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Euler's Method for Solving Logistic Growth Model Using MATLAB

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Abstract: This paper introduces Euler's explicit method for solving the numerical solution of the population growth model, logistic growth model. The Euler method is a first-order method, which means that the local error (error per step) is proportional to the square of the step size, and the global error (error at a given time) is proportional to the step size. Euler's method is a numerical method that you can use to approximate the solution to an initial value problem with a differential equation that can't be solved using a more traditional method, like the methods we use to solve separable, exact, or linear differential equations. To validate the applicability of the method on the proposed equation, a model example has been solved for different values of parameters. Using this balance law, we can develop the Logistic Model for population growth. For this model, we assume that we add population at a rate proportional to how many are already there. The numerical results in terms of point wise absolute errors presented in tables and graphs show that the present method approximates the exact solution very well. We discuss and explain the solution of logistic growth of population, the kinds of problems that arise in various fields of sciences and engineering. This study aims to solve numerically Euler's method for solving using the Matlab.

Keywords: Euler's Explicit Method, Logistic Growth Model, Least Square Method, 6th Order RK Method, MATLAB

1. Introduction

In this paper, we explore the Euler method for the numerical solution of first order differential equations. The Euler method is the simplest and most fundamental method for numerical integration. The differential equation to be solved has the form $\frac{dy}{dx} = f(x, y)$, with an initial condition of the form $y(x_0) = y_0$. From the idea of differentiation, we know that differentiation is an operation that allows us to find a function that outputs the rate of change of one variable with respect to another variable. The population growth models are some of the differential equation. The analytical solutions of some differential equations are not easily solvable. At a time of this numerical solutions or approximate solutions were extracted using different numerical methods, such as Euler's method and Runge Kutta method. Euler's method is the most elementary approximation technique for solving initial-value problems. Although it is seldom used in practice, the simplicity of its derivation can be used to illustrate the techniques involved in the construction of some of the more advanced techniques, without the cumbersome algebra that accompanies these constructions [1]. Differential Equations (DEs) form the basis of very many mathematical models of physical, chemical and biological phenomena, and more recently their use has spread into economics, financial forecasting, image processing and other fields. To investigate the predictions of DE models of such phenomena it is often necessary to approximate their solution numerically, commonly in combination with the analysis of simple special cases; while in some of the recent instances the numerical models play an almost independent role [2]. X(t) denotes the size of a population at time t, and dx/dt or x' (t) the rate of change of population size. We shall continue to assume that the population growth rate depends only on the population's size. Such an assumption appears to be reasonable for simple organisms such as micro-organisms. For more complicated organisms like plants, animals, or human beings this is obviously an oversimplification since it ignores intra-species competition for resources as well as other significant factors, including age structure (the mortality rate may depend on age rather than on population density, while the birth rate may depend on the adult population size rather than on total population size). Furthermore, the possibility that birth or death rates may be influenced by the size of populations that interact with the population under study must also be considered (competition, predation, mutualism) [3]. This paper aimed to solve one of the population growth models which was proposed by *Verhulst* called Logistic Growth model analytically and numerically using Euler explicit method and compare the two solutions.

2. Description of the Model

Consider, $P(t)$ be a population of the species at a time t, then we can describe as the rate of change of population with respect to time by the conservation equation mathematically, as:

$$\frac{dP}{dt} = births - deaths + immigration - migration$$

But, in simplest model (closed model), there is no immigration and migration. Thus the birth (b) and death (d) terms are proportional to P.

$$\frac{dP}{dt} = bP - dP = (b-d)P \tag{1}$$

Here, if we let $b - d = k$, Biologically k is known as the intrinsic growth rate.

$$\frac{dP}{dt} = kP \tag{2}$$

Thus, Eq. (2) is called simplest growth model or closed model.

Now, solving for P from Eq. (2) using separation of variable as the following way.

Integrating both sides of Eq. (2), we get:

$$\int \frac{dP}{P} = \int kdt$$

$$\ln|P| = kt + c, where\ c\ is\ constant\ of\ integration.$$

$$e^{ln|P|} = e^{kt+c}$$

$$\Rightarrow P(t) = e^c e^{kt} = Ce^{kt} \tag{3}$$

Where C is constant.

Eq. (3) is called exponential model and it is the solution of Eq. (2) and hence, we conclude that a population often increases exponentially in its early stages.

Example 1: consider the following population growth that describes the duplication of bacteria in a Laboratory. In this experiment, if we start with one cell of bacteria and if the cell doubles every minute. Then, the numbers of cell depend on a time t. For example:

At t = 0, there is 1 cell, and the corresponding point is (0, 1).

At t = 1, there are 2 cells, and the corresponding point is (1, 2).

At t = 2, there are 4 cells, and the corresponding point is (2, 4).

At t = 3, there are 8 cells, and the corresponding point is (3, 8), etc. This relationship is summarized in the following Table 1.

Table 1. *Relationship is summarized for time (t) and number of cells (p).*

Time (t)	0	1	2	3	4	5	6	⋯	*t*
Time (t) in Minute									
No. of cells (p)	$1 = 2^0$	$2=2^1$	$4=2^2$	$8=2^3$	$16=2^4$	$32=2^5$	$64=2^6$	⋯	2^t

Thus, we can drive the formula to estimate the number of bacteria cells after a time t minutes and let us introduce this formula as a function:

$$P(t) = 2^t \tag{4}$$

Eq. (4) is the exponential function that describes the population growth of bacteria cell in a laboratory.

Alternatively we can derive Eq. (4) for the above example by using the given conditions:

$$P(t) = P(0) = 1\ at\ t = 0, and\ P(t) = P(1) = 2\ at\ t = 1$$

Now, we can rewrite Eq. (3) by solving for k using $P(0) = 1$ and for C using $P(1) = 2$ as follow:

$$P(0) = 1 = Ce^{k.0} = Ce^0 = C.1 = C \qquad , P(1) = 2 = Ce^k$$

$$but\ \ C = 1$$

$$\Rightarrow C = 1 \qquad \Rightarrow 2 = e^k = \ln 2 = k$$

$$Now, P(t) = Ce^{kt} = 1.e^{\ln 2.t} = \left(e^{\ln 2}\right)^t = 2^t$$

$$\therefore\ \ P(t) = 2^t$$

Eventually, the population growth approaches its carrying capacity because of limited resources and when a population becomes too large another model proposed by *Verhulst* called the Logistic growth model as:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \tag{5}$$

Where,

P = p(t) is population size (density) at a time t.

k is positive intrinsic growth rate.

M is environmental carrying capacity.

Equilibrium point.

The equilibrium points of the Logistic growth model is obtained by making

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) = 0$$

$$\Rightarrow\ kP\left(1 - \frac{P}{M}\right) = 0 \qquad \Rightarrow \quad kP = 0\ or\ 1 - \frac{P}{M} = 0$$

$$\Rightarrow\ \ P = 0 (b/c\ k > 0)\ or\ P = M$$

Hence, $P = 0$ and $P = M$ are the equilibrium points of the logistic growth model.

Stability Analysis for Equilibrium Points

To analyze the stability of the equilibrium points of the logistic model, Setting

$$\frac{dP}{dt} = f(P) = kP\left(1 - \frac{P}{M}\right)$$

Here, the equilibrium points of the logistic model are the points in which:

$$f'(P) = k - \frac{2kP}{M}$$

$$f'(0) = k$$

- *if* $k > 0$, *then the equilibrium point* $P = 0$ *is unstable.*
- *if* $k < 0$, *then the equilibrium point* $P = 0$ *is stable*

$$f'(M) = -k$$

- *if* $k > 0$, *then the equilibrium point* $P = M$ *is stable.*
- *if* $k < 0$, *then the equilibrium point* $P = M$ *is unstable*

3. Solution of the Logistic Model

I. Analytical Solution of Logistic Model

To solve the logistic growth model in Eq. (5) we use separation of variable and rewrite as:

To integrate the left side, we can use integration by partial fraction.

$$\frac{M}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$

$$\Rightarrow M = A(M-P) + BP \Rightarrow A - B = 1 \text{ and } AM = M$$

$$\Rightarrow A = 1 \Rightarrow A = B = 1$$

$$\int \frac{M\,dP}{P(M-P)} = \int \left(\frac{A}{P} + \frac{B}{M-P}\right)dP = \int \frac{1}{P}dP + \int \frac{1}{M-P}dP \Rightarrow \ln|P| - \ln|M-P|$$

So,

$$\int \frac{dP}{P\left(1-\frac{P}{M}\right)} = \int \frac{M\,dP}{P(M-P)} = \ln|P| - \ln|M-P| = kt + C$$

$$\ln\left|\frac{M-P}{P}\right| = -kt - C =$$

$$\left|\frac{M-P}{P}\right| = e^{-kt-C} = e^{-C}.e^{-kt} = Ae^{-kt}, \text{ where } A = e^{-C} \quad (6)$$

$$\frac{M-P}{P} = Ae^{-kt}$$

At $t = 0$, we have the initial population $P = P_0$ and using these values we are going to find the value of A from Eq. (6).

$$P(0) = \frac{M}{1 + Ae^{-k.0}} = P_0 \Rightarrow \frac{M}{1+A} = P_0 \Rightarrow A = \frac{M - P_0}{P_0}$$

$$P(t) = \frac{M}{1 + Ae^{-kt}}, \text{ where } A = \frac{M - P_0}{P_0} \quad (7)$$

Example 2: Find a population in the ecosystem with k= 0.05, carrying capacity 1000, and initial population $P_0 = 1$ at a time of 5 years.

Solution: using Eq. (7);

$$P(t) = \frac{M}{1 + Ae^{-kt}}, \text{ where } A = \frac{M - P_0}{P_0}, \text{ where}$$

$$M = 1000,\ k = 0.05, P_0 = 1 \text{ and } t = 5$$

$$P(50) = \frac{1000}{1 + Ae^{-0.05.(5)}}, \text{ where } A = \frac{1000 - 1}{1} = 999$$

$$\frac{1000}{1 + 999.e^{-0.05.(5)}} = 1.28366082$$

Therefore, the number of population after 50 years will be 1.28366082.

II. Numerical Solution of Logistic Model by Euler's Method

Consider the initial value problem:

$$\frac{dP}{dt} = f(t, P), \quad \text{where } f(t, P) = kP\left(1 - \frac{P}{M}\right) \quad (8)$$

$P(0) = P_0$ (Initial conditions).

Here, our aim is to find $P(t)$ and P changes according to t. Thus we can find the approximate (numerical) solution of Eq. (8) using Euler's explicit Method by taking the step size (h) between points which is chosen by the user as follow:

$$t_{i+1} = t_i + h$$

$$P_{i+1} = P_i + hf(t_i, P_i) \quad (9)$$

(This is called Euler's Explicit Method).

In particular, $P_1 = P_0 + hf(t_0, P_0)$, $P_2 = P_1 + hf(t_1, P_1)$,

$P_3 = P_2 + hf(t_2, P_2)\ldots$

From above example 2, to find the number of population at time of $t = 5$ years numerically, we discretize the time at a step size of $h = 0.5$.

$$t_{i+1} = t + h,\ i = 0, 1, 2, 3, \ldots N+1 \text{ and}$$
$$N \text{ is the number of mesh points.}$$
$$t_0 = 0,\ t_1 = t_0 + 0.5 = 0.5,\ t_2 = t_1 + 0.5 = 1, \ldots,$$
$$\text{here the number of steps } N \text{ is given by}$$
$$\frac{(t_{end} - t_0)}{h} = \frac{(5-0)}{0.5} = 10, \text{ so for } h = 0.5$$
$$\text{there are } 10 \text{ steps, } P_0,\ P_1,\ P_2, \ldots,\ P_{10}$$
$$\text{Now, using Eq.}(9):$$
$$P_0 = P(t_0) = P(0) = 1$$
$$P_1 = P(t_1) = P(0.5) = P_0 + hf(t_0, P_0) = 1 + 0.5 \times 0.05 \times \left(1 - \frac{1}{1000}\right)$$
$$\approx 1.024975$$
$$P_2 = P(t_2) = P(1) = P_1 + hf(t_1, P_1) = 1.024975 + 0.5 \times 0.05 \times \left(1 - \frac{1.024975}{1000}\right)$$
$$\approx 1.049949$$
$$P_3 = P(t_3) = P(1.5) = P_2 + hf(t_2, P_2) = 1.049949 + 0.5 \times 0.05 \times \left(1 - \frac{1.049949}{1000}\right)$$
$$\approx 1.074923$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$P_{10} = P(t_{10}) = P(5)$$
$$\approx 1.2797$$

So, the population at a time of 5 years is $P(5) = P_{10} = P_9 + hf(t_9, P_9)$. After some successive approximations we get the population at a time of 5 years is $P(5) = P_{10} \approx 1.2797$

III. Implementation of the Method Using MATLAB Software

When we implement all the above numerical methods, using Euler's explicit method in the help of MATLAB software with $h = 0.5, t_0 = 0, t_{end} = 5, M = 1000, K = 0.05, P(0) = P_0 = 1$, then the result that we obtain was described by the following *Table 2*.

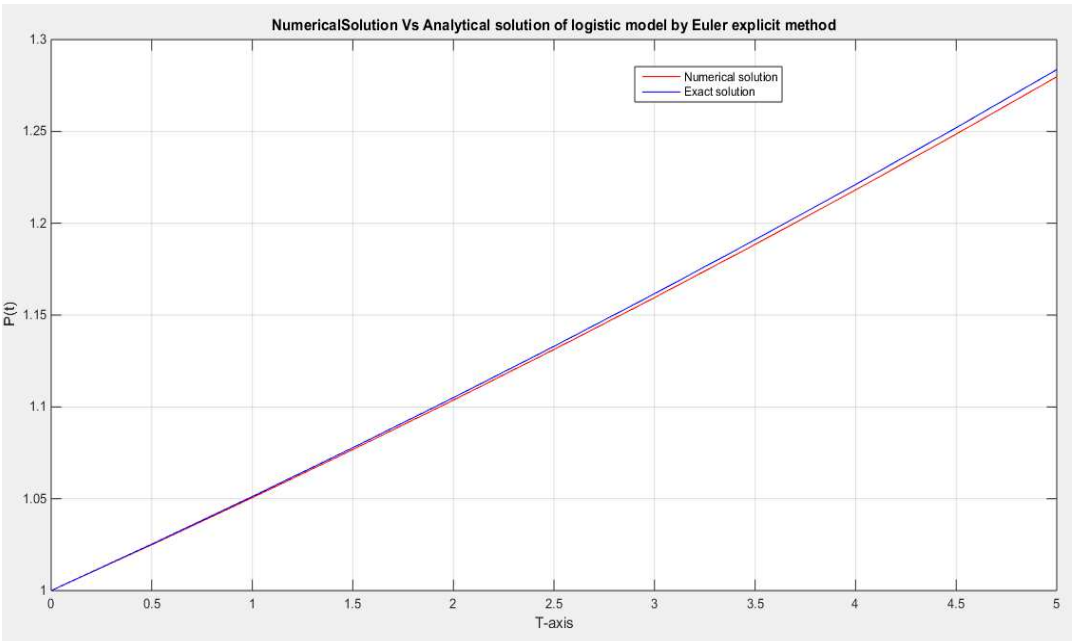

Figure 1. *Graphical solution of logistic growth model by Euler's explicit method.*

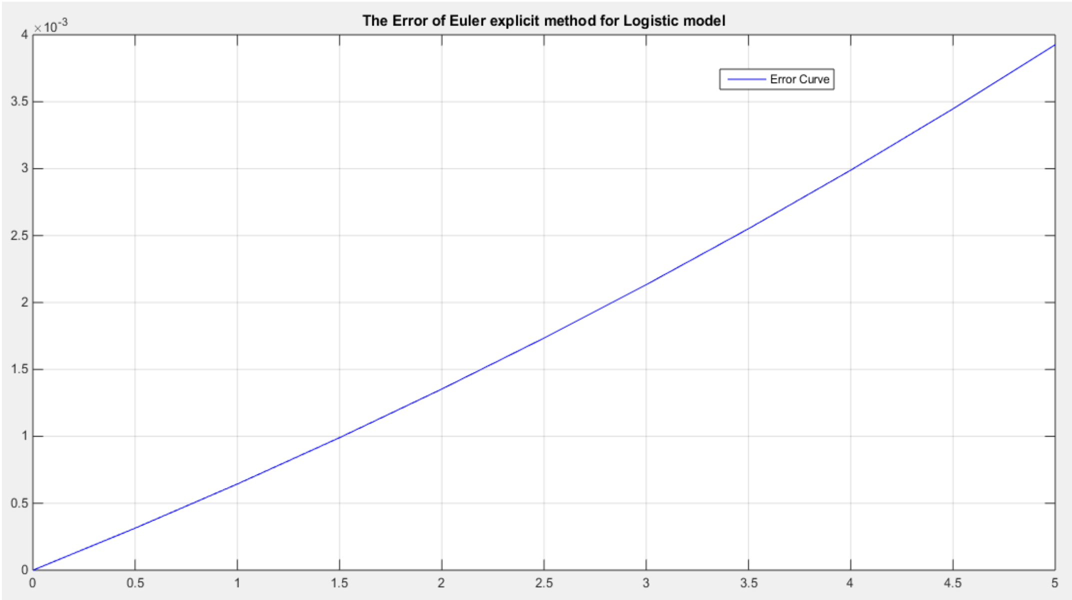

Figure 2. *The error of Euler's explicit method for logistic growth model.*

Table 2. *Experimental analysis of logistic growth model for example 2.*

Steps (i)	time	Exact Solution	Numerical Solution	Error
0	0	1.0000	1.0000	0
1	0.5000	1.0253	1.0250	0.0003
2	1.0000	1.0512	1.0506	0.0006
3	1.5000	1.0778	1.0768	0.0010
4	2.0000	1.1051	1.1037	0.0014
5	2.5000	1.1330	1.1313	0.0017
6	3.0000	1.1616	1.1595	0.0021
7	3.5000	1.1910	1.1885	0.0026
8	4.0000	1.2211	1.2181	0.0030
9	4.5000	1.2520	1.2486	0.0034
10	5.0000	1.2837	1.2797	0.0039

The solution of the above model becomes more accurate by making step size h small. For example taking $h = 0.05, 0.005, 0.001, \ldots$ makes our solution more accurate.

4. Conclusion

In this paper, the logistic growth model, which is described by a differential equation whose solutions vary according to time t was studied. This discussion is focused on analytical solution and numerical solution of the logistic growth model, using the method of separation of variable and integration techniques for analytical solution and Euler's explicit approximation for numerical solution. Even if, when the analytic solution is difficult, it is possible to approximate numerically to get efficient solutions and develop experience in how to implement and solve the differential equations using MATLAB. Generally, this project is helpful to solve differential equations, in particular population growth, logistic growth model which was described by DEs which are not solved analytically.

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