International Journal of Theoretical and Applied Mathematics
2022; 8(2): 45-50
http://www.sciencepublishinggroup.com/j/ijtam
doi: 10.11648/j.ijtam.20220802.13
ISSN: 2575-5072 (Print); ISSN: 2575-5080 (Online)

On Distributional Solutions of a Singular Differential Equation of 2-order in the Space K'

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To cite this article:
Abdourahman. On Distributional Solutions of a Singular Differential Equation of 2-order in the Space K'. *International Journal of Theoretical and Applied Mathematics*. Vol. 8, No. 2, 2022, pp. 45-50. doi: 10.11648/j.ijtam.20220802.13

Received: March 9, 2022; **Accepted**: April 1, 2022; **Published**: April 9, 2022

Abstract: The main purpose of this work is to describe all the zero-centered solutions of the second order linear singular differential equation with Dirac delta function (or it derivatives of some order) in the second right hand side in the space *K'*. All the coefficients and the exponents of the polynomials under the unknown function and it derivatives up to second order respectively, are real and natural numbers in the considered equation. We conduct investigations for both the euler case and left euler case situations of this equation, when it is fulfilled some particular conditions in the relationships between the parameters *A, B, C, m, n and r*. In each of these cases, we look for the zero-centered solutions and substitute the form of the particular solution into the equation. We then after, determinate the unknown coefficients and formulate the related theorems to describe all the solutions depending of the cases to be investigated.

Keywords: Test Functions, Generalized Functions, Dirac Delta Function, Fourier Transform, Zero-Centered Solutions

1. Introduction

The importance of differential equations is well known as these equations describe many physical phenomena in our daily life. One also understand and know that it is not easy, in some specific cases, to solve certain kind of differential equations even those of the first order. Solving differential equations in the spaces of generalized functions such as K' and $(S_\alpha^\beta)'$ is always challenging, and various scientific researches were devoted to such topic.

We recall that the theory of distributions was established by the imminent french mathematician Laurent Schwartz in 1945 posing the ideas which already in germ in the works of the great russian mathematician Sobolev in the 1930s. It is also well known that physics extended in space by functions of several variables and the expression of the laws of physics in terms of partial differential equations have been a great advance in the study of these phenomena. Talking to some specific notions, we know that sometimes even the concept of weak derivatives is not sufficient, and the need arises to define derivatives that are not functions, but are more general objects. Some measures and derivatives of measures will enter. We underline the fact that, for the purpose of setting up the rules for a general theory of differention where classical differentiability fails, Schwartz brought forward around 1950 the concept of distributions: a class of objects containing the locally integrable functions and allowing differentiations of any order.

In many scientific works it is mentioned that the distributional solutions, specifically as a series of Dirac delta function and its derivatives, have been used in several areas of applied mathematics such as the theory of partial differential equations, operational calculus, and functional analysis; in Physics such as quantum electrodynamics. We advice readers to see for more details the papers [15–16].

In our previous works, we have already used the Fourier Transform and its inverse applied to singular linear differential equation of some forms to describe and obtain all the generalized-function solutions of the considered equation, see [6, 11, 13]. In the same direction, we can quote among many others, for the generalized solutions, Nonlaopon et al. [17] used the Laplace transform technique to study some differential equations satisfying the differential equation.

$$t\,y^{(n)}(t) + my^{(n-1)}(t) + ty(t) = 0,$$

where $m, n \in \mathbb{Z}, with\ n \geq 2, and\ t \in \mathbb{R}$. Opio et al. [18] studied those of the differential equation with polynomial coefficients of the form.

$$t\, y^{(n)}(t) + (m-1)y^{(n-1)}(t) + py(t) = 0,$$

where $n, p, m \in \mathbb{N}, n \geq 2, and\ t \in \mathbb{R}$.

Here we consider the following singular linear differential equation.

$$\mathrm{A}x^m y''(x) + Bx^n y'(x) + Cx^r y(x) = \delta^{(s)}(x), \quad (1)$$

where $A, B, C \in \mathbb{R}, m, n \in \mathbb{N}, s, r \in \mathbb{N} \cup \{0\}, and\ y(x)$ is the unknown generalized function, in the space of generalized functions K'.

We focus our investigations in the situations called *Euler case* and *left Euler case* when it is realized respectively the conditions m=r+2, n=r+1 for the nonhomogenous equation and m=n+1, r>n-1 for the homogenous equation.

This is the first step we undertake to see in which way, we can generalize step by step, the results of researches obtained in the previous case studied for a linear singular differential equation of first order. See [6].

We structure this paper as follow: in section 2, we recall some fundamental well known concepts of distributions (generalized functions). Section 3 presenting the main results of the paper is firstly discribing, the *Euler case* and, secondly is devoted properly to the investigation of the solvability (existence of zero-centered solutions) of the considered homogenous equation in the situation called *the left Euler case*. We conclude our paper in section 4.

2. Preliminaries

Before we proceed to our main results, the following definitions and concepts well known from the theory of generalized functions are required. We also recall the notions of Fourier Transform and it inverse applied when looking for solutions of differential equations in our previous researches, for more details see [6].

By the way, we briefly review this important notions of Fourier transform, its properties, and generalized function centered at a given point (for a detailled study, we refer to [2, 6, 9, 14]). We recall that *K* is denoted the space of test functions, of finite infinitely differentiable on *R* functions and *K'* the space of generalized functions on *K*.

For the function $\varphi(t) \in K$, through $F\varphi = \widetilde{\varphi}$ we noted the Fourier transform defined by the formula.

$$(F\varphi)(x) = \widetilde{\varphi}(x) = \int_{-\infty}^{+\infty} \varphi(t) e^{ixt} dt, \quad (2)$$

We define the Fourier transform of the generalized function $f \in K'$ by the rule (Parseval equality)

$$(\widetilde{f}, \widetilde{\varphi}) = 2\pi(f, \varphi). \quad (3)$$

For the Fourier transform of generalized function, many properties are conserved as those taking place for Fourier transform for test functions, and particularly formulas of relationships between differentiability and decreasement.

From them in particular it follows that

$$F\left[\delta^{(s)}(t)\right] = (-ix)^s, s \in \mathbb{N} \cup \{0\}. \quad (4)$$

Next, we need the following assertions which can be found with their proofs in the books for theory of generalized functions, for example see [2, 3, 9].

Theorem 2.1 If $f, g \in K'$ and $f' = g'$ $then$ $f - g = C$.

Theorem 2.2 Let $A(x) \in C^\infty(\mathbb{R}^1)$. The differential equation $y' = A(x)y$ in the space *K'* does not admit other solutions which are not classical solutions.

Definition 2.1 Generalized function $f \in K'$ is called centered function at the point x_0 or x_0-centered function, if $(f, \varphi(x)) = 0$ for all $\varphi(x) \in K$ such $x_0 \,\overline{\in}\, Supp\varphi$.

Theorem 2.3 let $f \in K'$ zero-centered generalized function. Then there exist $m \in \mathbb{N} \cup \{0\}$ such that

$$f(x) = \sum_{j=0}^{m} C_j \delta^{(j)}(x), \quad (5)$$

where C_j are some constants.

Lemma 2.1. Let $\beta(x) \in C^\infty(\mathbb{R}^1)$. Then it takes place the following formula.

$$\beta(x)\delta^{(l)}(x) = \sum_{j=0}^{l} (-1)^j C_l^j \beta(0)^j \delta^{(l-j)}(x) \quad (6)$$

As consequence from lemma 2.1 when $\beta(x) = x^k$, we obtain the following assertion.

Lemma 2.2. Let $k, s \in \mathbb{N} \cup \{0\}$. Then it is holding place.

$$x^k \delta(x)^{(s)} = \begin{cases} 0, s < k; \\ \frac{(-1)^k s!}{(s-k)!} \delta(x)^{(s-k)}, s \geq k. \end{cases} \quad (7)$$

The proof of this lemma can be found in some special mathematical books related to the theory of distributions, see also [2, 3, 9].

Sometime we used in our investigations the following very important expression $\frac{\delta^{(s)}(x)}{x^n}$ which we are understanding in the sens of the following definition.

Definition 2.2 The quotient (the division) of an *sth-order* derivative of the Dirac delta function $\delta^{(s)}(x)$ by an *nth-power* of x $i.e$ x^n noted $\frac{\delta^{(s)}(x)}{x^n}$ is called a generalized function $y(x) \in K'$ that satisfies in the space K' the equality.

$$x^n y(x) = \delta^{(s)}(x) \quad (8)$$

i.e

$$\left(x^n y(x), \varphi(x)\right) = \left(\delta^{(s)}(x), \varphi(x)\right), \varphi(x) \in K \quad (9)$$

From lemma 2.2 and definition 2.2, we deduce the following formula for the computation of the generalized function $\frac{\delta^{(s)}(x)}{x^n}$.

Lemma 2.3 Let $n \in \mathbb{N}, s \in \mathbb{N} \cup \{0\}$. Then it is holding place the following formula.

$$\frac{\delta^{(s)}(x)}{x^n} = \frac{(-1)^n s!}{(s+n)!} \delta^{(s+n)}(x) + \sum_{k=1}^{n} C_k \delta^{(k-1)}(x) \quad (10)$$

where C_k, $k = 1, \ldots\ldots n$ are arbitrary constants.

For the proof of this lemma it is sufficient to apply the definition 2.2, the Fourier transform to both members of the equation (8) and, next applying the inverse Fourier transform, we reach to the needed result.

3. Mains Results

A. The Euler case.

In this section we consider the following second order linear singular differential equation:

$$\mathrm{A}x^m\, y''(x) + Bx^n\, y'(x) + Cx^r\, y(x) = \delta^{(s)}(x)$$

where $A, B, C \in \mathbb{R}, m, n \in \mathbb{N}, s, r \in \mathbb{N} \cup \{0\}, y(x) -$ the unknown generalized function from K'.

We have the most simple case of the equation (1) when the relationship between parameters $m, n\ and\ r$ holds:

$$m = r + 2, n = r + 1. \tag{11}$$

Here, we formulate the following theorem related to the equation (1) in the case called *Euler case*.

Theorem 3.1: Let $A \cdot B \cdot C \neq 0, m, n \in \mathbb{N}, s, r \in \mathbb{N} \cup \{0\}$. For the existence of the zero-centered solution of equation (1) in the space of generalized functions K', it is necessary and sufficient that

$$(r - n + 1)^2 + (r - m + 2)^2 + (C(r + s)! - B(r + s + 1)! + A(r + s + 2)!)^2 \neq 0$$

Proof: The proof of this theorem is deduced as a particular case of the proof of similar Theorem studied in [6].

Now, let stipulate this following result.

Theorem 3.2 Let$A \cdot B \cdot C \neq 0, m, n \in \mathbb{N}, s, r \in \mathbb{N} \cup \{0\}\ and\ fulfilled\ the\ condition$

$$m = r + 2, n = r + 1, C - B(s + r + 1) + A(s + r + 2)(s + r + 1) \neq 0. \tag{12}$$

Then the general zero-centered solution of the equation (1) has the following form:

$$y(x) = \frac{(-1)^r s!}{(r+s)![C - B(s+r+1) + A(s+r+2)(s+r+1)]}\delta^{(s+r)}(x) + \sum_{j=0}^{r-1} C_j \delta^{(j)}(x) \tag{13}$$

in the case, when

$$C - B(j + r + 1) + A(j + r + 2)(j + r + 1) \neq 0\ \forall j \in \mathbb{Z}_+. \tag{14}$$

If there exists at least one ${j_*}^1, {j_*}^2 \in \mathbb{Z}_+/\{s\}$ such, that $P_2(j_*^i) = 0$, $i \leq 2, then\ the\ solution\ has\ the\ following\ form$:

$$y(x) = \frac{(-1)^r s!}{(r+s)![C - B(s+r+1) + A(s+r+2)(s+r+1)]}\delta^{(s+r)}(x) + \sum_{j=0}^{r-1} C_j \delta^{(j)}(x) + \sum_{{j_*}^i \in Nul\, P_2(j)} C_{{j_*}^i + r}\, \delta^{\left({j_*}^i + r\right)}(x)\ . \tag{15}$$

The proof can be deduced analogously from Theorem 1.2 of a more general case see [6].

Next, we consider a more complicated case when it is violated at least one of the two conditions (12).

First of all let consider the case $m = r + 2\ and\ n \neq r + 1$ and, one of the situation to be investigated we call this case as following way.

B. The left Euler case.

In this section we consider the case $m = n + 1, r > n - 1$ and call this situation of the equation (1) *left Euler case*.

We pay attention to the possibility of the existence of the solutions of the homogeneous equation (1) different to the trivial $y(x)_j = \delta^{(j)}(x), j = 0, 1, \ldots\ldots, n - 2$.

The general form of the zero-centered solutions of the homogeneous equation (1) is brought downstair in the following theorem:

Theorem 3.3 Let $A \cdot B \cdot C \neq 0$, l_0- a number defined by (25), m, n $\in \mathbb{N}, r \in \mathbb{N} \cup \{0\}\ and$ fulfilled the condition.

$$m = n + 1, r > n - 1, \tag{16}$$

then the homogeneous equation (1) admit zero-centered solution of the following form:

$$y(x)_j = \sum_{l=1}^{l_0} (-1)^{(n-r-1)l}\ C^l \cdot \prod_{i=1}^{l} \frac{[\vartheta_* + i(n-1-r) + r]!}{[\vartheta_* + i(n-1-r) + n]!(B - A[\vartheta_* + s(n-1-r) + n + 1])}\delta^{(\vartheta_* + (l+1)(n-1) - lr)}(x) + \delta^{(\vartheta_* + n - 1)}(x), \tag{17}$$

Where ϑ_* is such that $r - (n - 1) - \vartheta_* \leq 0\ and$

$$B - A(\vartheta_* + n + 1) = 0. \tag{18}$$

Proof. We look for the solution in the following way:

$$y(x) = \sum_{l=1}^{l_0} R_l \delta^{(\vartheta_* + (l+1)(n-1) - lr)}(x) + \delta^{(\vartheta_* + n - 1)}(x) \tag{19}$$

with unknown coefficients R_l for the moment, and l_0- an interger number sufficiently large.

For simplicity of calculations let note as follow.

$$Ly = \mathrm{A}x^{n+1}\, y'' + Bx^n\, y' + Cx^r\, y = 0, \tag{20}$$

$$L_1 y = \mathrm{A}x^{n+1}\, y'' + Bx^n\, y', \tag{21}$$

$$L_2 y = C x^r y. \tag{22}$$

Next, let us calculate immediatly.

$$L\delta^{(\vartheta_*+n-1)}(x) = \mathrm{A}x^{n+1}\delta^{(\vartheta_*+n+1)}(x) + Bx^n\delta^{(\vartheta_*+n)}(x) + Cx^r\,\delta^{(\vartheta_*+n-1)}(x) = \left[(-1)^{n+1}A\frac{(\vartheta_*+n+1)!}{\vartheta_*!} + (-1)^n B\frac{(\vartheta_*+n)!}{\vartheta_*!}\right]\delta^{(\vartheta_*)}(x)+(-1)^r C\frac{(\vartheta_*+n-1)!}{(\vartheta_*+n-1-r)!}\delta^{(\vartheta_*+n-1-r)}(x)=(-1)^n\frac{(\vartheta_*+n)!}{\vartheta_*!}[B - A(\vartheta_* + n + 1)]\delta^{(\vartheta_*)}(x)+(-1)^r C\frac{(\vartheta_*+n-1)!}{(\vartheta_*+n-1-r)!}\delta^{(\vartheta_*+n-1-r)}(x).$$

Consequently, taking into account the condition (18) we obtain:

$$L\delta^{(\vartheta_*+n-1)}(x) = (-1)^r C\frac{(\vartheta_*+n-1)!}{(\vartheta_*+n-1-r)!}\delta^{(\vartheta_*+n-1-r)}(x), \tag{23}$$

Now let calculate the following $L\delta^{(\vartheta_*+(l+1)(n-1)-lr)}(x)$. For the beginning let start:

$$L_1\delta^{(\vartheta_*+(l+1)(n-1)-lr)}(x)=\left[A(-1)^{n+1}\frac{[(\vartheta_*+(l+1)(n-1)-lr+2)]!}{[\vartheta_*+l(n-1-r)]!} + B(-1)^n\frac{[\vartheta_*+(l+1)(n-1)-lr+1]!}{[\vartheta_*+l(n-1-r)]!}\right]\delta^{(\vartheta_*+l(n-1-r))}(x). \tag{24}$$

Now, let the most value of the number $l = l_0$. Then for the accomplishement of (24) for all $l = 1,\ldots\ldots, l_0$ it is necessary the fulfillement of the relationship:

$$\vartheta_* + (l_0+1)(n-1) - l_0 r + 2 - (n+1) \geq 0$$

or that is the same as

$$\vartheta_* + l_0(n-1-r) \geq 0 \Rightarrow l_0 \leq \frac{\vartheta_*}{r-(n-1)}$$

$$l_0 + 1 > \frac{\vartheta_*}{r-(n-1)}.$$

That means, the maximum value of

$$l_0 = \left[\frac{\vartheta_*}{r-(n-1)}\right]. \tag{25}$$

Next, when calculating $L_2 y$, verifiying the same assertion we may easily find that

$$\vartheta_* + (l+1)(n-1) - lr - r \geq 0, \tag{26}$$

so that we can define under which l it is true (3.16) which we can rewrite in another way as

$$l+1 \leq \frac{\vartheta_*}{r-(n-1)} \tag{27}$$

From (27) it is clear that the relation is accomplished by virtue of (26) for $l = 1,\ldots\ldots, l_0 - 1$ and not accomplished, for $l = l_0$, that means.

$$x^r\,\delta^{(\vartheta_*+(l_0+1)(n-1)-l_0 r)}(x) = 0, \tag{28}$$

for that value of l_0, and in the summation one term disappear. Therefore we have:

$$L_2\delta^{(\vartheta_*+(l+1)(n-1)-lr)}(x) = (-1)^r C\frac{[\vartheta_*+(l+1)(n-1)-lr]!}{[\vartheta_*+(l+1)(n-1-r)]!}\delta^{(\vartheta_*+(l+1)(n-1-r))}(x).$$

Next by combining all the calculations we obtain:

$$Ly = \sum_{l=1}^{l_0} R_l\left\{A(-1)^{n+1}\frac{[\vartheta_*+(l+1)(n-1)-lr+2]!}{[\vartheta_*+l(n-1-r)]!} + B(-1)^n\frac{[\vartheta_*+(l+1)(n-1)-lr+1]!}{[\vartheta_*+l(n-1-r)]!}\right\}\delta^{(\vartheta_*+l(n-1-r))}(x) + \sum_{l=1}^{l_0-1} R_l(-1)^r C\frac{[\vartheta_*+(l+1)(n-1)-lr]!}{[\vartheta_*+(l+1)(n-1-r)]!}\delta^{(\vartheta_*+(l+1)(n-1-r))}(x) + (-1)^r C\frac{(\vartheta_*+n-1)!}{(\vartheta_*+n-1-r)!}\delta^{(\vartheta_*+n-1-r)}(x).$$

Now performing in the second summation the changing of index by $l + 1 \longrightarrow l$, we find:

$$Ly = \sum_{l=1}^{l_0} R_l(-1)^n\frac{[\vartheta_*+(l+1)(n-1)-lr+1]!}{[\vartheta_*+l(n-1-r)]!}\cdot\left[B - A[\vartheta_* + (l+1)\,(n-1) - lr + 2]\right]\delta^{(\vartheta_*+l(n-1-r))}(x) +$$

$$\sum_{l=2}^{l_0} R_{l-1}(-1)^r C \frac{[\vartheta_*+l(n-1)-(l-1)r]!}{[\vartheta_*+l(n-1-r)]!} \delta^{(\vartheta_*+l(n-1-r))}(x) + (-1)^r C \frac{(\vartheta_*+n-1)!}{[\vartheta_*+(n-1-r)]!} \delta^{(\vartheta_*+n-1-r)}(x). \quad (29)$$

The equality Ly =0 by vritue of the linear independance of the delta distributions leads us to the recurrent algebraic system for the determination of the indertiminate coefficients R_l.

$$\begin{cases} R_l(-1)^n \frac{[\vartheta_*+l(n-1-r)+n]!}{[\vartheta_*+l(n-1-r)]!} \cdot [B - A(\vartheta_* + l(n-1-r) + n + 1)] = \\ (-1)^{r-1} C R_{l-1} \frac{[\vartheta_*+l(n-1-r)+r]!}{[\vartheta_*+l(n-1-r)]!}, l = 2, \ldots\ldots, l_0, \\ R_1(-1)^n \frac{[\vartheta_*+n-1-r+n]!}{[\vartheta_*+n-1-r]!} [B - A(\vartheta_* + n - 1 - r + n + 1)] = (-1)^{r-1} C \frac{(\vartheta_*+n-1)!}{(\vartheta_*+n-1-r)!}. \end{cases}$$

Finally, rewrite these relationships following way:

$$R_l = (-1)^{n-r-1} C \frac{[\vartheta_* + l(n-1-r) + r]!}{[\vartheta_* + l(n-1-r) + n]!} \cdot \left\{\frac{1}{B - A[\vartheta_*+l(n-1-r)+n+1]}\right\} R_{l-1}, \quad (30)$$

$l = 2, \ldots\ldots, l_0;$ *and*

$$R_1 = (-1)^{n-r-1} C \frac{[\vartheta_*+n-1-r+r]!}{[\vartheta_*+n-1-r+n]!} \cdot \frac{1}{[B-A(\vartheta_*+n-1-r+n+1)]}, \quad (31)$$

Immediatly note that the right hand side in (30) - (31) are similar and therefore from these recurrent relationships, it is easy to obtain the general form of the coefficients R_l. That allows us writing the result defined by formula (17).

The theorem is proved.

Theorem 3.4. Let $A \cdot B \cdot C \neq 0, m, n \in \mathbb{N}, r \in \mathbb{N} \cup \{0\}$ and fulfilled the condition (16) then the homogeneous equation (1) admit zero-centered solution of the following form:

$$y(x) = \sum_{j=0}^{n-2} C_j \delta(x)^{(j)} \quad (32)$$

when it is realized the condition $B - A(\vartheta_* + n + 1) \neq 0$

$$\forall \vartheta_* \in \mathbb{Z}_+; \quad (33)$$

or

$$y(x) = \sum_{j=0}^{n-2} C_j \delta(x)^{(j)} + C_{\vartheta_*+n-1} \delta(x)^{(\vartheta_*+n-1)}, \quad (34)$$

if $\exists \vartheta_* \in \mathbb{Z}_+$ such that,

$$B - A(\vartheta_* + n + 1) = 0, \quad (35)$$

and more $\vartheta_* + n - 1 - r < 0$;

Or

$$y(x) = \sum_{j=0}^{n-2} C_j \delta(x)^{(j)} + \alpha \left[\delta(x)^{(\vartheta_*+n-1)} + \sum_{l=1}^{l_0} (-1)^{(n-r-1)l} C^l \cdot \prod_{i=1}^{l} \frac{[\vartheta_*+i(n-1-r)+r]!}{[\vartheta_*+i(n-1-r)+n]!(B-A[\vartheta_*+s(n-1-r)+n+1])} \delta^{(\vartheta_*+(l+1)(n-1)-lr)}(x)\right],$$

if $\vartheta_* \in \mathbb{Z}_+$ such that it is realized (35), and more $\vartheta_* + n - 1 - r \geq 0$ with α an arbitrary constant.

The proof in the case c) is deduced from theorem 3.3, in the case b) from the fact that $L\delta^{(\vartheta_*+n-1)}(x) = 0$, by the realization (35) and $\vartheta_* + n - 1 - r < 0$ and finally in the case a) it is obvious.

4. Conclusion

In this paper we have completly investigated the existence of the zero-centered solutions of the equation (1) in both cases called: *Euler* and *left Euler cases*.

We have look for the wanted solutions by replacing initialy the particular solution expressed with unknown coefficients into the initial equation (1) and, therefore we obtain in theorem 3.1 the necessary and sufficient conditions for the existence of zero-centered solutions of the equation in the euler case. Next, it is described all the solutions in the previous case within theorem 3.2 in connexion of the two possibilities mentionned. Investigating left euler case, we bring out the existence of non-trivial solutions in the form of Dirac delta functions as well as it derivatives up to the order $n-2$. The main results obtained and concerning this case, depending of the relationships between the parameters, are formulated in theorems 3.3 and 3.4.

From the obtained results, it is clear that it is challenging to try to imagine how to generalize such investigations of similar type of differential equation up to the general cases call *left Euler case* or *right Euler case* when there are realized the conditions: $\sum_{i=0}^{l} a_i x^{k_i} y^{(i)}(x) = \delta^{(s)}(x)$ with $k_i = k_{i-1} + 1$; $i = 1 \ldots\ldots, l-1$ but $k_l \neq k_{l-1} + 1$ i.e $k_l > k_{l-1} + 1$ or $k_l < k_{l-1} + 1$.

Acknowledgements

The author is grateful to late Professor KARAPETYANTS N. K former Head of the Department of Differential and Integral Equations of the Rostov State University in Russia, for his invaluable support and helpful discussions by that time when conducting researches.

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