy neighborhood of \dot{F} therefore there must exists τ -open set ' θ ' such that $\dot{F} \subset \theta \subset J \subset G \implies \dot{F} \subset \theta \subset G \implies G$ is a neighborhood of \dot{F} . Hence $G \in N_{\dot{F}}$.

Theorem (3): Let (Z, τ) be a TSFP on Z . Let the fuzzy set \dot{F} be in Z then

- (i) \dot{F}^0 is τ -open as well as the largest τ -open fuzzy set contained in \dot{F} .
- (ii) \dot{F} is τ -open $\iff \dot{F}^0 = \dot{F}$.

Proof (i): Let (Z, τ) be a TSFP on Z . Suppose that fuzzy set \dot{F} is in Z . By fuzzy interior set definition, the fuzzy interior set of \dot{F} is again a fuzzy interior which we denote by \dot{F}^0 . Therefore there exists a τ -open set ' θ ' such that

$$\dot{F}^0 \subset \theta \subset \dot{F} \implies \dot{F}^0 \subset \theta \tag{4}$$

While the set θ is a fuzzy interior of \dot{F} therefore we have

$$\theta \subset \dot{F}^0 \tag{5}$$

from (3) and (4) we get $\dot{F}^0 = \theta \implies \dot{F}^0$ is τ -open as well as the largest τ -open fuzzy set that \dot{F} contains.

Proof (ii): Let the fuzzy set \dot{F} is τ -open then

$$\dot{F} \subset \dot{F}^0 \tag{6}$$

from part (i) of this theorem \dot{F}^0 is contained in \dot{F} which implies that

$$\dot{F}^0 \subset \dot{F} \tag{7}$$

Equations (5) and (6) we get $\dot{F}^0 = \dot{F}$.

Conversely, suppose that $\dot{F}^0 = \dot{F}$ so then by remark 3.7, union of all fuzzy interior sets of \dot{F} is \dot{F}^0 which is by itself a fuzzy interior of \dot{F} . Thus \dot{F} is a fuzzy neighborhood of \dot{F}^0 . Hence fuzzy set \dot{F} is open.

Theorem (4): Let (Z, τ) be a TSFP on Z . If every fuzzy set in (Z, τ) has a countable fuzzy neighborhood system then the fuzzy set \dot{F} is τ -open iff each sequence of fuzzy sets $\{\dot{F}_n; n \geq 1 \wedge n \in \mathbb{N}\}$ converging to fuzzy set G that \dot{F} contains is eventually contained in \dot{F} .

Proof: Let \dot{F} be an τ -open fuzzy set. Let each given sequence $\{\dot{F}_n; n \geq 1 \wedge n \in \mathbb{N}\}$ of fuzzy sets converges to G . Since \dot{F} is τ -open containing G which implies that \dot{F} is a neighborhood of $G \implies \dot{F}$ contains $\{\dot{F}_n; n \geq 1 \wedge n \in \mathbb{N}\}$.

Conversely, For every $G \subset \dot{F}$ we suppose that $H_1, H_2, \dots, H_n, \dots$ is a neighborhood system of G . Let $G_n = \cup_1^n \{H_i\}$. Then for the sequence $G_1, G_2, \dots, G_n, \dots$ contained in every fuzzy neighborhood of G there exists $i \in \mathbb{N}$ so that for $n > i$, $G_n \subset \dot{F}$. Thus G_n are the fuzzy neighborhoods of G . Hence \dot{F} is τ -open.

Definition (24): Let Ψ is a mapping from Z_1 to Z_2 and v is a fuzzy set in Z_2 then Ψ^{-1} is defined as

$$v_{\Psi^{-1}}(p) = v(\Psi(p)), \forall p \in Z_1$$

Suppose that μ is a fuzzy set in Z_2 then image of such a set

is given by

$$\mu(q) = \begin{cases} \sup \mu(p)_{p \in \Psi^{-1}(q)}, & \text{if } \Psi^{-1}(q) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

Definition (25): Let (Z_1, τ) and (Z_2, ω) are the TSFP on PU-algebras Z_1 and Z_2 respectively. Then the function Ψ from (Z_1, τ) to (Z_2, ω) is said to be fuzzy continuous function if every ω -open fuzzy set has its inverse as τ -open fuzzy set.

Theorem (5): Let (Z_1, τ) and (Z_2, ω) are the TSFP on PU-algebras Z_1 and Z_2 respectively then the function Ψ from (Z_1, τ) to (Z_2, ω) is fuzzy continuous function iff any closed fuzzy set has its inverse as a closed set.

Proof: Let Ψ is a fuzzy continuous function then by definition 3.18 inverse of any ω -open fuzzy set is τ -open fuzzy set. Suppose that the fuzzy set ω is closed in Z_2 then

$$\mu_{\Psi^{-1}(\omega)}(p) = \mu_{\omega}(\Psi(p)) = \mu_{\omega}(\Psi(p)) = 1 - \mu_{\omega}(\Psi(p)) = 1 - \mu_{\Psi^{-1}(\omega)}(p) = \mu_{\Psi^{-1}(\omega)}(p) \implies \Psi^{-1}(\omega) = \{\Psi^{-1}(\omega)\}, \forall p \in Z_1$$

Since Ψ is a fuzzy continuous therefore any closed fuzzy set has its inverse as a closed set.

Conversely, suppose that open fuzzy set in Z_2 is ω . Then

$$\mu_{\Psi^{-1}(\omega)}(p) = \mu_{\omega}(\Psi(p)), \forall p \in Z_1.$$

We know that any closed fuzzy set has its inverse as a closed set therefore each open fuzzy set has its inverse as an open set $\implies \Psi$ is a fuzzy continuous.

Theorem (6): Let (Z_1, τ) and (Z_2, ω) are the TSFP on PU-algebras Z_1 and Z_2 respectively and Ψ is a function from (Z_1, τ) to (Z_2, ω) then for every \dot{F} in Z_1 where \dot{F} is a fuzzy set the inverse of any neighborhood of $\Psi(\dot{F})$ is a neighborhood of \dot{F} iff for every fuzzy set \dot{F} in Z_1 and every neighborhood N of $\Psi(\dot{F})$ there is a neighborhood M of \dot{F} such that $\Psi(M) \subset N$.

Proof: Let \tilde{F} is a set that contains all fuzzy sets of Z_1 . Let \tilde{N} and \tilde{M} are the families of neighborhoods of fuzzy sets and their images. Suppose that $\dot{F} \in \tilde{F}$. Let $N \in \tilde{N}$, $M \in \tilde{M}$ are the neighborhoods of $\Psi(\dot{F})$ and \dot{F} respectively. Since the inverse of each neighborhood of $\Psi(\dot{F})$ is a neighborhood of \dot{F} . Therefore;

$$\Psi(M) = \Psi(\Psi^{-1}(N)) \tag{8}$$

If $\Psi^{-1}(q) \neq \emptyset$

Then

$$\mu_{\Psi(\Psi^{-1}(N))}(q) = \sup \mu_{\Psi^{-1}(N)}(r)_{r \in \Psi^{-1}(q)} = \sup \{\mu_N(\Psi(r))\}_{r \in \Psi^{-1}(q)} = \mu_N(q), \text{ for all } q \in Z_2. \text{ If } \Psi^{-1}(q) \neq \emptyset, \text{ then } \mu_{\Psi(\Psi^{-1}(N))}(q) = 0.$$

Hence $\mu_{\Psi(\Psi^{-1}(N))}(q) \leq \mu_N(q)$ for all $q \in Z_2$ and $\Psi(\Psi^{-1}(N)) \subset N$,

Equation (7) implies $\Psi(M) \subset N$.

Conversely, Let N is a neighborhood of $\Psi(\dot{F})$. Since there is a neighborhood M of \dot{F} such that $\Psi(M) \subset N$.

Hence;

$$\Psi^{-1}(\Psi(M)) \subset \Psi^{-1}(N) \tag{9}$$

Then for any $p \in Z_1$ we have

$$\mu_{\Psi^{-1}(\Psi(M))}(p) = \mu_{\Psi(M)}(\Psi(p)) = \sup \{\mu_M(r)\}_{r \in \Psi^{-1}(\Psi(p))} \geq \mu_M(p).$$

Thus $M \subset \Psi^{-1}(\Psi(M))$ and equation (2) implies $M \subset \Psi^{-1}(\Psi(M)) \subset \Psi^{-1}(N)$.

Hence $\Psi^{-1}(N)$ is a neighborhood of M .

Theorem (7): Let (Z_1, τ) and (Z_2, ω) are the TSFP on PU-algebras Z_1 and Z_2 respectively and Ψ is a function from (Z_1, τ) to (Z_2, ω) . If Ψ is a fuzzy continuous function then for every fuzzy set \hat{F} in Z_1 , inverse of each neighborhood of $\Psi(\hat{F})$ is a neighborhood of \hat{F} .

Proof.: Assume that \tilde{F} contains all fuzzy sets of Z_1 . Let \tilde{N} is a family of neighborhoods on \tilde{F} . Let $\hat{F} \in \tilde{F}$ and $N \in \tilde{N}$ then the set \hat{F} being a fuzzy in Z_1 and N being a neighborhood of $\Psi(\hat{F})$. There exists an open neighborhood M of $\Psi(\hat{F})$ so that $\Psi(\hat{F}) \subset M \subset N$

And;

$$\Psi^{-1}(\Psi(\hat{F})) \subset \Psi^{-1}(M) \subset \Psi^{-1}(N) \quad (10)$$

Since Ψ is a fuzzy continuous therefore $\Psi^{-1}(M)$ is open. Moreover $\mu_{\Psi^{-1}(\Psi(\hat{F}))}(p) = \mu_{\Psi(\hat{F})}(\Psi(p)) = \sup\{\mu_{\hat{F}}(r)\}_{r \in \Psi^{-1}(\Psi(p))} \geq \mu_{\hat{F}}(p)$ for all $p \in Z_1$. Therefore, $\hat{F} \subset \Psi^{-1}(\Psi(\hat{F}))$. Thus equation (1) implies $\hat{F} \subset \Psi^{-1}(\Psi(\hat{F})) \subset \Psi^{-1}(M) \subset \Psi^{-1}(N)$ and $\subset \Psi^{-1}(M) \subset \Psi^{-1}(N)$. Hence $\Psi^{-1}(N)$ is a neighborhood of \hat{F} .

4. Conclusion

In this manuscript various important results evaluated by inserting the concept of fuzzy topology in fuzzy PU-new ideal. It can also be expanded to interval valued fuzzy and intuitionistic fuzzy substructures on PU-algebras by combining with particular framework of fuzzy topology. It has no doubt that the research along this line can be kept up, and indeed, some results in this manuscript have already made up a foundation for further exploration concerning the further progression of PU-algebras. These definitions and main results can be similarly extended to some other algebraic systems such as BCH-algebras, Hilbert algebras, BF-algebras, J-algebras, WS-algebras, CI-algebras, BCL-algebras, BP-algebras and BO-algebras, Z- algebras and so forth. The main purpose of this research is that the future work is to further investigate the fuzzy topology in PU-algebras, which may have a lot of applications in different branches of theoretical physics and computer sciences.

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