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A Modified Non-Linear Ordinary Differential Equations Model for Unemployment Dynamics on Ghana's Economic Sectors

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Abstract: In this paper, we developed and analyzed a non-linear ordinary differential equations model for unemployment dynamics on Ghana's Economic Sectors. In the modelling process, seven dynamic variables were considered namely: i) number of unemployed persons in the agriculture sector, ii) number of unemployed persons in the industry sector, iii) number of unemployed persons in the service sector, iv) number of employed persons in the agriculture sector, v) number of employed persons in the industry sector, vi) number of employed persons in the service sector vii) newly created vacancies by all three sectors, We assumed that all entrants in the labour force as the total number of employed and unemployed persons of the working age group of 18 to 60 years. Entrants of the unemployment compartment are fully qualified to work at any time t, An increase in the number of unemployed is different across all three sectors. Vacancies are created collectively depending on the number of unemployed and employed persons in the agriculture, industry and service sectors. The model is studied using stability analysis by a system of differential equations. It established the behaviour of the system over time, which showed that the solution to the system is positive and bounded. The system of the equations has a non-negative equilibrium point. The Routh-Hurwitz stability criterion established the equilibrium point is locally asymptotically stable. It was observed that the stability of the model is feasible under certain conditions.

Keywords: Unemployment, Economic Sectors, Boundedness, Positivity, Stability, Equilibrium Point

1. Introduction

Unemployment is among the topmost global economic challenges and one of the major socioeconomic and developmental issues across the world [1, 2]. This has become a concern for governments around the world and Ghana is no exception. However, Ghana's unemployment rate has almost tripled in a little more than a decade as the country's population increased to 30.8 million in 2021 from 24.7 million in 2010 according to the 2021 PHC by the Ghana Statistical Service [3]. Anon [3] indicated that 1.5 million (13.4%) out of the 11.5 million economically active Ghanaians in the labour force are unemployed against the 2010 PHC where Ghana recorded a jobless rate of 5.3%. Employment is the state of work done, resulting in reimbursement. Work can be defined as activity/activities that are performed for the final use of others, in exchange for any monetary or non-monetary remuneration [4]. Ghana's economic drive depends on sector indicators for employment growth and on job creation. However, the country's economic sectors are grouped into three main sectors, thus, agriculture, industry, and services [5]. Employment is not only an economic condition, but a human rights issue [6].

2. Literature Review

Misra and Singh [7, 8] developed a nonlinear mathematical model that simulates the unemployment problem. They combined three variables, thus, the number of unemployed, number of temporarily employed and number of regularly employed and another three variables, number of unemployed persons, number of employed persons and number of newly created vacancies based on using some concepts of the model proposed by Nikolopoulos and Tzanetis [9]. Pathan and Bhathawala [1] analysed the effect of self-employment on the rate of unemployment, which focused on three dynamic variables as: number of unemployed persons, number of employed persons and the number of newly created vacancies by the intermediation of government and private sector. Pathan and Bhathawala [2] approached the unemployment from a different aspect, developed a nonlinear dynamic model of four variables for unemployment, namely: number of unemployed persons, the number of present jobs in the market, the number of employed persons and newly created vacancies. Their model only described the behaviour of the four variables.

A non-linear ordinary differential equation model which is an extension of Pathan and Bhathawala [1, 2] model was adapted and developed to suit this study. Using the concept of Pathan and Bhathawala [1, 2], we developed a new model with seven variables for unemployment and employment across the levels of the three main Ghana's economic sectors. We considered the following seven major variables in the problem, which are as follows:

1. The number of unemployed persons in the Agriculture Sector.
2. The number of unemployed persons in the Industry Sector.
3. The number of unemployed persons in the Service Sector.
4. Number of employed persons in the Agriculture Sector.
5. Number of employed persons in the Industry Sector.
6. Number of employed persons in the Service Sector.
7. Newly created vacancies by all three sectors.

The number of unemployed persons in the Agriculture Sector represented by U_A, number of unemployed persons in the Industry Sector denoted by U_I, number of unemployed persons in the Service Sector U_S, Number of employed persons in the Agriculture Sector be E_A, Number of employed persons in the Industry Sector E_I, Number of employed persons in the Service Sector be E_S, and Newly created vacancies by all three sectors C. We introduce the rate of newly created vacancies by the government and private sectors in relation to the unemployed and employed persons respectively.

3. Research Methodology

Due to this socioeconomic emergency, attempts have been made by many researchers to find solutions to the crisis in different areas some of which includes the use mathematical models which includes nonlinear ordinary differential equations. Here we used a nonlinear differential equation. An ordinary differential equation (ODE) is an equation that involves derivatives of an unknown function. Ordinary differential equations are used to model changeover a single independent variable (it is usually t for time). These equations do not involve any partial derivatives. Differential equations contain three types of variables: an independent variable, at least one dependent variable (these will be functions of the independent variable), and the parameters. ODE's can contain multiple iterations of derivatives and are named accordingly (i.e. if there are only first derivatives, then the ODE is called a first order ODE) [10]. Differential equations describe the evolution of systems in continuous time [11]. Once a solution to a system of ordinary differential equations has settled down, its limiting value is an equilibrium solution [12]. However, not all equilibria appear in this fashion. The only steady state solutions that one directly observes in a physical system are the stable equilibria. Equilibrium is a state of a system which does not change. The dynamics of a system are described by a differential equation (or a system of differential equations), then equilibria can be estimated by setting a derivative (all derivatives) to zero [13]. An equilibrium is considered stable (for simplicity we will consider asymptotic stability only) if the system always returns to it after small disturbances. If the system moves away from the equilibrium after small disturbances, then the equilibrium is unstable [13].

4. Model Formulation

In the process of making the model, we assumed that all entrants in the labour force as the total number of employed and unemployed persons of the working age group of 18 to 60 years. Entrants of the unemployment compartment are fully qualified to work at any time t, An increase in the number of unemployed is different across all three sectors. Vacancies are created collectively depending on the number of unemployed and employed persons in the agriculture, industry and service sectors. The rate of movement from unemployed class to the employed class is jointly proportional to U_A, U_I, U_S and A + C - E_A, I + C - E_I, S + C – E_S respectively. The model does not allow movement across different sectors. Thus, unemployed persons in a given sector cannot be employed in a different sector. The rate of death is same for unemployed and employed populations. Vacancies can be created due to death, resignation or retirement of persons across the sectors. Individuals in the employed compartment across the 3 $sectors$ can be fired (dismissed) by employers or resign from their jobs.

Entrants of the category unemployment are fully qualified to do any job at any time t. Available employment in the agriculture sector by both governments and private sectors denoted by A, available employment in the industry sector by both government and private sectors as I and available employment in the services sector by government and private sectors as S. The number of unemployed persons' in Agriculture, Industry and Service is denoted as U_A, U_I and U_S respectively. Likewise, the number of employed persons' in Agriculture, Industry and Service is denoted as E_A, E_I and E_S respectively. Newly created vacancies by governments and private sector in the agriculture, industry and services sectors denoted as C. The number of available vacancies in the market is $A + C - E_A$, $I + C - E_I$ and $S + C - E_S$. The number of unemployed persons in all sectors increases with a constant rate a_1. As some of the unemployed persons become employed, the rate of movement of unemployed persons joining the Agriculture Sector is denoted by a_2, the rate of movement of unemployed persons joining an Industry Sector denoted by a_3, rate of movement of unemployed persons joining Services Sector denoted by a_4. Where a_5 is a positive constant which represents the rate of retirement and death of employed persons from a sector. a_6 denote the rate of employed persons getting unemployed and the rate of resignation of employed persons out of a sector is denoted by a_7. Lastly, the creation of vacancies is highly dependent on the number of unemployed and employed persons in the given sectors. α and δ are the rate of newly created vacancies by the government and private sectors in relation to the unemployed and employed persons respectively. β represents the diminution of newly created vacancies by all sectors at time t.

$$\frac{dU_A}{dt} = a_1 - a_2U_A(A + C - E_A) - a_5U_A + a_6E_A \tag{1}$$

$$\frac{dU_I}{dt} = a_1 - a_3U_I(I + C - E_I) - a_5U_I + a_6E_I \tag{2}$$

$$\frac{dU_S}{dt} = a_1 - a_4U_S(S + C - E_S) - a_5U_S + a_6E_s \tag{3}$$

$$\frac{dE_A}{dt} = a_2U_A(A + C - E_A) - a_5E_A - a_6E_A - a_7E_A \tag{4}$$

$$\frac{dE_I}{dt} = a_3U_I(I + C - E_I) - a_5E_I - a_6E_I - a_7E_I \tag{5}$$

$$\frac{dE_S}{dt} = a_4U_S(S + C - E_S) - a_5E_S - a_6E_S - a_7E_S \tag{6}$$

$$\frac{dC}{dt} = \alpha(U_A + U_I + U_S) + \delta(E_A + E_I + E_S) - \beta C \tag{7}$$

5. Results and Discussions

5.1. Positivity and Boundedness of Solutions

To establish the behaviour of the system over time, we show that the solution to this system is nonnegative and bounded. Thus, this is because the model is about a human population and hence cannot be negative. We ascertain boundedness to show the solution stays in the defined region or domain prescribed but does not increase to infinity.

First, we need to show the following lemma which ensures that, under suitable conditions, all solutions of system (1)-(7) are nonnegative. We also identify a set Ω in $\mathbb{R}^7_+$ such that all solutions starting from Ω remain bounded.

Theorem 1. Let all parameters in (1)-(7) be positive and Ω be the region in $\mathbb{R}^7_+$ defined by

$$\Omega = \left\{(U_A, U_I, U_S, E_A, E_I, E_S, C) \in \mathbb{R}^7_+ \mid 0 \le U_A + U_I + U_S + E_A + E_I + E_S + C \le \frac{3a_1}{\gamma}\right\} \tag{8}$$

where $\gamma = min\{(a_5), (a_5 + a_7 - \delta), \beta\}$ is a region of attraction for the system (1)-(7) and it attracts all solutions initiating in the interior of the positive octant. Thus, the solution is positive and bounded.

5.2. Positivity

First, we establish the positive invariant part of the theorem. By the method of contradiction, let $[U_A(0), U_I(0), U_S(0), E_A(0), E_I(0), E_S(0), C(0)] \in \Omega$. Suppose $U_A(t)$ be non-positive. Then, there exists $t_0 > 0$ such that $U_A(t_0) = 0$ and $U_A(t) > 0$ for any t satisfying $0 \le t \le t_0$. Then,

$$\left.\frac{dU_A}{dt}\right|_{t=t_0} \le 0$$

This is a contradiction, because

$$\left.\frac{dU_A}{dt}\right|_{t=t_0} = a_1 - a_2U_A(t_0)\big(A + C(t_0) - E_A(t_0)\big) - a_5U_A(t_0) + a_6E_A(t_0) = a_1 > 0$$

Hence, $U_A(t)$ is positive $\forall t \ge 0$. Similarly, $U_I(t), U_S(t)$ are positive $\forall t \ge 0$.

Suppose $U_I(t)$ is nonpositive. Then, there exists $t_0 > 0$ such that $U_I(t_0) = 0$ and $U_I(t) > 0$ for any t satisfying $0 \le t \le t_0$. Then,

$$\left.\frac{dU_I}{dt}\right|_{t=t_0} \le 0$$

This is a contradiction, because

$$\left.\frac{dU_I}{dt}\right|_{t=t_0} = a_1 - a_3U_I(t_0)\big(I + C(t_0) - E_I(t_0)\big) - a_5U_I(t_0) + a_6E_I(t_0) = a_1 > 0$$

Hence, $U_I(t)$ is positive $\forall t \ge 0$.

Suppose $U_S(t)$ is nonpositive. Then, there exists $t_0 > 0$ such that
$U_S(t_0) = 0$ and $U_S(t) > 0$ for any t satisfying $0 \le t \le t_0$. Then,

$$\left.\frac{dU_S}{dt}\right|_{t=t_0} \le 0$$

This is a contradiction, because

$$\left.\frac{dU_S}{dt}\right|_{t=t_0} = a_1 - a_4U_S(t_0)\big(S + C(t_0) - E_S(t_0)\big) - a_5U_S(t_0) + a_6E_S(t_0) = a_1 > 0$$

Hence, $U_S(t)$ is positive $\forall t \ge 0$.
Next, suppose $E_A(t)$ is nonpositive. Then there exists $t_0 > 0$ such that
$E_A(t_0) = 0$ and $E_A(t) > 0$ for any $t \in [0, t_0]$. Then,

$$\left.\frac{dE_A}{dt}\right|_{t=t_0} \le 0$$

This is a contradiction, because

$$\left.\frac{dE_A}{dt}\right|_{t=t_0} \& = a_2U_A(t_0)\big(A + C(t_0) - E_A(t_0)\big) - a_5E_A(t_0) - a_6E_A(t_0) - a_7E_A(t_0) = a_2U_A(t_0)A > 0$$

Hence, $E_A(t)$ is positive $\forall t \ge 0$.
Suppose $E_I(t)$ is nonpositive. Then there exists $t_0 > 0$ such that $E_I(t_0) = 0$ and $E_I(t) > 0$ for any $t \in [0, t_0]$. Then,

$$\left.\frac{dE_I}{dt}\right|_{t=t_0} \le 0$$

This is a contradiction, because

$$\left.\frac{dE_I}{dt}\right|_{t=t_0} = a_3U_A(t_0)(A + C(t_o) - E_t(t_0) - a_5E_t(t_0) - a_6E_t(t_0) - a_7E_I(t_0) = a_2U_A(t_0)I > 0)$$

Hence, $E_I(t)$ is positive $\forall t \ge 0$.
Suppose $E_S(t)$ be non-positive. Then there exists $t_0 > 0$ such that $E_S(t_0) = 0$ and $E_S(t) > 0$ for any $t \in [0, t_0]$. Then,

$$\left.\frac{dE_S}{dt}\right|_{t=t_0} \le 0$$

This is a contradiction, because

$$\left.\frac{dE_S}{dt}\right|_{t=t_0} \& = a_4U_A(t_0)\big(A + C(t_0) - E_S(t_0)\big) - a_5E_S(t_0) - a_6E_S(t_0) - a_7E_S(t_0) = a_2U_A(t_0)S > 0$$

Hence, $E_S(t)$ is positive $\forall t \ge 0$.
Therefore, $E_A(t), E_I(t), E_S(t)$ are positive $\forall t \ge 0$. Lastly,

$$\left.\frac{dC}{dt}\right|_{t=t_0} \le 0$$

But

$$\begin{aligned}\left.\frac{dC}{dt}\right|_{t=t_0} &= \alpha(U_At_0 + U_It_0 + U_St_0) + \delta(E_At_0 + E_It_0 + E_St_0) - \beta Ct_0 \\ &= \alpha\big(U_A(t_0) + U_I(t_0) + U_S(t_0)\big) + \delta\big(E_A(t_0) + E_I(t_0) + E_S(t_0)\big)\big) > 0\end{aligned}$$

Hence, C_t is positive $\forall t \ge 0$. Thus, all solutions to the system (1)-(7) are positive.

5.3. Boundedness

Let

$$U_A + U_S + U_I = UE_A + E_S + E_I = E$$

thus, from our system (1)-(7) we obtain

$$
\begin{aligned}
\frac{d}{dt}[U_A + U_S + U_I + E_A + E_S + E_I + C] = \frac{d}{dt}[U + E + C] &= a_1 - a_2U_A(A + C - E_A) - a_5U_A + a_6E_A + \\
&\quad a_1 - a_3U_I(I + C - E_I) - a_5U_I + a_6E_I + \\
&\quad a_1 - a_4U_S(S + C - E_S) - a_5U_S + a_6E_S + \\
&\quad a_2U_A(A + C - E_A) - a_5E_A - a_6E_A - a_7E_A + \\
&\quad a_3U_I(I + C - E_I) - a_5E_I - a_6E_I - a_7E_I + \\
&\quad a_4U_S(S + C - E_S) - a_5E_S - a_6E_S - a_7E_S + \\
&\quad \alpha(U_A + U_I + U_S) + \delta(E_A + E_I + E_S) - \beta C
\end{aligned}
$$

which gives

$$
\begin{aligned}
\frac{d}{dt}[U + E + C] &= 3a_1 - a_5U - a_5E - a_7E + \alpha U + \delta E - \beta C \\
&= 3a_1 + (\alpha - a_5)U + (\delta - a_5 - a_7)E - \beta C \\
&= 3a_1 - (a_5 - \alpha)U - (a_5 + a_7 - \delta)E - \beta C \\
&\leq 3a_1 - \gamma(U + E + C)
\end{aligned}
$$

where $\gamma = min\{(a_5), (a_5 + a_7 - \tilde{o}), \beta\}$.

Taking the limit supremum

$$\lim_{t\to\infty} sup(U_A + U_S + U_I + E_A + E_S + E_I + C) \leq \frac{3a_1}{\gamma}$$

5.4. Equilibrium Point

The model system (1) - (7) has a non-negative equilibrium point as

$$U_A^1 = \frac{a_1a_2\delta + (Aa_2a_5 - a_1a_2 + a_5^2 + a_5a_6 + (Aa_2 + a_5)a_7)\beta + \sqrt{X_1}}{2(a_2a_5\beta - a_2a_5\delta + (a_2a_5 + a_2a_7)\alpha)}$$

$$U_l^1 = \frac{a_1a_3\delta + (Ia_3a_5 - a_1a_3 + a_5^2 + a_5a_6 + (Ia_3 + a_5)a_7)\beta + \sqrt{X_2}}{2(a_3a_5\beta - a_3a_5\delta + (a_3a_5 + a_3a_7)\alpha)}$$

$$U_S^1 = \frac{a_1a_4\delta + (Sa_4a_5 - a_1a_4 + a_5^2 + a_5a_6 + (Sa_4 + a_5)a_7)\beta + \sqrt{X_3}}{2(a_4a_5\beta - a_4a_5\delta + (a_4a_5 + a_4a_7)\alpha)}$$

$$E_A^1 = \frac{a_1a_2a_5\delta - 2(a_1a_2a_5 + a_1a_2a_7)\alpha - (Aa_2a_5^2 + a_1a_2a_5 + a_5^3 + a_5^2a_6 + (Aa_2a_5 + a_5^2)a_7)\beta - \sqrt{X_4}a_5}{2\left((a_2a_5^2 + 2a_2a_5a_7 + a_2a_7^2)\alpha + (a_2a_5^2 + a_2a_5a_7)\beta - (a_2a_5^2 + a_2a_5a_7)\delta\right)}$$

$$E_I^1 = \frac{a_1a_3a_5\delta - 2(a_1a_3a_5 + a_1a_3a_7)\alpha - (Ia_3a_5^2 + a_1a_3a_5 + a_5^3 + a_5^2a_6 + (Ia_3a_5 + a_5^2)a_7)\beta - \sqrt{X_5}a_5}{2\left((a_3a_5^2 + 2a_3a_5a_7 + a_3a_7^2)\alpha + (a_3a_5^2 + a_3a_5a_7)\beta - (a_3a_5^2 + a_3a_5a_7)\delta\right)}$$

$$E_S^1 = \frac{a_1a_4a_5\delta - 2(a_1a_4a_5 + a_1a_4a_7)\alpha - (Sa_4a_5^2 + a_1a_4a_5 + a_5^3 + a_5^2a_6 + (Sa_4a_5 + a_5^2)a_7)\beta - \sqrt{X_6}a_5}{2\left((a_4a_5^2 + 2a_4a_5a_7 + a_4a_7^2)\alpha + (a_4a_5^2 + a_4a_5a_7)\beta - (a_4a_5^2 + a_4a_5a_7)\delta\right)}$$

$$C^1 = \frac{X_7 + \sqrt{X_8}((a_5 + a_7)\alpha - a_5\delta)}{2\left((a_2a_5^2 + 2a_2a_5a_7 + a_2a_7^2)\alpha\beta + (a_2a_5^2 + a_2a_5a_7)\beta^2 - (a_2a_5^2 + a_2a_5a_7)\beta\delta\right)}$$

which were obtained by solving the following set of algebraic equations;

$$F_1 = a_1 - a_2U_A(A + C - E_A) - a_5U_A + a_6E_A = 0 \quad F_2 = a_1 - a_3U_I(I + C - E_I) - a_5U_I + a_6E_I = 0$$

$$F_3 = a_1 - a_4U_S(S + C - E_S) - a_5U_S + a_6E_S = 0$$

$$F_4 = a_2U_A(A + C - E_A) - a_5E_A - a_6E_A - a_7E_A = 0$$

$$F_5 = a_3U_I(I + C - E_I) - a_5E_I - a_6E_I - a_7E_I = 0$$

$$F_6 = a_4U_S(S + C - E_S) - a_5E_S - a_6E_S - a_7E_S = 0$$

$$F_7 = \alpha(U_A + U_I + U_S) + \delta(E_A + E_I + E_S) - \beta C = 0. \tag{9}$$

The equilibrium point will inform one about the dynamics of the system over time. In determining the stability analysis, we implore the linerization technique which gives the Jacobian matrix.

5.5. Jacobian Matrix

The matrix of partial derivatives of the system (9) is given as;

$$J = \begin{pmatrix} \frac{\partial F_1}{\partial U_A} & \frac{\partial F_1}{\partial U_I} & \frac{\partial F_1}{\partial U_S} & \frac{\partial F_1}{\partial E_A} & \frac{\partial F_1}{\partial E_I} & \frac{\partial F_1}{\partial E_S} & \frac{\partial F_1}{\partial C} \\ \frac{\partial F_2}{\partial U_A} & \frac{\partial F_2}{\partial U_I} & \frac{\partial F_2}{\partial U_S} & \frac{\partial F_2}{\partial E_A} & \frac{\partial F_2}{\partial E_I} & \frac{\partial F_2}{\partial E_S} & \frac{\partial F_2}{\partial C} \\ \frac{\partial F_3}{\partial U_A} & \frac{\partial F_3}{\partial U_I} & \frac{\partial F_3}{\partial U_S} & \frac{\partial F_3}{\partial E_A} & \frac{\partial F_3}{\partial E_I} & \frac{\partial F_3}{\partial E_S} & \frac{\partial F_3}{\partial C} \\ \frac{\partial F_4}{\partial U_A} & \frac{\partial F_4}{\partial U_I} & \frac{\partial F_4}{\partial U_S} & \frac{\partial F_1}{\partial E_A} & \frac{\partial F_4}{\partial E_I} & \frac{\partial F_4}{\partial E_S} & \frac{\partial F_4}{\partial C} \\ \frac{\partial F_5}{\partial U_A} & \frac{\partial F_5}{\partial U_I} & \frac{\partial F_5}{\partial U_S} & \frac{\partial F_2}{\partial E_A} & \frac{\partial F_5}{\partial E_I} & \frac{\partial F_5}{\partial E_S} & \frac{\partial F_5}{\partial C} \\ \frac{\partial F_6}{\partial U_A} & \frac{\partial F_6}{\partial U_I} & \frac{\partial F_6}{\partial U_S} & \frac{\partial F_6}{\partial E_A} & \frac{\partial F_6}{\partial E_I} & \frac{\partial F_6}{\partial E_S} & \frac{\partial F_6}{\partial C} \\ \frac{\partial F_7}{\partial U_A} & \frac{\partial F_7}{\partial U_I} & \frac{\partial F_7}{\partial U_S} & \frac{\partial F_7}{\partial E_A} & \frac{\partial F_7}{\partial E_I} & \frac{\partial F_7}{\partial E_S} & \frac{\partial F_7}{\partial C} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 & U_A a_2 + a_6 & 0 & 0 & -U_A a_1 \\ 0 & \lambda_2 & 0 & 0 & U_I a_3 + a_6 & 0 & -U_I a_3 \\ 0 & 0 & \lambda_3 & 0 & 0 & U_S a_4 + a_6 & -U_S a_4 \\ \lambda_7 & 0 & 0 & \lambda_4 & 0 & 0 & U_A a_2 \\ 0 & \lambda_8 & 0 & 0 & \lambda_5 & 0 & U_I a_3 \\ 0 & 0 & \lambda_9 & 0 & 0 & \lambda_6 & U_S a_4 \\ \alpha & \alpha & \alpha & \delta & \delta & \delta & -\beta \end{pmatrix}$$

where

$$\lambda_1 = -(A + C - E_A)a_2 - a_5$$

$$\lambda_2 = -(C - E_I + I)a_3 - a_5$$

$$\lambda_3 = -(C - E_S + S)a_4 - a_5$$

$$\lambda_4 = -U_A a_2 - a_5 - a_6 - a_7$$

$$\lambda_5 = -U_I a_3 - a_5 - a_6 - a_7$$

$$\lambda_6 = -U_S a_4 - a_5 - a_6 - a_7$$

$$\lambda_7 = (A + C - E_A)a_2$$

$$\lambda_8 = (C - E_I + I)\, a_3$$

$$\lambda_9 = (C - E_S + S)\, a_4$$

5.6. Local Stability Analysis

Theorem 2. The system (1)-(7) is locally asymptotically stable at $U_A^1, U_I^1, U_S^1, E_A^1, E_I^1, E_S^1, C^1$ if and only if
$i.\ A_0 > 0$
ii. $A_1 > 0$
iii. $A_0 A_3 > A_1 A_2$
$iv.\ B_1 A_3 > A_1 B_2$
$v.\ C_1 B_2 > B_1 C_2$
$vi.\ C_1 D_2 > D_1 C_2$
Proof. Evaluating the equilibrium point in the Jacobian matrix (J), we obtain

$$J_1 = \begin{pmatrix} \lambda_1 & 0 & 0 & U_A a_2 + a_6 & 0 & 0 & -U_A a_1 \\ 0 & \lambda_2 & 0 & 0 & U_I a_3 + a_6 & 0 & -U_I a_3 \\ 0 & 0 & \lambda_3 & 0 & 0 & U_S a_4 + a_6 & -U_S a_4 \\ \lambda_7 & 0 & 0 & \lambda_4 & 0 & 0 & U_A a_2 \\ 0 & \lambda_8 & 0 & 0 & \lambda_5 & 0 & U_I a_3 \\ 0 & 0 & \lambda_9 & 0 & 0 & \lambda_6 & U_S a_4 \\ \alpha & \alpha & \alpha & \delta & \delta & \delta & -\beta \end{pmatrix} \tag{10}$$

$$\lambda_1 = -(A + C^1 - E_A^1)a_2 - a_5$$
$$\lambda_2 = -(C^1 - E_I^1 + I)a_3 - a_5$$
$$\lambda_3 = -(C^1 - E_S^1 + S)a_4 - a_5$$
$$\lambda_4 = -U_A^1 a_2 - a_5 - a_6 - a_7$$
$$\lambda_5 = -U_I^1 a_3 - a_5 - a_6 - a_7$$
$$\lambda_6 = -U_S^1 a_4 - a_5 - a_6 - a_7$$
$$\lambda_7 = (A + C^1 - E_A^1)a_2$$
$$\lambda_8 = (C^1 - E_I^1 + I)a_3$$
$$\lambda_9 = (C^1 - E_S^1 + S)a_4$$

$$|\mathcal{J} - sI| = 0$$

$$\begin{pmatrix} \lambda_1 & 0 & 0 & U_A^1 a_2 + a_6 & 0 & 0 & -U_A^1 a_2 \\ 0 & \lambda_2 & 0 & 0 & U_I^1 a_3 + a_6 & 0 & -U_l^1 a_3 \\ 0 & 0 & \lambda_3 & 0 & 0 & U_S^1 a_4 + a_6 & -U_S^1 a_4 \\ \lambda_7 & 0 & 0 & \lambda_4 & 0 & 0 & U_A^1 a_2 \\ 0 & \lambda_8 & 0 & 0 & \lambda_5 & 0 & U_I^1 a_3 \\ 0 & 0 & \lambda_9 & 0 & 0 & \lambda_6 & U_S^1 a_4 \\ \alpha & \alpha & \alpha & \delta & \delta & \delta & -\beta \end{pmatrix} - s \begin{pmatrix} 1\,0\,0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,0\,1 \end{pmatrix} = 0$$

The characteristic equation is obtained as

$$A_0 s^7 + A_1 s^6 + A_2 s^5 + A_3 s^4 + A_4 s^3 + A_5 s^2 + A_6 s^1 + A_7 = 0 \tag{11}$$

$$s^7 - tr(J_1) s^6 + tr(J_1) s^5 - tr(J_1) s^4 + tr(J_1) s^3 - tr(J_1) s^2 + tr(J_1) s^1 - det(J_1) = 0 \tag{12}$$

Where

$$\mathrm{tr}(\mathcal{J}_1) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 - \beta$$

$$\begin{aligned} \det(\mathcal{J}_1) = {} & \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \left(\lambda_4 - \tfrac{\lambda_7 U_A^1 a_2 + a_6}{\lambda_1}\right) \cdot \left(\lambda_5 - \tfrac{\lambda_8 U_J^1 a_3 + a_6}{\lambda_2}\right) \\ & \cdot \left(\lambda_6 - \tfrac{\lambda_9 U_S^1 a_4 + a_6}{\lambda_3}\right) \cdot \left[-\beta - \tfrac{-\alpha a_2 U_A^1 - \alpha a_3 U_l^1}{\lambda_2} + \tfrac{\alpha U_S^1}{\lambda_3} + \left(\frac{\delta - \frac{\alpha U_A^1 a_2 + a_6}{\lambda_1}}{\lambda_4 - \frac{\lambda_7 U_A^1 a_2 + a_6}{\lambda_1}} \right) \right. \\ & \left(-a_2 U_A^1 - \tfrac{\lambda_7 a_2 U_A^1}{\lambda_2}\right) - \left(\frac{\delta - \frac{\alpha U_A^1 a_2 + a_6}{\lambda_1}}{\lambda_4 - \frac{\lambda_7 U_A^1 a_2 + a_6}{\lambda_1}} \right) \left(-a_3 U_l^1 - \tfrac{\lambda_8 a_2 U_A^1}{\lambda_2}\right) \\ & \left. - \left(\frac{\delta - \frac{\alpha U_S^1 a_4 + a_6}{\lambda_3}}{\lambda_3 - \frac{\lambda_9 U_S^1 a_4 + a_6}{\lambda_3}} \right) \left(a_3 U_S^1 - \tfrac{\lambda_9 a_4 U_S^1}{\lambda_3}\right) \right] \end{aligned}$$

By means of the Routh-Hurwitz criterion [14] to establish the local stability of the non-negative equilibrium points, we develop the Routh array from the characteristic equation (11) as,

Table 1. *Routh Array.*

A_0	A_2	A_4	A_6	0
A_1	A_3	A_5	A_7	0
B_1	B_2	B_3	0	0
C_1	C_2	A_7	0	0
D_1	D_2	0	0	0
e_1	A_7	0	0	0

Where

$$\begin{aligned} B_1 &= \tfrac{A_0 A_3 - A_1 A_2}{A_1}\, B_2 = \tfrac{A_1 A_4 - A_0 A_5}{A_1}\, B_3 = \tfrac{A_1 A_6 - A_0 A_7}{A_1} \\ C_1 &= \tfrac{B_1 A_3 - A_1 B_2}{B_1}\, C_2 = \tfrac{B_1 A_5 - A_1 B_3}{B_1} \\ D_1 &= \tfrac{C_1 B_2 - B_1 C_2}{C_1}\, D_2 = \tfrac{C_1 B_3 - B_1 A_7}{C_1} \\ e_1 &= \tfrac{C_1 D_2 - D_1 C_2}{D_1}. \end{aligned}$$

The Routh-Hurwitz stability criterion states a polynomial has all roots in the open left half plane if and only if all first-column elements of the Routh array have the same sign. Thus,

$$A_0 > 0,\ A_1 > 0, A_0 A_3 > A_1 A_2, B_1 A_3 > A_1 B_2, C_1 B_2 > B_1 C_2, C_1 D_2 > D_1 C_2 \tag{13}$$

Since the coefficients of the characteristic equation can be established and by algebraic manipulation, (13) are satisfied. By the Routh Hurwitz criteria all roots of the characteristic equation are negative or have negative real part. Therefore, the equilibrium point is locally asymptotically stable.

6. Conclusion

In this paper, a non-linear ordinary differential equations model is developed and analyzed for unemployment dynamics on Ghana's Economic sectors. It is found that the equilibrium point is locally asymptotically stable that any solutions that start close enough not only remain close enough but also eventually converge to the equilibrium point. Thus, around this point, there is a balance in the various compartments such that, the ratio of unemployed to employed are in relative proportions. Vacancies are optimal created to be filled. This behaviour continues over a certain period of time.

7. Recommendation

a) The model can be modified based on assumptions different from those considered here.
b) Movement within the three economic sectors should be studied to understand how they contribute to unemployment and the creation of jobs.

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