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Stability Analysis and Implementation of a New Fully Explicit Fourth-Stage Fourth-Order Runge-Kutta Method

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Abstract: The essence of this paper is to analyze the stability and implementation of a newly derived explicit fourth-stage fourth-order Runge-Kutta method. Efforts will be made to carry out a comparative analysis with an existing classical fourth stage fourth order explicit Runge Kutta method. The implementation on initial-value problems revealed that the method compared favorably well with the existing classical fourth stage fourth order explicit Runge Kutta method. The stability analysis revealed that the method is absolute stable, and capable in handling initial value problems in ordinary differential equations. The Taylor series expansion was carried out on the general explicit fourth stage fourth order Runge Kutta scheme, and then, parameters and coefficients were varied with the expansion to generate a set of linear / nonlinear equations which were resolved to generate the method. This approach has shown that when parameters are properly varied, and all equations in the set, whether lineaer / nonlinear, it will definitely give birth to a method that will improve results.

Keywords: Stability, Linear and Non-linear Equations, Taylor Series, Parameters, Initial-Value Problems, Implementation

1. Introduction

Runge-kutta methods are numerical (one-step) methods for solving initial value problems of the form:

$$y'(x) = f(x, y), y(x_0) = y_0. \quad (1)$$

According to Butcher [2, 5-9], in Ordinary Differential Equations, initial value problems are problems with subsidiary conditions which are called initial conditions and are applicable to solving real life problems like growth and decay problems, temperature problems, falling body problems, problems in chemical engineering, electrical circuit problems, dilution problems e.t.c. For example, a person opens an account with an initial amount that accrues interest compounded continuously and assuming no additional deposits or withdrawals, we can transform this problem to an initial-value first-order ordinary differential equation and solve for the amount that will be in the account after a period of time at a particular interest rate. Here, the initial amount will be the initial condition while the interest rate will be the constant of proportionality. The above problem is growth problem because it has to do with interest.

Explicit Runge-Kutta methods have proven to be one of the best methods for solving initial value problems in Ordinary Differential Equations. It has overtime been discovered that the method is subject to improvement, hence more research is still been carried out to get better efficiency and accuracy of the method. Many researchers have worked to improve on the accuracy of the method as can been seen in the work of Agbeboh, Barletti, Brugnano, Gear, Gianluca and Van Der Houwen [1, 3, 4, 10-13] e.t.c. This is what motivated us to work on a fourth–stage fourth-order method to find out how variation in parameters can yield further efficiency.

2. Methods of Derivation

i. Obtaining the Taylor series expansion of $k_{i's}^{,}$about the point (x_n, y_n), i = 2,3,4,

ii. Carry out substitution to ensure that all the $k_{i's}^{\prime}$ are in terms of k_1 only.
iii. Insert the $k_{i's}^{\prime}$ in terms of k_1 only into $b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4$
iv. Separate all partial derivatives involving only y with their coefficient from all partial derivatives involving x, y and their coefficients.
v. Compare the coefficients of all partial derivatives involving only y with Taylor series expansion involving only partial derivatives with respect to y of the form:

$$\emptyset_T(x,y,h) = f + \frac{h}{2!}ff_y + \frac{h^2}{3!}\left(ff_y^2 + f^2f_{yy}\right) + \frac{h^3}{4!}(4f^2f_yf_{yy} + ff_y^3 + f^3f_{yyy}) + \frac{h^4}{5!}(7f^3f_yf_{yyy} + 4f^3f_{yy}^2 + 11f^2f_y^2f_{yy} + ff_y^4 + f^4f_{yyyy})$$

vi. Arising from (v), a set of linear/nonlinear equations will be generated. Represent these equations and their partial derivatives on Butcher's rooted trees.
vii. Compare the coefficient of all partial derivatives involving x, y only with Taylor series expansion involving partial derivative of x, y only of the form:

$$\phi(x,y,h) = f + \frac{h}{2!}f_x + \frac{h^2}{3!}\left(f_{xx} + 2ff_{xy} + f_xf_y\right) + \frac{h^3}{4!}(f_{xxx} + 3ff_{xxy} + 3f^2f_{xyy} + 3f_xf_{xy} + 5ff_yf_{xy} + 3ff_xf_{yy} + f_{xx}f_y + f_xf_y^2)$$

viii. As a result, also, a set of linear/non-linear equations will be generated. Represent those equations and their x, y partial derivatives on Butcher's rooted trees.
ix. Vary the parameters from the two different sets of equations generated above. Two new fourth-stage fourth-order explicit Runge-Kutta formulae will be birthed with their various Butcher's rooted trees.

From the steps above, we present here the derivation of our proposed formula whose basis is predicated on the scheme in (1), hence, the fourth-stage fourth–order method is:

$$y_{n+1} = y_n + h\,(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4)$$
$$k_1 = f\,(x_n, y_n)$$
$$k_2 = f\,(x_n + c_2, y_n + ha_{21}k_1)$$
$$k_3 = f\,(x_n + c_3h, y_n + h(a_{31}k_1 + a_{32}k_2))$$
$$k_4 = f\,(x_n + c_4h, y_n + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)) \quad (2)$$

Using Taylor's series expansion for $k_i's$, we have:

$$k_1 = f\,(x_n, y_n)$$
$$k_2 = \sum_{r=o}^{\infty}\frac{1}{r!}(c_2h\frac{\partial}{\partial x} + ha_{21}k_1\frac{\partial}{\partial y})^r f(x_n, y_n)$$
$$k_3 = \sum_{r=o}^{\infty}\frac{1}{r!}(c_3h\frac{\partial}{\partial x} + h(a_{31}k_1 + a_{32}k_2)\frac{\partial}{\partial y})^r f(x_n, y_n)$$
$$k_4 = \sum_{r=o}^{\infty}\frac{1}{r!}(c_4h\frac{\partial}{\partial x} + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\frac{\partial}{\partial y})^r f(x_n, y_n) \quad (3)$$

Hence, we have:

$$k_1 = f$$
$$k_2 = f + \left(c_2hf_x + ha_{21}k_1f_y\right) + \frac{1}{2!}\left(c_2hf_x + ha_{21}k_1f_y\right)^2 + \frac{1}{3!}\left(c_2hf_x + ha_{21}k_1f_y\right)^3 + \frac{1}{4!}\left(c_2hf_x + ha_{21}k_1f_y\right)^4 + 0(h^5)$$
$$k_3 = f + \left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)\,f_y\right) + \frac{1}{2!}\left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)\,f_y\right)^2 + \frac{1}{3!}\left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)\,f_y\right)^3 + \frac{1}{4!}\left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)\,f_y\right)^4 + 0(h^5)$$
$$k_4 = f + \left(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\,f_y\right) + \frac{1}{2!}\left(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\,f_y\right)^2 + \frac{1}{3!}(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\,f_y)^3 + \frac{1}{4!}\left(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\,f_y\right)^4 + 0(h^5) \quad (4)$$

Expanding fully and substituting the various k_i' s, i = 2, 3, 4 into their various positions in terms of k_1 only and collecting like terms, in terms of y derivatives and (x, y) derivatives separately, we have:

$$k_1 = f$$

$$k_2 = f + ha_{21}ff_y + \frac{h^2}{2!}a_{21}^2f^2f_{yy} + \frac{h^3}{3!}a_{21}^3f^3f_{yyy} + \frac{h^4}{4!}a_{21}^4f^4f_{yyyy} + hc_2f_x + \frac{h^2}{2!}c_2^2f_{xx} + h^2c_2a_{21}ff_{xy}\frac{h^3}{3!}c_2^3f_{xxx} + \frac{h^3}{2!}c_2^2a_{21}ff_{xxy} + \frac{h^3}{2!}c_2a_{21}^2f^2f_{xyy} + \frac{h^4}{4!}c_2^4f_{xxxx} + \frac{h^4}{3!}c_2^3a_{21}ff_{xxxy}\frac{h^4}{2!2!}c_2^2a_{21}^2f^2f_{xxyy} + \frac{h^4}{3!}c_2a_{21}^3f^3f_{xyyy} + 0\,(h^5)$$

$$k_3 = f + h\,(a_{31} + a_{32})ffy + h^2a_{21}a_{32}ff_y^2 + \frac{h^2}{2!}(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_{yy} + \frac{h^3}{3!}a_{21}a_{32}\big(a_{21} + 2(a_{31} + a_{32})\big)f^2f_yf_{yy} + \frac{h^3}{3!}(a_{31}^3 + 3a_{31}^2a_{32} + 3a_{31}a_{32}^2 + a_{32}^3)f^3f_{yyy} + \frac{h^4}{3!}(a_{32}a_{21}^3 + 3a_{31}^2a_{32}a_{21} + 3a_{32}^2a_{21} + 6a_{31}a_{32}^2a_{21})f^3f_yf_{yyy} + \frac{h^4}{2!}a_{21}^2a_{32}(a_{31} + a_{32})f^3f_{yy}^2 + \frac{h^4}{2!}a_{32}^2a_{21}^2f^2f_y^2f_{yy} + \frac{h^4}{4!}(a_{31}^4 + 4a_{31}^3a_{32} + 6a_{31}^2a_{32}^2 + 4a_{31}a_{32}^3 + a_{32}^4)f^4f_{yyyy} + hc_3f_x + \frac{h^2}{2!}c_3^2f_{xx} + h^2c_3(a_{31} + a_{32})ff_{xy} + h^2c_2a_{32}f_xf_y + \frac{h^3}{3!}c_3^3f_{xxx} + \frac{h^3}{2!}c_3^2(a_{31} + a_{32})ff_{xxy} + \frac{h^3}{2!}c_3(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_{xyy} + h^3c_2a_{32}(a_{31} + a_{32})ff_xf_{yy} + h^3a_{21}a_{32}(c_2 + c_3)ff_yf_{xy} + \frac{h^3}{2!}c_2^2a_{32}f_yf_{xx} + h^3c_2c_3a_{32}f_xf_{xy} + \frac{h^4}{4!}c_3^4f_{xxxx} + \frac{h^4}{3!}c_2^3a_{32}f_{xxx}f_y + \frac{h^4}{2!}c_3^2c_2a_{32}f_xf_{xxy} + \frac{h^4}{2!}a_{21}a_{32}(c_2^2 + c_3^2)ff_yf_{xxy}\frac{h^4}{3!}a_{21}a_{32}(2c_2a_{31} + 3c_2a_{21} + 6c_3a_{31})f^2f_yf_{xyy} + \frac{h^4}{2!}c_3a_{32}c_2^2f_{xx}f_{xy} + \frac{h^4}{2!}c_2^2a_{32}(a_{31} + a_{32})ff_{xx}f_{yy} + \frac{h^4}{2!}a_{21}a_{32}(2c_2a_{31} + 2c_2a_{32} + c_3a_{21})f^2f_{xy}f_{yy} + h^4c_3a_{32}c_2a_{21}ff_{xy}^2 + \frac{h^4}{2!}a_{32}^2c_2^2f_x^2f_{yy} + h^4a_{32}^2a_{21}c_2ff_xf_yf_{yy} + \frac{h^4}{2!}c_3c_2a_{32}(6a_{31} + 2a_{32})ff_xf_{xyy} + \frac{h^4}{2!}c_2a_{32}(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_xf_{yyy} + \frac{h^4}{3!}c_3^3(a_{31} + a_{32})ff_{xxxy} + \frac{h^4}{2!2!}c_3^2(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_{xxyy} + \frac{h^4}{3!}c_3(a_{31}^3 + 3a_{31}^2a_{32} + 3a_{31}a_{32}^2 + a_{32}^3)f^3f_{xyyy} + 0\,(h^5)$$

$$k_4 = f + h\,(a_{41} + a_{42} + a_{43})ff_y + h^2(a_{21}a_{42} + a_{31}a_{43} + a_{32}a_{43})ff_y^2 + \frac{h^2}{2!}(a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43} + 2a_{42}a_{43} + a_{42}^2 + a_{43}^2)f^2f_{yy} + \frac{h^3}{2!}(a_{21}^2a_{42} + a_{31}^2a_{43} + 2a_{31}a_{32}a_{43} + a_{32}^2a_{43} + 2a_{21}a_{41}a_{42} + 2a_{31}a_{41}a_{43} + 2a_{32}a_{41}a_{43} + 2a_{31}a_{42}a_{43} + 2a_{32}a_{42}a_{43} + 2a_{21}a_{42}a_{43} + 2a_{21}a_{42}^2 + 2a_{31}a_{43}^2 + 2a_{32}a_{43}^2)\,f^2f_yf_{yy} + h^3a_{21}a_{32}a_{43}ff_y^3 + \frac{h^3}{2!}(a_{41}^3 + 3a_{41}^2a_{42} + 3a_{41}^2a_{43} + 3a_{41}a_{42}^2 + 6a_{41}a_{42}a_{43} + 3a_{42}^2a_{43} + 3a_{41}a_{43}^2 + 3a_{42}a_{43}^2 + a_{42}^3 + a_{43}^3)f^3f_{yyy} + \frac{h^4}{3!}(a_{31}^3a_{43} + 3a_{31}^2a_{32}a_{43} + 3a_{31}a_{32}^2a_{43} + 3a_{21}a_{41}^2a_{42} + 3a_{31}a_{41}^2a_{43} + 3a_{32}a_{41}^2a_{43} + 6a_{21}a_{41}a_{42}^2 + 6a_{31}a_{41}a_{42}a_{43} + 6a_{32}a_{41}a_{42}a_{43} + 6a_{21}a_{41}a_{42}a_{43} + 3a_{31}a_{41}^2a_{43} + a_{42}a_{21}^3 + 3a_{32}a_{42}^2a_{43} + 6a_{21}a_{42}^2a_{43} + 6a_{31}a_{41}a_{43}^2 + 6a_{32}a_{41}a_{43}^2 + 6a_{31}a_{42}a_{43}^2 + 6a_{32}a_{42}a_{43}^2 + 3a_{21}a_{42}a_{43}^2 + 3a_{21}a_{43}^3 + 3a_{31}a_{43}^3 + 3a_{32}a_{43}^3) + \frac{h^4}{2!}(a_{21}^2a_{32}a_{43} + 2a_{21}a_{31}a_{32}a_{43} + 2a_{21}a_{32}^2a_{43} + 2a_{21}a_{32}a_{41}a_{43} + 2a_{21}a_{32}a_{42}a_{43} + 2a_{21}a_{31}a_{42}a_{43} + 2a_{21}a_{32}a_{42}a_{43} + a_{21}^2a_{42}^2 + a_{21}a_{32}a_{43}^2 + a_{31}^2a_{43}^2 + 2a_{31}a_{32}a_{43}^2 + a_{32}^2a_{43}^2)f^2f_y^2f_{yy} + \frac{h^4}{2!}(a_{21}^2a_{41}a_{42} + a_{31}^2a_{41}a_{42} + 2a_{31}a_{32}a_{41}a_{43} + a_{32}^2a_{41}a_{43} + a_{31}^2a_{41}a_{43} + 2a_{31}a_{32}a_{42}a_{43} + a_{32}^2a_{42}a_{43} + \frac{a_{21}^2a_{42}^2}{2!} + \frac{a_{31}^2a_{43}^2}{2!} + a_{31}a_{32}a_{43}^2 + \frac{a_{31}^2a_{43}^2}{2!})f^3f_{yy}^2 + \frac{h^4}{4!}(a_{41}^4 + 4_{41}^3a_{42} + 4a_{41}^3a_{43} + 6a_{41}^2a_{41}^2 + 12a_{41}^2a_{42}a_{43} + 6a_{42}^2a_{43}^2 + 4a_{41}a_{43}^3 + 4a_{42}a_{43}^3 + 12a_{41}a_{42}^2a_{43} + 2a_{41}a_{42}a_{43}^2 + 6a_{41}^2a_{43}^2 + 4a_{42}^3a_{43} + 4a_{41}a_{42}^3 + a_{42}^4 + a_{43}^4)\,f^4f_{yyyy} + \mathrm{h}c_4f_x + h^2(c_4a_{42} + c_3a_{43})f_xf_y + \frac{h^2}{2!}c_4^2f_{xx} + h^2c_4(a_{41} + a_{42} + a_{43})ff_{xy} + \frac{h^3}{2!}(c_2^2a_{42} + c_3^2a_{43})f_{xx}f_y + h^2(c_4a_{21}a_{42} + c_3a_{31}a_{43} + c_3a_{32}a_{43})ff_{xy}f_y + h^3c_2a_{32}a_{43}f_xf_y^2 + h^3(c_2c_4a_{42} + c_3c_4a_{43})\,f_xf_{xy} + h^3c_2a_{32}a_{43}f_xf_y^2 + h^3(c_2c_4a_{42} + c_3c_4a_{43})\,f_xf_{xy} + h^3(c_2a_{21}a_{42} + c_4a_{31}a_{43} + c_4a_{32}a_{43})ff_yf_{xy} + h^3(c_2a_{41}a_{42} + c_3a_{41}a_{43} + c_3a_{42}a_{43} + c_2a_{42}a_{43} + c_2a_{42}^2 + c_3a_{43}^2)ff_xf_{yy} + \frac{h^3}{3!}c_4^3f_{xxx} + \frac{h^3}{2!}(c_4^2a_{41} + c_4^2a_{42} + c_4^2a_{43})ff_{xxy} + \frac{h^3}{2!}c_4(a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43} + a_{42}^2 + 2a_{42}a_{43} + a_{43}^2)f^2f_{xyy} + \frac{h^4}{3!}(c_2^3a_{42} + c_3^3a_{43})f_{xxx}f_y + \frac{4}{3!}(3c_2^2a_{21}a_{42} + c_3^2a_{31}a_{43} + 3c_3^2a_{32}a_{43} + 3c_4^2a_{21}a_{42} + 3c_4^2a_{31}a_{43} + 3c_4^2a_{32}a_{43})ff_{xxy}f_y + \frac{h^4}{2!}(c_2a_{21}^2a_{42} + 2c_3a_{31}a_{32}a_{43} + c_3a_{31}^2a_{43} + c_3a_{32}^2a_{43} + 2c_4a_{21}a_{41}a_{42} + 2c_4a_{31}a_{41}a_{43} + 2c_4a_{32}a_{41}a_{43} + 2c_4a_{21}a_{42}^2 + 2c_4a_{31}a_{42}a_{43} + 2c_4a_{32}a_{42}a_{43} + 2c_4a_{21}a_{42}a_{43} + 2c_4a_{31}a_{43}^2 + 2c_4a_{32}a_{43}^2)f^2f_yf_{xyy} + \frac{h^4}{3!}(c_2^2a_{32}a_{43})f_{xx}f_y^2 + h^4(c_2a_{21}a_{32}a_{43} + c_3a_{21}a_{32}a_{43} + c_4a_{21}a_{32}a_{43})ff_{xy}f_y^2 + \frac{h^4}{2!}(2c_2a_{31}a_{32}a_{43} + 2c_2a_{32}^2a_{43} + 2c_2a_{32}a_{41}a_{43} + 2c_2a_{32}a_{42}a_{43} + 2c_2a_{31}a_{42}a_{43} + 2c_2a_{32}a_{42}a_{43} + 2c_3a_{21}a_{42}a_{43} + 2c_2a_{21}a_{42}^2 + c_2a_{32}a_{43}^2 + c_3a_{31}a_{43}^2 + c_3a_{32}a_{43}^2 + c_3a_{31}a_{43}^2 + c_3a_{32}a_{43}^2)ff_xf_yf_{yy} + h^4(c_2c_3a_{32}a_{43} + c_2c_4a_{32}a_{43})f_xf_yf_{xy} + \frac{h^4}{2!}(c_2^2c_4a_{42} + c_3^2c_4a_{43})f_{xx}f_{xy} + h^4(c_2c_4a_{21}a_{42} + c_3c_4a_{31}a_{43} + c_3c_4a_{32}a_{43})ff_{xy}^2 + \frac{h^4}{2!}(c_4a_{21}^2a_{42} + c_4a_{31}^2a_{43} + 2c_4a_{31}a_{32}a_{43} + c_4a_{32}^2a_{43} + 2c_2a_{21}a_{41}a_{42} + 2c_3a_{31}a_{41}a_{43} + 2c_3a_{32}a_{41}a_{43} + 2c_3a_{31}a_{42}a_{43} + 2c_3a_{32}a_{42}a_{43} + c_2a_{21}a_{42}^2 + c_3a_{31}a_{43}^2 + c_3a_{32}a_{43}^2)\,f^2f_{xy}f_{yy} + \frac{h^4}{2!}(2c_2c_3a_{42}a_{43} + c_2^2a_{42}^2 + c_3^2a_{43}^2)f_x^2f_{yy} + \frac{h^4}{2!}(c_2c_4^2a_{42} + c_3c_4^2a_{43})f_xf_{xxy} + h^4(c_2c_4a_{41}a_{42} + c_3c_4a_{41}a_{43} + c_2c_4a_{42}^2 + c_3c_4a_{42}a_{43}) + c_2c_4a_{42}a_{43} + c_3c_4a_{43}^2)ff_xf_{xyy} + \frac{h^4}{2!}(c_2c_{41}^2a_{42} + c_3c_{41}^2a_{43} + 2c_2a_{41}a_{42}^2 + 2c_3a_{41}a_{42}a_{43} + 2c_2a_{41}a_{42}a_{43} + c_3c_{42}^2a_{43} + 2c_2c_{42}^2a_{43} + 2c_3a_{41}a_{43}^2 + 2c_3a_{42}a_{43}^2 + c_2a_{42}a_{43}^2 +$$

$$c_2c_{42}^3+c_3a_{43}^3)f^2f_xf_{yyy}+\frac{h^4}{4!}c_4^4f_{xxxx}+\frac{h^4}{3!}(c_4^3a_{41}+c_4^3a_{42}+c_4^3a_{43})ff_{xxxy}+\frac{h^4}{2!2!}c_4^2(a_{41}^2+2a_{41}a_{42}+2a_{41}a_{43}+a_{42}^2+2a_{42}a_{43}+a_{43}^2)f^2f_{xxyy}+\frac{h^4}{3!}c_4(a_{41}^3+3a_{41}^2a_{42}+3a_{41}^2a_{43}+3a_{41}a_{42}^2+6a_{41}a_{42}a_{43}+3a_{42}^2a_{43}+3a_{42}a_{43}^2+a_{42}^3+a_{43}^3)f^3f_{xyyy}+\frac{h^4}{2!2!}(2c_2^2a_{41}a_{42}+2c_3^2a_{41}a_{43}+2c_3^2a_{42}a_{43}+c_2^2a_{42}^2+c_3^2a_{43}^2)ff_{xx}f_{yy.}+0(h^5). \quad (5)$$

Putting the $k_{i's}^{;}$ (y derivatives only) into $y_{n+1}=y_n+h(b_1k_1+b_2k_2+b_3k_3+b_4k_4$ where $\phi(x,y,h)=b_1k_1+b_2k_2+b_3k_3+b_4k_4$, we have:

$$y_{n+1}=y_n+h(b_1f+b_2(f+ha_{21}ff_y+\frac{h^2}{2!}a_{21}^2f^2f_{yy}+\frac{h^3}{3!}a_{21}^3f^3f_{yyy}+\frac{h^4}{4!}a_{21}^4f^4f_{yyyy})+b_3(f+h(a_{31}+a_{32})ffy+h^2a_{21}a_{32}ff_y^2+\frac{h^2}{2!}(a_{31}^2+2a_{31}a_{32}+a_{32}^2)f^2f_{yy}+\frac{h^3}{3!}a_{21}a_{32}\big(a_{21}+2(a_{31}+a_{32})\big)f^2f_yf_{yy}+\frac{h^3}{3!}(a_{31}^3+3a_{31}^2a_{32}+3a_{31}a_{32}^2+a_{32}^3)f^3f_{yyy}+\frac{h^4}{3!}(a_{32}a_{21}^3+3a_{31}^2a_{32}a_{21}+3a_{32}^3a_{21}+6a_{31}a_{32}^2a_{21})f^3f_yf_{yyy}+\frac{h^4}{2!}a_{21}^2a_{32}(a_{31}+a_{32})f^3f_{yy}^2+\frac{h^4}{2!}a_{32}^2a_{21}^2f^2f_y^2f_{yy}+\frac{h^4}{4!}(a_{31}^4+4a_{31}^3a_{32}+6a_{31}^2a_{32}^2+4a_{31}a_{32}^3+a_{32}^4)f^4f_{yyyy})+b_4(f+h(a_{41}+a_{42}+a_{43})ff_y+h^2(a_{21}a_{42}+a_{31}a_{43}+a_{32}a_{43})ff_y^2+\frac{h^2}{2!}(a_{41}^2+2a_{41}a_{42}+2a_{41}a_{43}+2a_{42}a_{43}+a_{42}^2+a_{43}^2)f^2f_{yy}+\frac{h^3}{2!}(a_{21}^2a_{42}+a_{31}^2a_{43}+2a_{31}a_{32}a_{43}+a_{32}^2a_{43}+2a_{21}a_{41}a_{42}+2a_{31}a_{41}a_{43}+2a_{32}a_{41}a_{43}+2a_{31}a_{42}a_{43}+2a_{32}a_{42}a_{43}+2a_{21}a_{42}a_{43}+2a_{21}a_{42}^2+2a_{31}a_{43}^2+2a_{32}a_{43}^2)f^2f_yf_{yy}+h^3a_{21}a_{32}a_{43}ff_y^3+\frac{h^3}{2!}(a_{41}^3+3a_{41}^2a_{42}+3a_{41}^2a_{43}+3a_{41}a_{42}^2+6a_{41}a_{42}a_{43}+3a_{42}^2a_{43}+3a_{41}a_{43}^2+3a_{42}a_{43}^2+a_{42}^3+a_{43}^3)f^3f_{yyy}+\frac{h^4}{3!}(a_{31}^3a_{43}+3a_{31}^2a_{32}a_{43}+3a_{31}a_{32}^2a_{43}+3a_{21}a_{41}^2a_{42}+3a_{31}a_{41}^2a_{43}+3a_{32}a_{41}^2a_{43}+6a_{21}a_{41}a_{42}^2+6a_{31}a_{41}a_{42}a_{43}+6a_{32}a_{41}a_{42}a_{43}+6a_{21}a_{41}a_{42}a_{43}+3a_{31}a_{41}^2a_{43}+a_{42}a_{21}^3+3a_{32}a_{42}^2a_{43}+6a_{21}a_{42}^2a_{43}+6a_{31}a_{41}a_{43}^2+6a_{32}a_{41}a_{43}^2+6a_{31}a_{42}a_{43}^2+6a_{32}a_{42}a_{43}^2+3a_{21}a_{42}a_{43}^2+3a_{21}a_{43}^3+3a_{31}a_{43}^3+3a_{32}a_{43}^3)+\frac{h^4}{2!}(a_{21}^2a_{32}a_{43}+2a_{21}a_{31}a_{32}a_{43}+2a_{21}a_{32}^2a_{43}+2a_{21}a_{32}a_{41}a_{43}+2a_{21}a_{32}a_{42}a_{43}+2a_{21}a_{31}a_{42}a_{43}+2a_{21}a_{32}a_{42}a_{43}+a_{21}^2a_{42}^2+a_{21}a_{32}a_{43}^2+a_{31}^2a_{43}^2+2a_{31}a_{32}a_{43}^2+a_{32}^2a_{43}^2)f^2f_y^2f_{yy}+\frac{h^4}{2!}(a_{21}^2a_{41}a_{42}+a_{31}^2a_{41}a_{42}+2a_{31}a_{32}a_{41}a_{43}+a_{32}^2a_{41}a_{43}+a_{31}^2a_{41}a_{43}+2a_{31}a_{32}a_{42}a_{43}+a_{32}^2a_{42}a_{43}+\frac{a_{21}^2a_{42}^2}{2!}+\frac{a_{31}^2a_{43}^2}{2!}+a_{31}a_{32}a_{43}^2+\frac{a_{31}^2a_{43}^2}{2!})f^3f_{yy}^2+\frac{h^4}{4!}(a_{41}^4+4_{41}^3a_{42}+4a_{41}^3a_{43}+6a_{41}^2a_{41}^2+12a_{41}^2a_{42}a_{43}+6a_{42}^2a_{43}^2+4a_{41}a_{43}^3+4a_{42}a_{43}^3+12a_{41}a_{42}^2a_{43}+2a_{41}a_{42}a_{43}^2+6a_{41}^2a_{43}^2+4a_{42}^3a_{43}+4a_{41}a_{42}^3+a_{42}^4+a_{43}^4)f^4f_{yyyy})) \quad (6)$$

The Taylor series expansion for y derivatives only is:

$$\emptyset_T(x,y,h)=f+\frac{h}{2!}ff_y+\frac{h^2}{3!}(ff_y^2+f^2f_{yy})+\frac{h^3}{4!}(4f_y^2f_yf_{yy}+ff_y^3+f^3f_{yyy})+\frac{h^4}{5!}(4f^3f_yf_{yyy}+4f^3f_{yy}^2+11f^2f^2f_{yy}+ff_y^4+f^4f_{yyyy}) \quad (7)$$

Equating the coefficients, we have the following equations:

$$b_1+b_2+b_3+b_4=1 \quad (8)$$

$$b_2a_{21}+b_3(a_{31}+a_{32})+b_4(a_{41}+a_{42}+a_{43})=½ \quad (9)$$

$$b_3a_{21}+b_4(a_{21}a_{41}+a_{43}(a_{31}+a_{32})=1/6 \quad (10)$$

$$b_2a_{21}^2+b_3(a_{31}^2+2a_{31}a_{32}+a_{32}^2)+b_4(a_{41}^2+2a_{41}a_{42}+2a_{41}a_{43}+2a_{42}a_{43}+a_{42}^2+a_{43}^2)=\frac{1}{3} \quad (11)$$

$$b_2a_{21}^3+b_3(a_{31}^3+3a_{31}^2a_{32}+3a_{31}a_{32}^2+a_{32}^3)+b_4(a_{41}^3+3a_{41}^2a_{42}+3a_{42}^2a_{43}+3a_{41}a_{42}^2+6a_{41}a_{42}a_{43}+3a_{42}^2a_{43}+3a_{41}a_{43}^2+3a_{42}a_{43}^2+a_{42}^3+a_{43}^3)=¼. \quad (12)$$

$$b_3a_{21}a_{32}\big(a_{21}+2(a_{31}+a_{32})\big)+b_4(a_{41}^2a_{42}+a_{43}(a_{31}+a_{32})^2+2a_{21}a_{42}(a_{41}+a_{42}+a_{43})+2a_{31}4_3(a_{41}+a_{42}+a_{43})+2a_{32}a_{43}(a_{41}+a_{42}+a_{43}))=1/3 \quad (13)$$

$$b_4a_{21}a_{32}a_{43}=1/24 \quad (14)$$

Solving, we have, let A = a_{21}, B = $a_{31}+a_{32}$,

$$\text{P}=a_{41}+a_{42}+a_{43}$$

From the above, we have:

$$\text{A}b_2+\text{B}b_3+\text{P}b_4=½ \quad (15)$$

$$Aa_{32}b_3 + Aa_{42}b_4 + Ba_{43}b_4 = 1/6 \tag{16}$$

$$A^2b_2 + B^2b_3 + P^2b_4 = 1/3 \tag{17}$$

$$A^3b_2 + B^3b_3 + P^3b_4 = 1/4 \tag{18}$$

$$A^2a_{32} + 2BAa_{32}b_3 + A^2a_{42}b_4 + B^2a_{43}b_4 + 2APa_{42}b_4 + 2Pa_{31}a_{43}b_4 + 2Pa_{32}a_{43}b_4 = 1/3 \tag{19}$$

$$Aa_{32}a_{43}b_4 = 1/24 \tag{20}$$

Now from (9), setting $b_1 = b_4 = 1/6$, $b_2 = b_3 = 2/6$

$$\text{(16) becomes: } 2A + 2B + P = 3 \tag{21}$$

$$\text{(18) becomes: } 2A^2 + 2B^2 + P^2 = 2 \tag{22}$$

$$\text{(19) becomes: } 2A^3 + 2B^3 + P^3 = 3/2 \tag{23}$$

From (22), (23), (24), we have:

$$A = 1/2,\ B = 1/2,\ P = 1 \tag{24}$$

Hence, (17) becomes:

$$2a_{32} + a_{42} + a_{43} = 2 \tag{25}$$

$$\text{(21) becomes: } a_{32}a_{43} = 1/2 \tag{26}$$

$$\text{(20) becomes: } 6a_{32} + 5a_{42} + a_{43} + 8a_{43}a_{43} + 8a_{42}a_{43} = 8 \tag{27}$$

From (27), let $a_{43} = \frac{1}{2}$, then $a_{32} = 1$, From (26), $a_{42} = -\frac{1}{2}$, But A = ½, B = ½, P = 1, therefore, $a_{21} = \frac{1}{2}, a_{31} + a_{32} = \frac{1}{2}$,

$$\rightarrow a_{31} = -\frac{1}{2}, a_{41} + a_{42} + a_{43} = 1,\ a_{41} = 1 \tag{28}$$

In conclusion, $b_1 = 1/6$, $b_2 = 2/6$, $b_3 = 2/6$, $b_4 = 1/6$

$$a_{21} = \frac{1}{2},\ a_{31} = -\frac{1}{2},\ a_{32} = 1, a_{41} = 1, a_{42} = -\frac{1}{2}, a_{43} = \frac{1}{2}.$$

The fourth-stage fourth-order explicit method ($f(y)$ functional derivatives) becomes:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(y_n)$$

$$k_2 = f(y_n + \frac{h}{2}k_1)$$

$$k_3 = f(y_n + \frac{h}{2}(-k_1 + 2k_2))$$

$$k_4 = f(y_n + \frac{h}{2}(2k_1 - k_2 + k_3))$$

The Butchers Tableau for the above fourth-stage fourth-Order method is:

0				
0	½			
0	- ½	1		
0	1	-½	½	
	1/6	1/3	1/3	1/6

3. Stability Function of Our Fourth Stage Fourth Order Method

Theorem 3.0: The explicit fourth-stage fourth-order method ($f(y)$ only) is absolutely stable.
Proof: Applying the test equation $y' = \lambda y$ on the above method, we have:

$$k_1 = \lambda y, k_2 = f\left(y_n + \frac{h}{2}k_1\right) = \lambda(y_n + \frac{h\lambda y}{2})$$

$$k_2 = \lambda y\left(1 + \frac{\lambda h}{2}\right)$$

$$k_3 = f\left(y_n + \frac{h}{2}(-k_1 + 2k_2)\right) = \lambda\left(y_n - \frac{h\lambda y}{2} + \frac{2h}{2}\left(\lambda y\left(1 + \frac{\lambda h}{2}\right)\right)\right)$$

$$k_3 = \lambda\left(y_n - \frac{\lambda y h}{2} + h\lambda y + \frac{h^2\lambda^2 y}{2}\right)$$

$$k_3 = \lambda y\left(1 + \frac{h\lambda}{2} + \frac{h^2\lambda^2 y}{2}\right)$$

$$k_4 = f\left(y_n + \frac{h}{2}(2k_1 - k_2 + k_3)\right) + \lambda(y_n + h\lambda y - \frac{\lambda y h}{2}\left(1 + \frac{h\lambda}{2}\right) + \frac{\lambda y h}{2}(1 + \frac{h\lambda}{2} + \frac{h^2\lambda^2}{2}))$$

$$k_4 = \lambda\left(y_n + h\lambda y - \frac{\lambda y h}{2} - \frac{h^2\lambda^2 y}{4} + \frac{\lambda y h}{2} + \frac{h^2\lambda^2 y}{4} + \frac{h^3\lambda^3 y}{4}\right)$$

$$k_4 = \lambda\left(y_n + h\lambda y + \frac{h^3\lambda^3 y}{4}\right), k_4 = \lambda y\left(1 + \lambda h + \frac{h^3\lambda^3 y}{4}\right)$$

$$then\ y_{n+1} - y_n = \frac{h}{6}\left[\lambda y + 2\lambda y\left(1 + \frac{h\lambda}{2}\right) + 2\lambda y\left(1 + \frac{h\lambda}{2} + \frac{h^2\lambda^2}{4}\right) + \lambda y\left(1 + \lambda y + \frac{h^3\lambda^3}{4}\right)\right]$$

$$y_{n+1} - y_n = \frac{\lambda y h}{6}\left[1 + 2\left(1 + \frac{h\lambda}{2}\right) + 2\left(1 + \frac{h\lambda}{2} + \frac{h^2\lambda^2}{2}\right) + 1\left(1 + \lambda y + \frac{h^3\lambda^3}{4}\right)\right]$$

$$y_{n+1} - y_n = \frac{\lambda y h}{6}\left[6 + 3\lambda h + h^2\lambda^2 + \frac{h^3\lambda^3}{4}\right]$$

Dividing by y and setting $\mu = \lambda h, we\ have$:

$$\frac{y_{n+1} - y_n}{y_n} = \frac{\mu}{6}\left[6 + 3\lambda h + h^2\lambda^2 + \frac{h^3\lambda^3}{4}\right]$$

$$\frac{y_{n+1}}{y_n} - 1 = \left[\mu + \frac{\mu^2}{2} + \frac{\mu^3}{6} + \frac{\mu^4}{24}\right]$$

$$\frac{y_{n+1}}{y_n} = 1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \frac{\mu^4}{4!} = 0$$

Hence, we have the stability polynomial, which is the same as the Classical fourth-order method.
Resolving the above polynomial using MAPLE, we have the complex roots as follows:
-1.72944423106770545660-0.88897437612186582717i,
-1.72944423106770545660+0.88897437612186582717i,
-0.27055576893229454343-2.50477590436243448970i,
-0.27055576893229454343+2.50477590436243448970i.

Plotting the complex roots on a graph (the real parts on the x-axis and the imaginary parts on the y-axis) using MATLAB CODE, we have the absolute stability region seen in the diagram below:

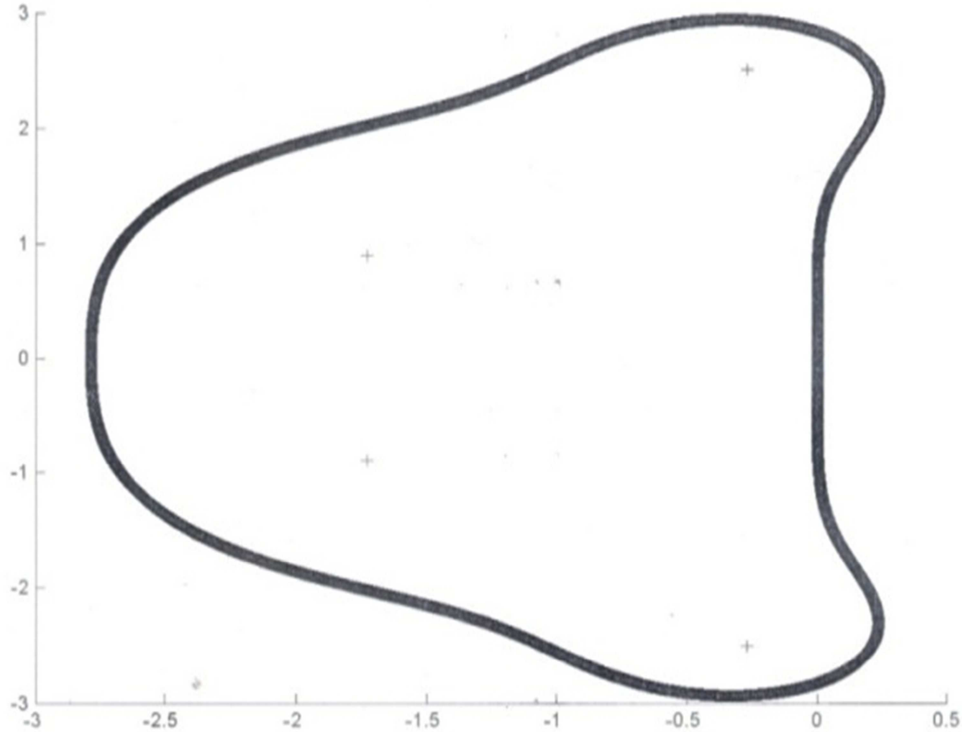

Figure 1. *Region of Absolute Stability For fourth-Stage fourth-order (y derivatives).*

4. Implementation on Initial Value Problems

The results are presented in the table below:

Table 1. *Table of Results.*

(i) $y' = y - y^2, y(0) = 0.5, 0 \leq x \leq 1$. *Theoretical Solution is* $y(x_n) = \frac{1}{1+e^{-x_n}}$

XN	TSOL	YN (4^{th}-stage)	Error (4^{th})	YN (Classical)
.1D+00	0.5249791D+00	0.5249791D+00	-.194D-08	0.52497D+00
.2D+00	0.5498340D+00	0.5498340D+00	-.380D-08	0.54983D+00
.3D+00	0.5744425D+00	0.5744425D+00	-.542D-08	0.57444D+00
.4D+00	0.5986876D+00	0.5986876D+00	-.665D-08	0.59868D+00
.5D+00	0.6224593D+00	0.6224593D+00	-.737D-08	0.62245D+00
.6D+00	0.6456563D+00	0.6456563D+00	-.749D-08	0.64565D+00
.7D+00	0.6681877D+00	0.6681877D+00	-.695D-08	0.66818D+00
.8D+00	0.6899744D+00	0.6899744D+00	-.573D-08	0.68997D+00
.9D+00	0.7109495D+00	0.7109495D+00	-.386D-08	0.71094D+00
.1D+01	0.7310585D+00	0.7310585D+00	-.137D-08	0.73105D+00

(ii) $y' = -y, y(0) = 1, 0 \leq x \leq 1$. *Theoretical Solution is* $y(x_n) = \frac{1}{e^{x_n}}, h = 0.1$

XN	TSOL	YN (4^{th}-stage)	Error (4^{th}-stage)	YN (Classical)	Error (classical)
.1D+00	0.90483D+00	0.90483D+00	-.8196D-07	0.90483D+00	-.8196404044369D-07
.2D+00	0.81873D+00	0.81873D+00	-.1483D-06	0.81873D+00	-.1483282683346D-06
.3D+00	0.74081D+00	0.74081D+00	-.2013D-06	0.74081D+00	-.2013194597694D-06
.4D+00	0.67032D+00	0.67032D+00	-.2428D-06	0.67032D+00	-.2428818514089D-06
.5D+00	0.60653D+00	0.60653D+00	-.2747D-06	0.60653D+00	-.2747107467060D-06
.6D+00	0.54881D+00	0.54881D+00	-.2982D-06	0.54881D+00	-.2982822888686D-06
.7D+00	0.49658D+00	0.49658D+00	-.3148D-06	0.49658D+00	-.3148798197183D-06
.8D+00	0.44932D+00	0.44932D+00	-.3256D-06	0.44932D+00	-.3256172068089D-06
.9D+00	0.40656D+00	0.40656D+00	-.3314D-06	0.40656D+00	-.3314594766990D-06
.1D+01	0.36787D+00	0.36787D+00	-.3332D-06	0.36787D+00	-.3332410563051D-06

(iii) $y' = y, y(0) = 1, 0 \leq x \leq 1$. *Theoretical Solution is* $y(x_n) = e^{x_n}, h = 0.1$

XN	TSOL	YN(4th-stage)	Error (4th-stage)	YN (Classical)	Error (class.)
.1D+00	0.1105D+01	0.1105D+01	0.847D-07	0.1105D+01	0.847D-07
.2D+00	0.1221D+01	0.1221D+01	0.187D-06	0.1221D+01	0.187D-06
.3D+00	0.1349D+01	0.1349D+01	0.310D-06	0.1349D+01	0.310D-06
.4D+00	0.1491D+01	0.1491D+01	0.457D-06	0.1491D+01	0.457D-06
.5D+00	0.1648D+01	0.1648D+01	0.632D-06	0.1648D+01	0.632D-06
.6D+00	0.1822D+01	0.1822D+01	0.838D-06	0.1822D+01	0.838D-06
.7D+00	0.2013D+01	0.2013D+01	0.108D-05	0.2013D+01	0.108D-05
.8D+00	0.2225D+01	0.2225D+01	0.136D-05	0.2225D+01	0.136D-05
.9D+00	0.2459D+01	0.2459D+01	0.169D-05	0.2459D+01	0.169D-05
.1D+01	0.2718D+01	0.2718D+01	0.208D-05	0.2718D+01	0.208D-05

5. Conclusion

It is clearly seen from the tables of results that the new fourth stage fourth order explicit Runge Kutta method compares favorably well with the classical fourth stage fourth order explicit method. The error columns in our tables of results justify our claim. Also, from our stability analysis it shows that the method is absolutely stable.

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