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Comparative Study of Weak Galerkin and Discontinuous Galerkin Finite Element Methods

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Abstract: FEM is a valuable approximation tool for the solution of Partial Differential Equations when the analytical solutions are difficult or impossible to obtain due to complicated geometry or boundary conditions. The Project work involved collecting facts related to WG and DG-FEMs. WG-FEM is a numerical method that was first proposed and analyzed by Wang and Ye (2013) for general second-order elliptic BVPs on triangular and rectangular meshes. DG-FEMs as developed by Cockburn et al. (1970) uses a discontinuous function space to approximate the exact solution of the equations. The comparison and numerical examples demonstrated that WG-FEMs are viable and hold some advantages over DG-FEMs, due to their properties. Numerical examples demonstrated that WGM generates a smaller linear system to solve than the DGMs. WG-FEM have less unknowns, no need for choosing penalty factor and normal flux is continuous across element interfaces compared to DG-FEMs and the implementation of WG-FEMs is easier than that of DG-FEMs based on error and convergence rate. The computations were done by hand and with the help of MATLAB 2021Rb.

Keywords: Finite Element Method (FEM), Galerkin Method, Weak Galerkin FEM, Discontinuous Galerkin FEM

1. Introduction

1.1. Background of the Study

Many problems in science and engineering can be reduced to the problem of solving differential equations. Only a few of these equations can be solved by analytical methods which are also complicated requiring use of advanced mathematical techniques. In most of these cases, it is easier to develop approximate solutions by numerical methods. The term Finite Element Method (FEM) was first coined by clough in 1960. After then, in the 1960s and 1970s mathematicians founded a theoretical basis for finite elements. In the late 1960s and early 1970s, the FEM was applied to a wide variety of engineering problems. Most of the real problems are defined on domains that are geometrically complex, may have different boundary conditions on different portions of the boundary. In simple terms, FEM is a method for dividing up a very complicated problem in to small element that can be solved in relation to each other. The advantages of dividing a big element in to small ones allow that every small has a simpler shape, which leads to a good approximate for the analysis.

The basic idea of FEM is to approximate the solution of a given differential equation with a set of algebraically simple function. In mathematics, in the area of numerical analysis, Galerkin methods are a class of methods for converting a continuous operator problem (such as a differential equation) to a discrete problem. The GM is an old numerical technique, classically used to solve differential equations not ameanable to analytical techniques. For solving a general differential equation, which is based on seeking an (approximate) solution in a finite dimensional space, we use FEMs known as the G-FEM. The area of numerical analysis comprises of several methods, GMs are classes of methods for converting a differential equation to a discrete problems. It is in principle the same as applying the method of variation of parameters to a function space, by converting the equation to a weak formulation.

In this Project we describe a FEM that is a combination of processes used in the WG and DG methods for solving second order Laplace /Poisson equations. There have been a variety of numerical methods for the model problem of WG and the DG-FEMs. All these numerical methods result in large-scale discrete linear systems, which are solved directly or iteratively.

WG-FEMs are numerical methods for solving PDEs that were first introduced by Wang and Ye [1] for solving general second order elliptic PDEs. WG refers to general FEMs for PDEs in which differential operators are approximated by weak forms through the usual integration by parts. In this Project we are concerned with computation and numerical accuracy issues for the WG method that was recently introduced [2]. They rely (depend) on numerical concepts such as weak functions, the weak gradient operator, discrete weak functions, and discrete weak gradients. There are two basic ways to construct DG methods for an elliptic problems. The first way is to add a penalty term into the bilinear form, penalizing the interelement discontinuity. This was followed up a few years later by Baker [4] who proposed the first modern DGM for elliptic problems, later followed by Wheeler et al. [3]. We have shown that how WG-FEMs and DG-FEMs compared for solving second-order elliptic BVPs using MATLAB based on convergence and error.

1.2. Statement of the Problem

In engineering situations today it is necessary to obtain approximate numerical solutions to problems due to the unavailability of exact closed-form solutions [5]. This project focused on a comparative study of the WG and DG-FEMs. The comparison has been made by tables and graphs using MATLAB.

We consider a self-adjoint linear partial differential operator of second-order which is applied to a scalar real valued function u, and the BVPs for the general second-order elliptic equation

$$\begin{cases} Lu = f, for\ x \in \Omega \\ u = 0, for\ x \in \partial\Omega \end{cases} \tag{1}$$

where, L is the elliptic operator given by

$$\mathrm{L}[u](\mathrm{x}) = -\sum_{i,j=1}^{n}\left(a_{ij}\frac{\partial^2 \mathrm{u(x)}}{\partial x_i\, \partial x_j}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_{\mathrm{i}}} + cu(x) = f(x)$$

The coefficient functions $a_{ji}(x)$ are assumed to be a sufficient smooth satisfying $a_{ij}(x) = a_{ji}(x)$ for all $i, j = 1, \dots\dots\dots\dots\dots\dots\dots, n, x \in \Omega$ and Ω is a polygonal or polyhedral domain in $\mathbb{R}^d (d = 2{,}3)$. (Partial differential operator (1) is the elliptic equaiton typed by the Poisson's operator is

$$\mathrm{L}[u](\mathrm{x}) = -\Delta \mathrm{u(x)} = -\sum_{i,j=1}^{n}\left(a_{ij}\frac{\partial^2}{\partial x_i\, \partial x_j}\right)\mathrm{u(x)} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_{\mathrm{i}}} + cu(x) = f(x) \tag{2}$$

The essence of the Project is focused on the second-order elliptic equation by considering the above equation (1) with the boundary condition of Dirichlet problem.

We compared WG and DG-FEMs using boundary value problem (BVP) for second-order elliptic equations directly:

$$\begin{cases} -\nabla.(\nabla \mathrm{u}) = \mathrm{f}, \mathrm{x} \in \Omega \\ u = 0, x \in \Gamma = \partial\Omega \end{cases} \tag{3}$$

All these FEMs are based on variational (weak) formulation: seek $u \in H_0^1(\Omega)$ such that $\forall v \in H_0^1(\Omega)$

$$\int_\Omega \nabla u . \nabla v dx = \int_\Omega f v dx$$

Finite Elements Methods:

WG-FEM: - weak continuity and aproximation ∇_w by weak gradients may good features.

Discrete weak gradient + stabilization in the solution u to enforce weak continuities.

DG-FEM: $V_h \not\subset H_0^1(\Omega)$, non-conforming FEM, penalty factor, no continuity/jump.

1.3. Objectives

The main objective of this Project is deal a Comparative study on the Weak Galerkin and Discontinuous Galerkin Finite Element Methods for solving second-order elliptic PDEs.

Hence, this project study is intended to explore the following specific objective:

1. To explain detail study of WG-FEMs and DG-FEMs.
2. To solve second-order PDEs of elliptic boundary value problem with the two methods.
3. To compare implementation issues, convergence and error of the methods.

2. Literature Review

Nowadays, FEMs are widely used in almost every field of engineering and industrial analysis. Since Courant R. [18] formulated the essence of what is now called a finite element in 1943, this method has been getting more and more attractive with the development of computers and is now recognized as one of the most versatile and powerful methods for approximating the solutions of BVPs, especially for problems over complicated domains. Among the different FEMs, the conforming FEM with continuous, piecewise polynomial approximating spaces, has long been employed to approximate solutions for PDEs. Within the past few decades, however, a number of researchers have investigated GMs based on fully discontinuous approximating spaces, such as the DG methods and the WG methods.

Today it has become a powerful tool for solving partial differential equations. By FEM, we denote a family of approaches developed to compute an approximate solution to PDEs.

In the FEM, the domain of integration is subdivided into a number of smaller regions called elements and over each of these elements the continuous function is approximation by a suitable piecewise polynomial. Its development can be traced back to the work by Alexander Hrennikoff (1941) and Richard Courant (1942). Hrennikoff's work discretizes the domain by using a lattice analogy while Courant's approach divides the domain into finite triangular subregions for solution of second order elliptic PDEs that arise from the problem of torsion of a cylinder. The FEM is a good choice for solving partial differential equations over complex domains (like cars and oil pipelines), when the domain changes (as during a solid state reaction with a moving boundary), when the desired precision varies over the entire domain, or when the solution lacks smoothness. FEM algorithms are implemented in Finite Element Analysis and engineering problems are solved using software like MATLAB. One of the approximation methods GM, invented by Russian Mathematician Boris Grigoryevich in (1915).

The work by Boris Grigoryevich [X] on the approximate solution of differential equations appears in the literature for the first time in 1915. The GM can be used to approximate the solution to ODEs, PDEs and integral equations. Deriving the governing dynamics of physical processes is a complicated task in itself; finding exact solutions to the governing partial differential equations is usually even more formidable.

Two such methods, the WG and the DG-FEMs, are typically used in the literature and are referred to as classical variational methods. According to Reddy (1993), when solving a DE by a variational method, the equation is first put into a weighted-integral form, and then the approximate solution within the domain of interest is assumed to be a linear combination ($\sum_i C_i\phi_i$) of appropriately chosen approximation functions ϕ_i and undetermined coefficients, C_i . The coefficients $C_i$are determined such that the integral statement of the original system dynamics is satisfied. Various variational methods, like WGM and DGM differ in the choice of integral form, or approximating functions. Many researchers have been using FEMs for the solutions of PDEs since Gerisch et al. (1956) have used high-order linearly implicit two-step peer - finite element methods for time independent PDEs successfully. Since the WGM is a brand new method, we would like to comment on its relation with some existing methods in literature. Due to the “discontinuous” nature of the functions in the WG-FEM space, it is sensible to make comparisons with methods employing discontinuous functions, such as DGM. WG refers to general FEMs for partial differential equations in which differential operators are approximated by weak forms through the usual integration by parts. Several variations of DGMs for second order elliptic boundary value problems have been proposed during recent years, which exhibit special convergence, conservation, error estimates and local approximation properties attractive for parallel adaptive approximations. A comprehensive account of several types of DGMs can be found in the volume edited by Cockburn et al. [17].

Two commonly used DG schemes are investigated: the original average flux proposed by Bassi and Rebay [7]. Several variations of DGMs for second order elliptic boundary value problems have been proposed during recent years, which exhibit special convergence, conservation and local approximation properties attractive for parallel adaptive hp-approximations.

Numerical treatment of WG and DG-FEMs

Iterative solvers like WG and DG are among the methods used numerically to solve elliptic BVPs. G-FEM was developed as a result of efforts to achieve this best of both worlds combination. In practical cases we often apply approximation. When referring to a GM, we also need to give the name along with typical approximation methods used, such as DG, WG or Ritz-G.

In applied mathematics, DG methods form a class of numerical methods for solving differential equations. The name of DG appears to have started to be used in the early 1980's, and to the authors knowledge the name first appears in a paper by Delfour and Trochu in 1978. Viewed from the current ideas it is the opinion of the authors that it represents a method of linking separate domains in which finite element, series, or whatever other current procedures of solution is used for approximation. We provide a common framework for the understanding, comparison, and analysis of several DGMs that have been proposed for the numerical treatment of elliptic problems. In this Project we analyze a DG-FEMs recently introduced by Bassi and Rebay [7] for the approximation of elliptic problems. Here test problems were selected for comparing the WG and DG-FEMs. The problem is the Laplace equation (Poisson’s equation), which is obviously an elliptic equation, but has the advantages of sharing a number of properties with a simple parabolic equation, an analytical solution, and a focus on spatial error, not temporal error.

3. Materials and Methods

The Project work involved collecting facts related to Weak Galerkin and Discontinuous Galerkin Finite Element Methods. Sources in the web and libraries were used to collect all the pieces of information about comparison of Weak Galerkin and Discontinuous Galerkin Finite Element Method together with the methods and recorded subsequently. The overall procedure of this project was:

1. The collected material and the techniques or methods that relates with Weak Galerkin and Discontinuous Galerkin Finite Element Method were examined.
2. Important preliminary concepts and facts were discussed to make the high light concept of Weak Galerkin Finite Element Methods and Discontinuous Galerkin Finite Element Method clearly and simply.
3. Symbolic softwares, presented MATLAB suitably to ease the computations by the methods.
4. The collected information (definition, examples, and solution methods) have been taken and organized in proper manner and examined in detail.

4. Basic Preliminaries

4.1. Finite Element Method

It is of interest to solve linear second-order elliptic boundary value problem and assume that u is a solution of our original boundary value problem, so that

$$-u'' + p(x)u' + q(x)u = f(x), \text{ for } a < x < b \quad (4)$$

$$u(a) = \alpha, u(b) = \beta,$$

where a, b, α and β are given constants p, q and f are given functions. Partition the domain $I = [a, b]$ into n parts as:

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n < x_{n+1} = b,$$

Then try to find numerical solutions on the grid:

$$u_j \approx u(x_j) \text{ for } j = 0, 1, \dots \dots \dots \dots n, n+1.$$

From the boundary conditions, we have $u_0 = \alpha, u_{n+1} = \beta$. Thus the unknowns are $c_1, c_2, \dots \dots c_n$. The FEM relies on an integral relation derived from the differential equations. Let φ be a differentiable function of x satisfying $\varphi(a) = \varphi(b) = 0$, we can multiply the differential equation of an arbitrary test function φ, obtaining

$$-u''(x)\varphi(x)dx + p(x)u'(x)\varphi(x)dx + q(x)u(x)\varphi(x)dx = f(x)\varphi(x)dx,$$

and integrating differential equation of u we get:

$$\Rightarrow -\int_a^b u'' p(x)\varphi(x)dx + \int_a^b p(x)u(x)\varphi(x)dx = \int_a^b f(x)\varphi(x)dx$$

We use integration by parts for the term involving u'', we obtain

$$\int_a^b [-\varphi' u' + p(x)\varphi u' + q(x)\varphi u - f(x)\varphi]dx = 0, \quad (5)$$

For $i = 0, \dots \dots \dots \dots \dots \dots \dots \dots \dots, n$ set $h_i = x_{i+1} - x_i$ is the length of element j.

Now, we consider the basis functions $\phi_j(x)$ defined as the continuous piecewise linear function satisfying $\phi_j(x_j) = 1$, $\phi_j(x_k) = 0$ if $k \neq j$.

More precisely, we have

$$\phi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h_i}, if x_{j-1} \le x \le x_j \\ \dfrac{x_{j+1} - x}{x_{j+1} - x_j}, if x_j \le x \le x_{j+1} \\ 0, otherwise. \end{cases}$$

The piecewise linear function obtained by connecting (x_j, u_j) by line segments is

$$u_n(x) = \sum_{j=0}^{n+1} u_j \phi_j(x)$$

$$u_n(x) = u_0\phi_0(x) + u_1\phi_1(x) + \cdots + u_{n+1}\phi_n(x) \quad (6)$$

The stiffness matrix for above equation can be given and these equations can be written as

$$\sum_{j=0}^{n+1} a_{kj} u_j = b_k$$

where, $a_{kj} = \int_a^b [-\phi_k'(x)\phi_j'(x) + p(x)\phi_k(x)\phi_j'(x) + q(x)\phi_k(x) - \phi_j(x)]dx = 0$

The load vector is

$$b_k = \int_a^b \phi_k'(x) f(x)dx.$$

This gives a tridiagonal matrix A and a load vector b such that

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & 0 & a_{n,n+1} & a_{nn} \end{bmatrix}, b = \begin{bmatrix} b_1 - a_{nn} \\ b_{21} \\ \vdots \\ a_{n-1} \end{bmatrix}$$

For $k = 1, 2, \dots \dots \dots \dots \dots \dots, n$. If $|j - k| > 1$, we observe that ϕ_k and ϕ_j are non-zero only on intervals that do not overlap. This leads to $a_{kj} = 0$ if $|j - k| > 1$.

Therefore, we have matrix-vector equation including boundary condititon

$$\begin{pmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & & & \\ & & \ddots & & \\ \vdots & \vdots & & \ddots & a_{n-1,n} \\ \cdots & \cdots & \cdots & a_{n,n+1} & a_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} b_1 - a_{10} \dots . . u_0 \\ b_2 \\ \vdots \\ \vdots \\ b_n - a_{n,n+1} \dots \dots u_{n+1} \end{pmatrix}$$

The linear system has a symmetric (tri-diagonal) matrix and can be solved using standard numerical linear algebra techniques.

Example 4.1: Solve the boundary value problem defined by

$$u''(x) - 3u(x) + x^2 = 0, \quad (7)$$

Over $x \in \Omega = 0, \dots . . , 1$ with boundary conditions $u(0) = u(1) = 0$ and using piecewise linear trial functions.

Solution

Let the trial function be the linear function $u(x) = c_1 x + c_2$. Multiplying the differential equation (7) by a test function $v(x)$ and integrating over the domain gives the weak formulation of the problem

$$\int_{\Omega} (u''(x)v(x) - 3u(x)v(x) + x^2 v(x)dx) = 0 \quad (8)$$

Because v also has to fulfill the boundary condtions, integration by parts of (8) yields:

$$\int_{x_i}^{x_{i+1}} -\frac{du}{dx}\frac{dv}{dx} - 3u(x)v(x) + x^2 v(x)dx + \int_{x_i}^{x_{i+1}} v\,(x)\frac{du}{dx}dx \quad (9)$$

$$\text{Or } \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \left(-\frac{du}{dx}\frac{dv}{dx} - 3u(x)v(x) + x^2 v(x)dx + \int_{x_i}^{x_{i+1}} v\,(x)\frac{du}{dx}dx\right)$$

where, n is the number of elements. In our case we divide the domain into n equal subdomins.

The above is the n equations we need to solve for the unknown u_i at the internal nodes.

Let us denote the basis function by ϕ. Then we can define discretized u and v as:

$$u_h(x) = \sum_{i=1}^{n} u_i \phi_i(x), v_h(x) = \sum_{i=1}^{n} v_i \phi_i(x)$$

Because in the weak formulation the functions only have to be differentiable once, we can use picewise linear basis functions:

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_i}, x \in [x_{i+1} - x_i] \\ \frac{x_{i+1} - x}{h}, x \in [x_i, x_{i+1}] \\ 0, otherwise. \end{cases}$$

Rewriting the triangle $u(x)$ interms of unknown values at nodes results in

$$u(x) = \phi_1 u_i(x) + \phi_2 u_{i+1}(x)$$

where, $\phi_1(x) = \frac{x_{i+1} - x}{h}$ and $\phi_2(x) = \frac{x - x_{i-1}}{h}$. where, h is the length of each element.

We have as unknowns $\{u_1, u_2, \dots\dots\dots, u_n\}$ and basis function is given by $\{\phi_0, \phi_1, \dots\dots\dots, \phi_n, \phi_{n+1}\}$, write in matrix form

$$\sum_{i,j}^{n} A_{ij} U_j = F_i, i = 1, \dots\dots, n$$

which we write as AU=F and $A_{ij} = \int_0^1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx$ and $F_i = \int_0^1 1\phi_i dx$

$$\frac{1}{h}\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_1 \end{bmatrix} = h\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving the system of linear equations result in:

$$\begin{bmatrix} u_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} 2h^2 \\ 3h^3 \end{bmatrix}$$

Let $n = 2$ then $h = \frac{1}{n} = \frac{1}{2} = 0.5$

$x0$	$x1$	$x2$	$x3$	$x4$	
	0	0.25	0.5	0.75	1

We derive a linear system of equations for the coefficients by substituting the approximate solution $u_1 = 0.125, u_2 = 0.1875$

$$\text{i.e., } \begin{bmatrix} u_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0.1875 \end{bmatrix} = \begin{bmatrix} 1/8 \\ 3/16 \end{bmatrix}$$

Approximate solution is a linear combination of trial functions $u_h(x) = \sum_{i=1}^{n} u_i \phi_i(x)$

In order to obtain a GM approximation to the solution of the proceeding BVP, written u_h as

$$u_h(x) = \sum_{i=1}^{2} u_i \phi_i(x) \quad (10)$$

Error: the exact solution of (4.8) is $u(x) = 2x - x^2/2$

Those from (7), $u_h(x) = \frac{1}{4}\phi_1(x) - \frac{9}{2}\phi_2(x)$.

The following graph briefly demonstrated that from example 4.1 if FEM partitioned number of element into $(n = 9)$.

Then, MATLAB presented for FEM approximate numerical solutions to BVPs.

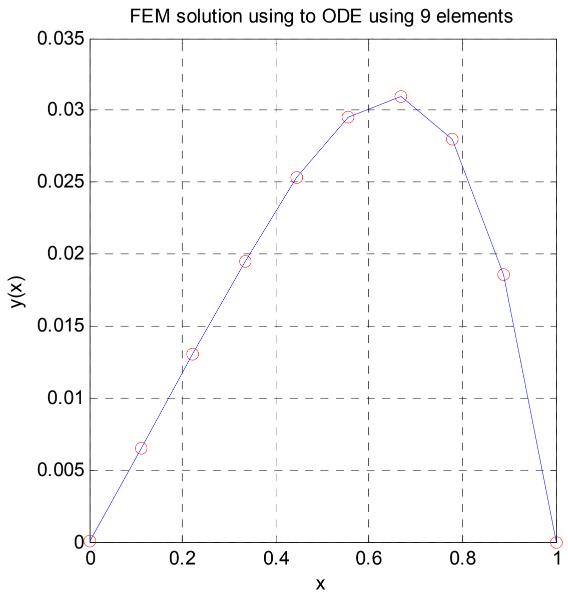

***Figure 1.** FEM solutions using ODEs for 9 elements.*

The citation of FEM can be found at: (http://www.mathworks.ch/ch/help/pde/ug/basics-of-the-finite-element-method.html).

4.2. Galerkin Methods

In this section we will discuss the procedures and methods for solving the one dimensional elliptic equation given by

$$-u''(x) = f(x),$$

Subject to the conditions given in $0 \le x \le h$, u (0) = u_0, u (h) = u_h

We illustrate the FE method for the one dimension two-point boundary value problem

$$-u''(x) = f(x), 0 < x < 1,$$

$$u(0) = 0, u(1) = 0$$

in the Galerkin approach as follows.

First construct a variational or weak formulation, by multiplying both sides of the differential equation by a test function $v(x)$ satisfying the boundary conditions $v(1) = 0, v(0) = 0$ to get

$$-u''v = fv$$

and then integrating from 0 to 1 (using integration by parts): thus

$$\int_0^1 (-u''v)dx = -u'v\Big|_0^1 + \int_0^1 u'v'dx$$

$$= \int_0^1 u'v'dx$$

$$\rightarrow \int_0^1 u'v'dx = \int_0^1 fvdx \text{, the weak form}$$

Then generate mesh, e.g., a uniform Cartesian mesh $x_i = ih$, $i = 0,1,\dots,n$, where $h = \frac{1}{n}$, defining the intervals $[x_{i-1}, x_i]$, $i = 1,2,\dots,n$. Construct a set of basis functions based on the mesh, such as the piecewise linear functions $(i = 1, 2, \dots, n-1)$.

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h} & if\ x_{i-1} \le x < x_i \\ \frac{x_{i+1} - x}{h} & if\ x_i \le x \le x_{i+1} \\ 0 \text{ otherwise} \end{cases}$$

often called the hat functions. The approximate finite element solution by the linear combination of basis functions is

$$u_h(x) = \sum_{j=1}^{n-1} c_j \phi_j(x),$$

where the coefficients c_j are the unknowns to be determined.

Or the matrix – vector form;

$$\begin{bmatrix} a(\phi_1,\phi_1) & a(\phi_1,\phi_2) & \dots & a(\phi_1,\phi_{n-1}) \\ a(\phi_2,\phi_1) & a(\phi_2,\phi_2) & \dots & a(\phi_2,\phi_{n-1}) \\ \vdots & \vdots & & \vdots \\ a(\phi_2,\phi_{n-1}) & a(\phi_2,\phi_{n-1}) & \dots & a(\phi_{n-1},\phi_{n-1}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} (f,\phi_1) \\ (f,\phi_2) \\ \vdots \\ (f,\phi_{n-1}) \end{bmatrix}$$

Where

$$a(\phi_i, \phi_j) = \int_0^1 \phi'_i \phi'_j dx, \ (f, \phi_i) = \int_0^1 f\phi_i\, dx$$

The term *a (u,v)* is called a bilinear form since it is linear with each variable, and (*f, v*) is called a linear form. If ϕ_i are the hat functions, then in particular we get

$$\begin{bmatrix} \frac{2}{h} & \frac{-1}{h} & & & & \\ \frac{-1}{h} & \frac{2}{h} & \frac{-1}{h} & & & \\ & \frac{-1}{h} & \frac{2}{h} & \frac{-1}{h} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{-1}{h} & \frac{2}{h} & \frac{-1}{h} \\ & & & & \frac{-1}{h} & \frac{2}{h} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} \int_0^1 f\phi_1\, dx \\ \int_0^1 f\phi_2\, dx \\ \int_0^1 f\phi_3\, dx \\ \vdots \\ \int_0^1 f\phi_{n-2}\, dx \\ \int_0^1 f\phi_{n-1}\, dx \end{bmatrix}$$

Finally solve the linear system of equations for the coefficients and hence obtain the approximate solution

$$u_h(x) = \sum_i c_i \phi_i(x)$$

Example 4.2. Let us consider the simple differential equation problems

$$\frac{d^2u}{dx^2} = x + 1, 0 < x < 1 \tag{11}$$

Boundary condition $u(0) = 0$ and $u(1) = 0$
Solution:-(using Galerkin Method)
Take the weak form

$$\Rightarrow \int_0^1 u'v'dx = \int_0^1 (x+1)vdx,$$

Let $n = 4$ then $h = \frac{1}{n} = \frac{1}{4} = 0.25$
In matrix representation

$$\begin{bmatrix} a(\phi_1,\phi_1) & a(\phi_1,\phi_2) & a(\phi_1,\phi_3) \\ a(\phi_2,\phi_1) & a(\phi_2,\phi_2) & a(\phi_2,\phi_3) \\ a(\phi_3,\phi_1) & a(\phi_3,\phi_2) & a(\phi_3,\phi_3) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \int_0^1 (x+1)\phi_1\,dx \\ \int_0^1 (x+1)\phi_2\,dx \\ \int_0^1 (x+1)\phi_3\,dx \end{bmatrix}$$

$$\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \frac{23}{48} \\ \frac{-85}{48} \\ \frac{89}{48} \end{bmatrix}$$

Using Gauss elimination i.e.,

We have been the triangular system. This triangular system can be solved now by the back substitution. We obtain that

$c_1 = -\frac{1}{64} \approx -0.01563, c_2 = -\frac{29}{192} \approx -0.15105$ and $c_3 = \frac{5}{32} \approx 0.15625$.

Obtain the approximation solution

$$u_h(x) = \sum_{i=1}^{3} c_i\phi_i(x) \tag{12}$$

Error: -The exact solution of (4.10) is

$$u(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{2}{3}x$$

Those from (8)

$$u_h(x) = -\frac{1}{64}\phi_1(x) - \frac{29}{192}\phi_2(x) + \frac{5}{32}\phi_3(x)$$

The approximate and exact solution of Galerkin Method citation can be found at:

(http://www4.ncsu.edu/~zhilin./TEACHIM/MA587/chap6.pdf)

Table 1. Numerical result for example 4.2.

x	$u(x)$	$u_h(x)$	$\lvert u - u_h \rvert$
0.2	-0.112006	-0.0125	0.099506
0.4	-0.17599	-0.096875	0.079115
0.6	-0.184	-0.08854	0.09545
0.8	-0.12006	0.125	0.24506

The comparison between the approximate solution and exact solution is illustrated in the above table 2. Approximate solution according to the GM as a function of coordinate

$$u_h(x) = -\frac{1}{64}\phi_1(x) - \frac{29}{192}\phi_2(x) + \frac{5}{32}\phi_3(x)$$

5. Discussion of Weak Galerkin and Discontinuous Galerkin Finite Element Methods

5.1. WG-FEMs

The WG-FEMs were first developed based on discrete weak gradients, which approximate the weak gradient operator. The key WG-FEM is to define the weak derivative operators from integration by parts, such as: weak gradient (∇_w), weak divergence (∇_w) and weak Laplacian (Δ_w). Related notation, definitions and the concept of weak gradient and its approximations result in discrete weak gradients, which will play an important role in the WG-FEMs for solving elliptic BVPs.

WG-FEM is making use of discontinuous basis functions for approximation. It divides a function v to the following two parts:

Define weak function $v = \{v_0, v_b\}$ such that

$$v = \begin{cases} v_0, inK^0 \\ v_b, on\partial K \end{cases}$$

Although a WG finite element space can be constructed for any combination of $p, l, r,$ good approximate solutions to PDEs.

Define weak Galerkin finite scheme with homogeneous boundary value

$$V_b = \{v = \{v_0, v_b\}: v_0|_K \in P_j(K), v_b \in P_l(e), e \subset \partial K, v_b = 0 \text{ on}\partial\Omega\}$$

Properties of of WG-FEMs

For the WG-FEMs to work well, two properties should be satisfied using definition WG finite element spaces

1. For any $v \in V_h$ and any $K \in T_h$, if $\nabla_{w,d} v = 0$ on K, then there must be v_0 =constant on K_0 and v_b = constant on ∂K.
2. For $u \in L^2(\Omega)$, $k \geq 1$, let $\nabla_w u = V_h(l, r)$ be an interpolation of projection of u. should be a good approximation of ∇_w.

WG-FEM for Second-order Elliptic BVPs: For simplicity, we demonstrate the idea of optimality for polynomials by using a linear second-order elliptic equation as our model problem that seeks an unknown function $u(x) = u(x, y)$ satisfying:

$$\begin{cases} -\nabla.(a\nabla u) = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{13}$$

where, Ω is apolytopal domain in $\mathbb{R}^d (d = 2{,}3)$, a is symmetric matrix-valued function in Ω and ∇u denotes weak gradient operator of the function.

Therefore, WG method that that uses discontinuous elements and has a formulation:

$$(a\nabla_w u_h, \nabla_w v) = (f, v), \forall v \in V_h \tag{14}$$

The primal formulation of WG is obtained by adding discrete weak gradient with stabilization (to ensure a weak continuous of u_h).

Now, we can write the WG formulation for equation (13) as follows: find $u_h = \{u_0, u_b\} \in V_h$ and WG-FEMs satisfying:

$$(a\nabla_w u_h, \nabla_w v) + s(u_h, v) = (f, v), \forall v \in V_h \tag{15}$$

where, $s(u_h, v) = \sum_K h_K^{-1} \langle u_0 - u_b,\ v_0 - v_b \rangle_{\partial K}$,

$s(.,.)$ is stabilizer (enforcing weak continuity) and measures the discontinuity of the finite element solution.

To compute WG using (16), we get

$$(a\nabla_w u_h, \nabla_w v) = \sum_K (a\nabla_w u_h, \nabla_w v)K$$

Then, a WG-FEM approximation is defined as $u_h = \{u_0, u_b\} \in V_h$ such that

$$(a\nabla_w u_h, \nabla_w v) = \langle f, v_0 \rangle, \forall v = \{v_0, v_b\} \in V_h^0 \tag{16}$$

The weak formulation for our second-order elliptic BVP is given: find $u_h \in V_h$ such that

$$u_h | e = \nabla_w v \text{for } e \in \partial\Omega, \text{ and}$$

$$a(u_{WG}, v) = \sum_K (\nabla_w u_{WG}, \nabla_w v)K + \sum_e h^{-1} \langle u_0 - u_b, v_0 - v_b \rangle_e = (f, v), \forall v = \{v_0, v_b\} \in V_h \tag{17}$$

Let $\{\phi_{0,i}(x), i = 1, 2, \dots, N_0\}$ and $\{\phi_{b,i}(x), i = 1, 2, \dots, N_b\}$ be basis of functions in $\mathrm{V_h}$ associated with i^{th} interior and boundary edges, respectively.

i.e., $\mathrm{V_h} = span\{\phi_{0,i}(x), \phi_{b,i}(x)\}, i = 1, 2, \dots, N_0, i = 1, 2, \dots, N_b$. Then $u_h = \{u_0, u_b\} \in V_h$ can be expressed as

$$u_h = \sum_{i=1}^{N_0} u_{0,i} \phi_{0,i} + \sum_{i=1}^{N_b} u_{b,i} \phi_{b,i} \tag{18}$$

We define the vector representation of u_h as $u = \begin{bmatrix} u_0 \\ u_b \end{bmatrix}, u_0 = \begin{bmatrix} u_{0,1} \\ u_{0,2} \\ \vdots \\ u_{0,} N_0 \end{bmatrix}, u_b = \begin{bmatrix} u_{b,1} \\ u_{b,2} \\ \vdots \\ u_b, N_b \end{bmatrix}$

The numerical approximate solution for WG method is given by solving the linear system of (17) as follows:

$$\begin{bmatrix} a(\nabla_w \phi_{0,j}, \nabla_w \phi_{0,i})K & a(\nabla_w \phi_{0,j}, \nabla_w \phi_{0,i})K \\ a(\nabla_w \phi_{0,j}, \nabla_w \phi_{0,i})K & a(\nabla_w \phi_{0,j}, \nabla_w \phi_{0,i})K \end{bmatrix} \begin{bmatrix} u_0 \\ u_b \end{bmatrix} = \begin{bmatrix} \langle a(f(x, y), \phi_{0,1} \rangle \\ \langle a(f(x, y), \phi_{0,2} \rangle \end{bmatrix}$$

Solution:- (using WG-FEM)

The goal of this section is to numerically verify the convergence theory for the WG-FEM (17) through in example (5.1). In particular, the following issues shall be examined: rate of convergence and accuracy for WG solutions on triangular meshes.

Let $u(x, y) = x(1 - x)y(1 - y) = (x - x^2)(y - y^2)$, then

$$\nabla u = \big((1 - 2x)(y - y^2), (x - x^2)(1 - 2y)\big)$$

$$a\nabla_w u = (xy(1 - 2x)(y - y^2), (x - x^2)(1 - 2y))$$

When we choose $K = 1, (\nabla_w. v, u)|_K = (1, 1) = a \in P_0(K), K \in T_h$. We have

$$\frac{a}{2} = (\nabla_w. u, 1)_K = -(u, \nabla.1)_K + \sum_{i=1}^{3} \langle \{\{u\}\}, n \rangle_{e_i} = 0 + \int_{e_1} 0 ds + {}^1\!/_2 \int_{e_2} 1 ds - {}^1\!/_2 \int_{e_3} 1 ds = 0$$

We derive a linear system of WG equation for the coefficients by substitution the approximation solution is

$$\nabla_w u_{WG}(x,y) = \sum_{i=1}^{n} = c_j \nabla_w \phi_{i,j}(x,y) = c_1 \nabla_{\mathrm{w}} \phi_{i,j}(x,y) + c_2 \nabla_{\mathrm{w}} \phi_{i,j}(x,y) + c_3 \nabla_{\mathrm{w}} \phi_{i,j}(x,y)$$

$$\nabla_w u_{WG}(x,y) = (2x - x^2)(2y - 3y^2) \tag{19}$$

To compute the error and convergence rates using the following norms:

$$\|\nabla_w u_{WG} - \nabla u\| = \left(\sum_{K \in T_h} \int_{\mathrm{K}} |\nabla_w u_{WG} - \nabla u|^2 \, dxdy\right)^{1/2} \tag{20}$$

Table 2 clearly suggests that the following rates of convergence hold for smooth solution problems: $\|u - u_{WG}\| L^2(\Omega)$ =O(h^{r+1}), where r is the number of degrees of freedom. The error of the computed solution and rates of convergence for WG and DG method of the above equation is obtained here based on MATLAB.

***Table 2** Comparison of convergence for WG-FEM P_0P_0 and WG-FEM P_1P_1 for a Degenerate Elliptic Problem.*

Mesh size $n = 1/h = h^{-1}$	WG-FEM P_1P_1	WG-FEM P_0P_0
	$\|\nabla_w u_{WG} - \nabla u\| L^2$-error	$\|\nabla_w u_{WG} - \nabla u\| L^2$-error
$1/8$	1.46-03	2.01e-03
$1/16$	3.74e-03	9.30e-04
$1/32$	9.47e-05	4.02e-04
$1/64$	2.39e-05	1.70e-04
$1/128$	6.04e-06	7.17e-04
$\mathrm{O}(h^r), r =$	1.98e+00	1.21+00

Discussion and results

The PDE under consideration is thus elliptic, but some degeneracy near the origin. The coloumn corresponging to WG-FEM P_1P_1 refers to the computational results obtained from the numerical scheme (15) with piecewise linear functions on each element and its edges. The coloumn corresponding to WG-FEM P_0P_0 refers to the computational results obtained from the numerical scheme (15) with piecewise constants. The computational results indicate that the WG-FEM scheme (15) presented and analyzed in the table 2. Here, WG-FEM P_1P_1 refers high optimal order of convergence in L^2-error than WG-FEM P_0P_0.

Example 5.2: using WG-FEMs, show that the discrete weak gradient $\nabla_w.\mathrm{v}$ is the same as the L^2 projection (an estimate) of the continuous divergence $\nabla.\mathrm{v}$ for a piecewise smooth vector-valued function v.

Solution:

Let K is a triangular element ΔABC whose nodes are $A(0,0), B(1,1), C(0,1)$, and $\overline{AB} = e_1, \overline{BC} = e_2, \overline{CA} = e_3$.

We have

$$\frac{a}{2} = (\nabla_w.u, 1)_K = -(u, \nabla.1)_K + \sum_{i=1}^{3} \langle \{\{u\}\}, n \rangle_{e_i} = 0 + \int_{e_1} 0ds + 1/2 \int_{e_2} 1ds - 1/2 \int_{e_3} 1ds = 0$$

Advantage and Disadvantage of the WG-FEM

WG methods use discontinuous approximations. The WG methods keep the advantages:

1. The finite element partition can be of polytopal (the simultaneous use of two or more keys relating composition) type;
2. The WG methodology provide a general framework for deriving new methods and simplifying the exisiting methods;
3. and minimize the disadvantages.
4. Simple formulations.
5. Comparable number of unknowns to the continuous FEMs if implemented appropriately.

The disadvantage of WG-FEMs is:

Difficult to construct high order continuous elements.

5.2. DG-FEMs

5.2.1. DG-FEM for Second-Order Elliptic BVPs

In this section, we discuss the DG-FEMs for approximating the solutions of second-order linear elliptic boundary value problems. DG methods in mathematics form a class of numerical methods for solving PDEs. The most recent technique for the numerical solution of PDEs is the DGM, which uses ideas of both the finite element and finite volume methods.

Properties of DG-FEMs for solving second-order elliptic BVPs.

The DG-FEM is a numerical technique for solving partial differential equations when there are discontinuities or jumps in the solution or highly advective flows. According to Heshaven et al. 1969, the DGM has several important properties:

1. The mesh does not have to be regular, hanging-nodes can be handled easily,
2. Conservation laws can be achieved by the numerical solutions.

The main ingredients of a DG-FEM are:

The use of a Galerkinelement-by-elementweak formulation;

Since 1970s, there has been a substantial amount of work in the literature on so-called non-conforming FEM, where by $V_{DG} \not\subset V = H_0^1(\Omega)$.

5.2.2. The Model Problem and DG Weak Formulation

A DG type nonconforming element method and a local flux matching nonconforming element method for the second-order elliptic BVPs.

We consider the homogeneous Dirichlet boundary-value problem whose solution u satisfies

$$\begin{cases} -\nabla.\left(a\nabla u(x)\right) = f(x), x \in \Omega \\ u(x) = 0, x \in \partial\Omega \end{cases} \tag{21}$$

where, ∇ is the standard elliptic gradient operator, coefficients $a = a(x, y), x \in \Omega$, is a piecewise constant strictily scalar function, is symmetric, positive definite, bounded smooth matrix function and $f \in L^2(\Omega)$.

In order to derive the weak form for the Poisson equation we multiplying both sides of equations (21) by with a test function $v = 0$ on $\partial\Omega$ and integrating both sides, we have

$$\int_\Omega (-\nabla. a\nabla u)v dx = \int_\Omega f v dx \tag{22}$$

and decomposing (22) over K, we obtain

$$-\sum_{K\in T_h} \int_K (\nabla. au)v dx = \sum_{K\in T_h} \int_K f v dx$$

The integration part gives us

$$\sum_{K\in T_h} \int_K (\nabla.(\nabla a u)\nabla v dx - \sum_{K\in T_h} \int_{\partial K} (a\nabla u.n)v ds = \sum_{K\in T_h} \int_K f v dx = \int_\Omega f v ds = \langle f, v\rangle \tag{23}$$

where, n denotes the outward normal to each element edge. The unit normal vector outward from K (respectively K_i) is denoted by n (respectively n_i).

Step II: using jump and averages

We introduce the following bilinear form $B(.,.)$ defined on $H^2(T_h)xH^2(T_h)$ and the linear form $F(.)$ defined on $H^2(T_h)$ such as:

$$B(u, v) = \sum_{K\in T_h} \int_K (\nabla.(\nabla a u)\nabla v dx \tag{24}$$

and

$$F(v) = \sum_{K\in T_h} \int_K f v dx = \int_\Omega f v ds \tag{25}$$

Thus, we define the weak formulation for DG finite element discretizations for elliptic problems (21) finally can be written as follows:

$$B(u_{DG}, v) = \int_\Omega f v ds, \forall u_{DG}, v \in V_{DG} \tag{26}$$

where,

$$B(u_{DG}, v) = \sum_{K\in T_h} \int_K (\nabla.(\nabla a u)\nabla v dx - \int_{\Gamma_{int}} \{n. a(\nabla u)\}[v] ds$$

where, h is a measure for the average of the size defines as $h = \frac{(h^+ + h^-)}{2}$ for the two cells (meshes) $K_+$and K_- given in the interiror faces, and Γ^{int} is interior faces.

Step V: Forming and solving the linear system to obtain a DG method for numerical approximation to the primal variable.

To form the linear system, firstly, rewrite the discrete DG scheme for the primal variable of (25) is:

$$B_h(u_{DG}, v) = A_h(u_{DG}, v) = l_h(u_{DG}, v), \forall v \in V_{DG} \tag{27}$$

Corresponding to poisson's parts of the problem, respectively:

$$A_h(u_{DG}, v) = \sum_{K\in T_h} \int_K (\nabla.(\nabla a u)\nabla v dx, \text{ and } l_h(u_{DG}, v) = \sum_{K\in T_h} \int_K f v dx = \int_\Omega f v ds$$

For a set of basis functions $\{\phi_i\}_{i=1}^N$ spanning the space V_{DG}, the discrete solution $u_{DG} \in V_{DG}$ is of the form

$$u_{DG} = \sum_{j=1}^N v_j \phi_j \tag{28}$$

where, $v = (v_1, v_2, \dots \dots, v_N)^T$ is the uniform coefficients of vector. After substituting (27) and (28) and taking $u_{DG} = \phi_j$, we get the linear system of equations

$$\sum_{j=1}^N v_j A_h(\phi_j, \phi_i) = l_h(\phi_j), j = 1, \dots \dots, N.$$

Choosing a basis $\phi_{DG} = \{\phi_1, \phi_2, \dots \dots \dots, \phi_n\}$ for the DG approximation space $V_{DG}$and expanding the solution in this basis as $u = \sum_{j=1}^n u_j \phi_j$, the discrete variational problem of (25) is equivalent to a system of linear equation

$$AU = L \tag{29}$$

for the coeffiecients$u$ with $a_{i,j} = a_h(\phi_j, \phi_i)$ and $l_i = f_h(\phi_i)$.

Using this partition and imposing an ordering $T_h = \{T_1, \dots\dots\dots, T_n\}$ on the mesh elements, the linear system (5.26) can be written in block matrix form

$$\begin{pmatrix} a_h(\phi_1^i, \phi_1^j) & a_h(\phi_2^i, \phi_1^j) & \cdots & a_h(\phi_m^i, \phi_1^j) \\ \vdots & \ddots & & \vdots \\ \vdots & & & \vdots \\ a_h(\phi_1^i, \phi_m^j) & \cdots & \dots & a_h(\phi_m^i, \phi_m^j) \end{pmatrix} \begin{pmatrix} u_1^{(i)} \\ \vdots \\ \vdots \\ u_m^{(i)} \end{pmatrix} = \begin{pmatrix} L(\phi_1^{(j)}) \\ \vdots \\ \vdots \\ L(\phi_m^{(j)}) \end{pmatrix}, \forall i, j = 1, 2, \dots., N$$

Definition 5.1: (Convergence rate) The quantity of flux errors with exact solution $u^{(j)}$ based on the poisson model problem on Ω_j and its flux approximations $\nabla \mathrm{u}$ and $\nabla \mathrm{u}_{\mathrm{DG}}$ based on the triangulation T_h is

$$\frac{\|u - u_{DG}\|}{\|\nabla u - \mathrm{a}\nabla u_{DG}\|}$$

Example 5.3: [Arnold and Cockburn, 2002] Compute DG approximations solution u_{DG} of u ; compute error and convergence rate to the following elliptic PDEs:

$$-\Delta u = 0, in\ \Omega \tag{30}$$

$$u = 0 \text{ on } \partial\Omega,$$

where, $\Omega = (0,1) \text{ x } (0,1) = (0,1)^2$ for the above second-order elliptic PDEs. We choose a function f in such a way that numerical tests, that the analytical solution takes the form $u = xy(1-x)(1-y)$, which corresponds to a homogeneous Dirichlet boundary condition. The function $f = f(x,y)$ is given to match the exact solution.

Solution:- (Using DG-FEMs)

We mean where $u = 0$ on $\partial\Omega$ and u is apolynomial. We choose polynomial of degree two, i.e, $K = 2.$,

The DG-FEMs with respective solutions is u_{DG} based on shape-regular triangulation of Ω.

Using equation (30) and we will formulate the given PDE as a variational BVP by multiplying by a test function $v \in V$ and apply greeen's identity, we have

$$u(x,y) = xy(1-x)y(1-y) = f(x), x \in \Omega = (0,1)^2 \Rightarrow \left[-\nabla.\left(a\nabla u(x,y)\right) - f(x,y)\right]vdxdy = 0, \forall\ v \in V \tag{31}$$

Given, $u(x,y) = xy(1-x)y(1-y) = (xy - xy^2)(1-y)$, we get

$$\nabla u = ((y - 2xy)(1-y), (-xy + x^2y))$$

Substitute $u(x,y), \nabla u$ in equation (32) then we have

$$\int_\Omega (-\nabla. a\nabla u)vdxdy = \int_\Omega f(x,y)v(x,y)dxdy \tag{32}$$

Integration by parts using jump and averages and using by green's formula, we get

$$\sum_{K\in T_h} \int_\Omega (\nabla.(a\nabla u)\nabla vdxdy = \int_{\Gamma int}\{n. a(\nabla u)\}[v] = \sum_{K\in T_h} \int_{\mathrm{K}} fvdx \tag{33}$$

where, $\partial\Omega$ is the boundary of Ω integrated counter clock wise.

let $\gamma \in \Omega_h$ be an interior edge shared by two elements e_1 and e_2. By convenction for mesh data structure, the unit normal vector n points to e_1 and e_2, and since, using $\gamma = p[u_{DG}]$ we obtain, $[u_{DG}] = u_{DG}|e_1 - u_{DG}|e_2$.

where as,

$$\{a\nabla u_{DG}.n\} = \frac{1}{2}a|e_1.\nabla u_{DG}.n + \frac{1}{2}a|e_1.\nabla u_{DG}.n \tag{34}$$

The stiffness matrix A is given from equation (33) for

$$A_{ij} = \int_{\Omega_h}\left[a(\phi_i'\phi_j' + \phi_i'\phi_j')\right]dxdy + \int_{\partial\Omega}\{n. a(\nabla u)[v]\}ds \tag{35}$$

and the load vector F_i is

$$F_i = \int_{\partial\Omega} f\phi_i dxdy \tag{36}$$

Using partition and imposing an ordering a triangulation $T_h = \{T_1, T_2, T_3\}$ on the mesh elements, the linear – can be written in block matrix form

$$\begin{pmatrix} a(\phi_1^i, \phi_1^j) & a(\phi_2^i, \phi_1^j) & a(\phi_3^i, \phi_1^j) \\ a(\phi_1^i, \phi_2^j) & a(\phi_2^i, \phi_2^j) & a(\phi_3^i, \phi_2^j) \\ a(\phi_1^i, \phi_3^j) & a(\phi_2^i, \phi_3^j) & a(\phi_3^i, \phi_3^j) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} f(\phi_1^j) \\ f(\phi_2^j) \\ f(\phi_3^j) \end{pmatrix} \tag{37}$$

The rest of calculations are similar, and the result is the following stiffness matrix:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

Continuing in this manner, we find $F = \begin{bmatrix} \frac{7h^3}{2} \\ h^3 \\ 3h^3 \end{bmatrix}$ and

we can now solve $A\alpha = F$ to obtain $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0.01324 \\ 0.02712 \\ 0.01132 \end{bmatrix}$

So, the numerical approximate solution of DG method u_{DG} is

$$u_{DG}(x,y) = \sum_{i=1}^{3} \alpha_j \phi_j(x,y) = \alpha_1 \nabla\phi_1(x,y) + \alpha_2 \nabla\phi_2(x,y) + \alpha_3 \nabla\phi_3(x,y) \tag{38}$$

$$u_{DG}(x,y) = 0.01324\left(1/_h, 1/_h\right) + 0.02712\left(1/_h, 0\right) + 0.01132(1/_h, 1/_h)$$

From the above, we obtain

$$\nabla u_{DG}(x,y) = ((y - 2xy)(1-y), (-xy + x^2 y) \tag{39}$$

The DG-FEMs with respective solutions is u_{DG} based on shape-regular triangulation of Ω. Inorder to evaluate the L^2-norm of approximate solution $u \in V_{DG}$, which are respectively defined as follows

$$\|u - u_{DG}\| L^2(\Omega) = \left(\int_\Omega (u(x,y) - u_{DG}(x,y))^2 \, d\Omega\right)^{1/2}$$

Here, MATLAB is given for the L^2-norm of error to approximate solution $\|u - u_{DG}\| L^2(\Omega$.

Table 3. *Numerical error and convergence order of DG-FEM for domain $\Omega = (0,1)^2$.*

Mesh (h)	DG-FEMs $\|u - u_{DG}\|L^2$ =O(h^2)	$\nabla u_{DG}(x,y)$	$\|\nabla u - \nabla u_{DG}\|$
1	0.0186579226	0.150872	0.1074034465
2	0.0057148929	0.682464	0.0588911882
3	0.0015087244	0.959512	0.0301758912
4	0.000382464	0.740651	0.0115826243
5	0.000959512	0.3074065	0.00763263
6	0.0000240008	0.2074065	0.003803128

The order of convergence is the ratio of two meshes with respect to the largest edge length. Since the mesh is decreasing $1/2$ each time, the order of convergence converges to 2.

Advantage and Disadvantage of the DG-FEMs

The DG methods have the following main advantages:

1. like any finite element method, the maximum order of accuracy a DG method can attain solely depends on the regularity of the exact solution;
2. DG methods of arbitrarily high order of accuracy can be obtained by suitably choosing the degree of the approximating polynomials;

The disadvantage of DGM is:

1. No connection among functions on individual elements;
2. DG-FEM does not require continuity of the solution along edges;
3. DG-FEM is not well suited for problems with direction (global statement);
4. Unknown inside elements;

5.3. Illustrative Examples for WG and DG-FEMs

Example 5.5: Wang and Ye, (2012); Arnold and Brezzi, [11] consider a simple two dimensional Poisson equation with homogeneous boundary. Compute the following degenerate elliptic problem for WG and DG approximations solution u_{DG} and u_{WG} of u and estimate the error $\|u - u_{DG}\|L^2$and $\|u - u_{WG}\|L^2$ =O(h^2) to the following second-order elliptic PDEs:

$$\begin{cases} -\nabla(a\nabla a) = f(x,y), in\ \Omega, \mathrm{a} = \mathrm{xy} \\ u(x,y) = 0, on\ \partial\Omega \end{cases} \tag{40}$$

where $\Omega = (0,1)\mathrm{x}(0,1) = (0,1)^2$ for the above second-order elliptic PDEs. We choose a function f in such a way that numerical tests, that the analytical/exact solution takes the form $u(x,y) = x(1-x)y(1-y)$, which corresponds to a homogeneous Dirichlet boundary condition. The function $f = f(x,y)$ is given to match the exact solution.

Solution: (Using DG-FEMs)

Using equation (40) and we will formulate the given PDE as a variational BVP by multiplying by a test function $v \in V$ and apply greeen's identity, we have

$$-\nabla.(a\nabla u) = x(1-x)y(1-y) = f(x), x \in \Omega = (0,1)^2$$

$$\Rightarrow \left[-\nabla.\left(a\nabla u(x,y)\right) - f(x,y)\right] v dx dy = 0, \forall\, v \in V \tag{41}$$

From, $u(x,y) = x(1-x)y(1-y) = (x - x^2)(y - y^2)$, we get

$$\nabla u = \left((1-2x)(y-y^2), (x-x^2)(1-2y)\right)$$

$$\mathrm{a}\nabla u = xy\nabla u = (xy(1-2x)(y-y^2), (x-x^2)(1-2y))$$

Substitute $u(x,y)$, ∇u and $a\nabla u$ in equation (41), then we have

$$\Rightarrow \left[-\nabla.\left((xy(1-2x)(y-y^2),(x-x^2)(1-2y))\right)-(x-x^2)(y-y^2)\right]vdxdy=0 \tag{42}$$

Integrating over the domain $\Omega=(0,1)^2$, we obtain

$$\int_\Omega \left[-\nabla.\left((xy(1-2x)(y-y^2),(x-x^2)(1-2y))\right)-(x-x^2)(y-y^2)\right]vdxdy=0 \tag{43}$$

In order to get everything in terms of first derivative using product rule for differentiation to show that

$$v.\nabla.(a\nabla u)=\nabla.(va\nabla \mathrm{u})-\mathrm{a}\nabla\mathrm{u}.\nabla\mathrm{v}$$

which can be inserted into equation (43) to give

$$\int_\Omega [(\mathrm{a}\nabla\mathrm{u}).\nabla\mathrm{v}-fv]dxdy-\int_\Omega \nabla.(va\nabla\mathrm{u})dxdy=0$$

$$\Rightarrow \int_\Omega \left[\left((xy(1-2x)(y-y^2),(x-x^2)(1-2y)\right).\nabla\mathrm{v}-(x-x^2)(y-y^2)v\right]dxdy=0 \tag{44}$$

where, $\partial\Omega$ is the boundary of Ω integrated counter clock wise (ccw)

Then, we have

$u_{DG}=\sum_{j=1}^{n}u_j\phi_j(x,y)$such that $u_j=\hat{u}_j$ at the nodes on Ω_h and

$$\int_{\Omega_h}\left[a(\frac{\partial u_h}{\partial x}.\frac{\partial v_h}{\partial x}+\frac{\partial u_h}{\partial y}.\frac{\partial v_h}{\partial y})\right]dxdy+\int_{\partial\Omega}\{n.a(\nabla u)[v]\}ds\ =\int_{\Omega_h}fuvdxdy+ \tag{45}$$

To compute its numerical flux $u_{DG}=-a|e_1\nabla u_{DG}$, which is a constant vector on each triangular element.
let$\gamma\in\Omega_h$ be an interior edge shared by two elements e_1 and e_2. By convenction for mesh data structure, the unit normal vector n points to e_1 and e_2, and since, using $\gamma=p[u_{DG}]$ we obtain, $[u_{DG}]=u_{DG}|e_1-u_{DG}|e_2$.
where as,

$$\{a\nabla u_{DG}.n\}=\frac{1}{2}a|e_1.\nabla u_{DG}.n+\frac{1}{2}a|e_1.\nabla u_{DG}.n$$

The area of the triangle $|e_i(x,y)|=\frac{1}{2}|detA_i(x)|$

The stiffness matrix A is given from equation (33)

$$A_{ij}=\int_{\Omega_h}\left[a(\phi_i'\phi_j'+\phi_i'\phi_j')\right]dxdy+\int_{\partial\Omega}\{n.a(\nabla u)[v]\}ds \tag{46}$$

and the load vector F_i is

$$F_i=\int_{\partial\Omega}f\phi_i dxdy \tag{47}$$

So, the solution of DG method is

$$u_{DG}(x,y)=\sum_{i=1}^{3}\alpha_j\phi_j=\alpha_1\nabla\phi_1(x,y)+\alpha_2\nabla\phi_2(x,y)+\alpha_3\nabla\phi_3(x,y)$$

$$u_{DG}(x,y)=\sum_{i=1}^{3}\alpha_j\nabla\phi_j(x,y)=\alpha_1\nabla\phi_1(x,y)+\alpha_2\nabla\phi_2(x,y)+\alpha_3\nabla\phi_3(x,y) \tag{48}$$

$$u_{DG}(x,y)=0.015904\left(0,{}^{1}\!/_{h}\right)+0.027344-\left({}^{1}\!/_{h},0\right)+0.02064(-{}^{1}\!/_{h},{}^{1}\!/_{h})$$

Then, we substitute mesh size for $h={}^{1}\!/_{n}$ in equation (48), to get numerical approximate solution of DG method u_{DG}based on shape-regular triangulation domain $\Omega=(0,1)^2$as follows:

For $h=1$,$u_{DG}(x,y)=0.50872$, for $h=2$,$u_{DG}(x,y)=0.2353$

$$\Longrightarrow \nabla u_{DG}(x,y)=\left((1-2x)(y-y^2),(x-x^2)(1-2y)\right)= \tag{49}$$

and $\nabla u=\left((1-2x)(y-y^2),(x-x^2)(1-2y)\right)$

Since, the mesh size is denoted $h={}^{1}\!/_{n}$.

For fixed n, $\| u(x,y) - u_{DG}(x,y)\|$, $h = {}^{1}/_{n} = 2, 4, 6, 8, 16, 32, 64$ since, the error of the DG-FEM from error is: $e := u - u_{DG}$ and we solve edge-based L^2-norm of the error based on MATLAB as follows

$$\|u - u_{DG}\| L^2 = \left(\int_0^1 (u(x,y) - u_{DG}(x,y))^2 \, dx\right)^{1/2}$$

Solution: (using WG-FEMs)

The goal of this section is to numerically verify the convergence theory for the WG-FEM (5.5) through in example (5.58). In particular, the following issues shall be examined: rate of convergence and accuracy for WG solutions on triangular meshes.

Let $u(x,y) = x(1-x)y(1-y) = (x - x^2)(y - y^2)$, then

$$\nabla u_{WG} = \big((1-2x)(y-y^2), (x-x^2)(1-2y)\big)$$

$$a\nabla_w u_{WG} = (xy(1-2x)(y-y^2), (x-x^2)(1-2y))$$

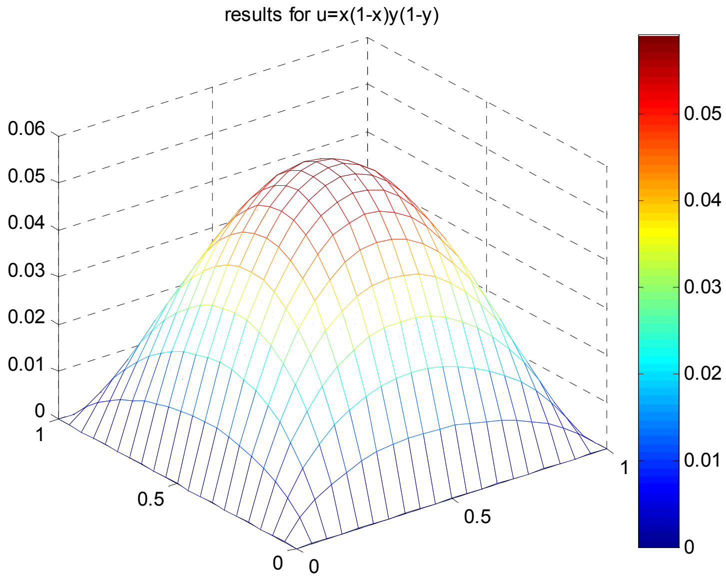

Figure 2. *Numerical solution of the homogeneous Dirichlet problem for the Laplace operator and surface plot of the approximate solution u_{DG}.*

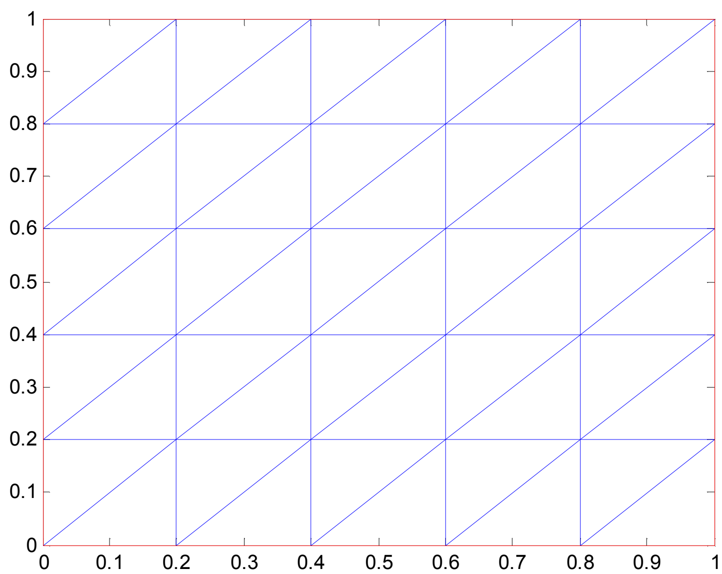

Figure 3. *Uniform triangular mesh for 2-D Laplace equation of example (5.58) for $\Omega = (0,1)^2$.*

$$\nabla_w u_{WG}(x,y) = \sum_{i=1}^{n} = c_j \nabla_w \phi_j = c_1 \nabla_{\mathrm{w}} \phi_{i,j}(x,y), + c_2 \nabla_{\mathrm{w}} \phi_{i,j}(x,y) + \nabla_{\mathrm{w}} \phi_{i,j}(x,y \tag{50}$$

The error of the computed solution and rates of convergence for WG and DG method of the above equation is obtained below based on MATLAB.

Table 4. *Comparison of convergence for WG-FEM P_1P_1, WG-FEM P_0P_0 and $DG - FEMP_1$ schemes for a Degenerate Elliptic Problem.*

Mesh size $n = {}^{1}/_{h} = h^{-1}$	**WG-FEM P_1P_1**	**WG-FEM P_0P_0**	**DG-FEM P_1**
	$\|\|\nabla_w u_{WG} - \nabla u\|\| L^2$-error	$\|\|\nabla_w u_{WG} - \nabla u\|\| L^2$-error	$\|\|u - u_{DG}\|\| L^2$
$^{1}/_{8}$	1.46-03	2.01e-03	6.29e-03
$^{1}/_{16}$	3.74e-03	9.30e-04	2.92e-03
$^{1}/_{32}$	9.47e-05	4.02e-04	1.32e-03
$^{1}/_{64}$	2.39e-05	1.70e-04	5.89e-04
$^{1}/_{128}$	6.04e-06	7.17e-04	2.64e-04
$O(h^r), r =$	1.98e+00	1.21+00	1.15e+00

Discussion and Result.

The three methods were chosen for comparison because they have the same rate of convergence in theory when the error is measured between the Finite Element solutions and a certain interpolation of the exact solution. The column corresponding to DG-FEM P_1 refers to the computational results obtained from the numerical scheme of with piecewise linear functions on each element and its edges. It must exchange information with all nighbouring triangles through jumps and averages on edges. Therefore, the computational results indicate that the WG-FEM scheme presented and has better optimal order of convergence in L^2.

6. Summary, Conclusions and Recommendations

6.1. Summary

In this Project, we have conducted a comparative study on the newly introduced WG-FEMs; the widely accepted DG-FEMs. Compared to DG-FEMs, WG- FEMs are easier to use and performs better with respect to continuous normal flux across element interfaces, less unknowns, and no need for choosing penalty factors. The main objective of this project is to deal with a Comparative study on the WG and DG-FEMs for solving second-order elliptic boundary value problems. This Project presents the basic understanding of FEM and the methodology to solve any problem of differential equation.

The WG-FEMs represent advanced methodology for handling discontinuous functions in finite element procedure. The WG methodology provides a general framework for deriving new methods and simplifying the existing methods.

6.2. Conclusions

In this Project, we present the numerical comparison of WG and DG methods for a model second-order elliptic problems. The main features of the WG-FEMs compared to DG-FEMs have been clearly observed. Such as WG-FEMs have less unknowns, no need for choosing penalty factor and normal flux is continuous across element interfaces. We have developed a MATLAB code that includes WG-FEMs and DG FEMs with post-processing.

The WG-FEMs are known as lacking of “local conservation”, even though they are conceptually simple and have relatively less unknowns. The DG-FEMs are locally conservative but there is no continuity in the DG flux.

As to implementation, the weak formulation of the DGM involves jumps of the primal variable and averages of the flux on mesh edges. One motivation for using DG discretization is that for deterministic elliptic problems with discontinuous coefficients, the DG solution is more accurate on a fixed mesh than the classical finite element solution. Therefore, DG methods emerge as a very attractive class of arbitrary order methods for the numerical solution of various classes of PDE problems where classical FEM are not applicable.

6.3. Recommendations

Based on the comparative study on the Weak Galerkin Finite Element Methods and Discontinuous Galerkin Finite Element Methods the following basic recommendations suggested:

1. The effectiveness of the program Laplace to deal with solutions to other second-order elliptic BVPs than, hyperbolic problems needs to be validated to compare the WG-FEMs with DG-FEMs.
2. A comparative study on the WG, DG and mixed FEM is best and optimal to solve more PDEs applications using MATLAB based on error and convergence rates.
3. In addition, we would like to try higher-order elements in both FEMs. Because of comparison of these WG-FEMs and DG-FEMs approaches using higher order elements would be interesting because the disadvantages associated with the DG-FEMs (specifically, the larger number of degrees of freedom) would be minimized with the higher order elements.

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