International Journal of Theoretical and Applied Mathematics
2022; 8(5): 85-95
http://www.sciencepublishinggroup.com/j/ijtam
doi: 10.11648/j.ijtam.20220805.11
ISSN: 2575-5072 (Print); ISSN: 2575-5080 (Online)

Cycles and Divergent Trajectories for a Class of Permutation Sequences

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To cite this article:
John Simons. Cycles and Divergent Trajectories for a Class of Permutation Sequences. *International Journal of Theoretical and Applied Mathematics*. Vol. 8, No. 5, 2022, pp. 85-95. doi: 10.11648/j.ijtam.20220805.11

Received: September 15, 2022; **Accepted:** October 13, 2022; **Published:** November 2, 2022

Abstract: Let f be a permutation from $\mathbb{N}_0$ onto $\mathbb{N}_0$. Let $x \in \mathbb{N}_0$ and consider a (finite or infinite) sequence $s = (x, f(x), f^2(x), \cdots)$. We call s a permutation sequence. Let D be the set of elements of s. If D is a finite set then the sequence s is a cycle, and if D is an infinite set the sequence s is a divergent trajectory. We derive theoretical and computational bounds for cycles and divergent trajectories for a defined class of permutations. This class contains generalizations of the original Collatz permutation $f(2n) = 3n,\ f(4n+1) = 3n+1,\ f(4n+3) = 3n+2$ and is based on a generalization of the covering system of congruences of Erdös. For the derivation of theoretical cycle bounds from transcendental number theory we adjust the cycle concept and the approach of Simons and de Weger for the $3x+1$ problem. For the derivation of computational cycle bounds we use continued fraction approximation methods. For a defined subclass the existence of divergent trajectories is proved.

Keywords: Permutation Sequences, Cyclic Trajectories, Divergent Trajectories, Collatz Conjecture, Transcendental Number Theory, Complete Coverage Sets

1. Introduction

We consider a class of "linear" permutations in $\mathbb{N}_0$ i.e. if $x = a_i n + b_i$ then $f(x) = c_i n + d_i$ for $i = 0 \ldots N$. After introducing some examples, this class will be defined formally by imposing restrictions on a_i, b_i, c_i, d_i. There exist permutations outside this "linear" class, see the section Closing remarks 1.

1.1. Examples

Consider the function $f(2n) = 3n,\ f(4n+1) = 3n+1,\ f(4n+3) = 3n+2$ with $n \in \mathbb{N}_0$. Lothar Collatz (the inventor of the famous Collatz conjecture) studied this permutation. See the section Closing remarks 2. There is computational evidence (no proof) that the only cycles are (0), (1), (2, 3), (4, 6, 9, 7, 5), (44, 66, 99, 74, 111, 83, 62, 93, 70, 105, 79, 59) and that the numbers 8, 14, 40, 64, 80, 82, $\cdots$ are minima of different divergent trajectories e.g. $\cdots$, 97, 73, 55, 41, 31, 23, 17, 13, 10, 15, 11, 8, 12, 18, 27, 20, 30, 45, 34, 51, 38, 57, $\cdots$. We call this function the Collatz permutation and the associated sequences Collatz permutation sequences.

Consider the function $f(4n) = 3n,\ f(8n+1) = 9n+1,\ f(8n+2) = 9n+2,\ f(8n+3) = 9n+4,\ f(8n+5) = 9n+5,\ f(8n+6) = 9n+7,\ f(8n+7) = 9n+8$. There are (computational evidence, no proof) at least 12 cycles with length varying form 1 to 93 and many apparently divergent trajectories.

Consider the function $f(2n) = n$ for $n > 0$, $f(0) = 1$, $f(2n+1) = 2n+3$ for $n \geq 0$. Now only the double divergent trajectory $\cdots, 16, 8, 4, 2, 0, 1, 3, 5, 7, \cdots$ exists.

Consider the function $f(6n) = 3n,\ f(12n+1) = 15n+1,\ f(12n+2) = 15n+2,\ f(12n+3) = 15n+4,\ f(12n+4) = 15n+5,\ f(12n+5) = 15n+7,\ f(12n+7) = 15n+8,\ f(12n+8) = 15n+10,\ f(12n+9) = 15n+11,\ f(12n+10) = 15n+13,\ f(12n+11) = 15n+14$. Next to the cycles $(0),\ (1),\ (2)$ there exists an infinite number of double divergent trajectories.

1.2. Definitions

Let $f : \mathbb{N}_0 \to \mathbb{N}_0$ be a permutation (bijection). Let $x_0 \in \mathbb{N}_0$. Consider a sequence $(x_0, x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), \cdots)$ associated with f. Then either there exists a smallest $k \in \mathbb{N}$ such that $x_k = x_0$ or such a $k \in \mathbb{N}$ does not exist. If k exists then x_0 is an element of a cycle of length k. If k does not exist, then x_0 is an element of a divergent trajectory with $\lim_{k\to\infty} f^k(x_0) = \infty$ and also $\lim_{k\to\infty} f^{-k}(x_0) = \infty$. Guy (problem $E17$) calls this a double infinite chain [7]. Let D be the set of (different) x_j-values of such a sequence. Then f is also a permutation from D onto D. We call sequences that are associated with f permutation sequences. We denote the set of these sequences by $PS(f)$.

Let $a, b, c, d \in \mathbb{N}$. Assume that $b > 1, d > 1$ and $N = a(b-1) = c(d-1)$. Let $R = \{r_i > 0 | \ i = 1 \cdots N\}$ be the set of different residues (mod ab) with $r_i \not\equiv 0$ (mod b) and let $S = \{s_i > 0 | \ i = 1 \cdots N\}$ be the set of different residues (mod cd) with $s_i \not\equiv 0$ (mod d).

Lemma 1 Let a, b, c, d, r_i, s_i be defined as above. Let $n \in \mathbb{N}_0$. Let f be a function from $\mathbb{N}_0$ onto $\mathbb{N}_0$, defined by $f(bn) = dn, \ f(abn + r_i) = cdn + s_i | i = 1, \cdots, N$. Then $f : \mathbb{N}_0 \to \mathbb{N}_0$ is a bijection (permutation).

Proof Let $x \in \mathbb{N}_0$. Then either $x \equiv 0$ (mod b) or there exists exactly one i with $x \equiv r_i$ (mod ab). Hence f is a surjective function from $\mathbb{N}_0$ onto $\mathbb{N}_0$. Let g be a function from $\mathbb{N}_0$ onto $\mathbb{N}_0$, defined by $g(dn) = bn, \ g(cdn + s_i) = abn + r_i | i = 1, \cdots, N$. Similarly g is a surjective function from $\mathbb{N}_0$ onto $\mathbb{N}_0$. Since g is the inverse function of f, f is a permutation (bijection) from $\mathbb{N}_0$ onto $\mathbb{N}_0$.

Lemma 1 is independent of the order of the residues. We arbitrarily choose $1 = r_1 < r_2 < \cdots < r_N = ab - 1$. If also $1 = s_1 < s_2 < \cdots < s_N = cd - 1$ we write f as $P(a, b, c, d)$. We write $P(a, b, c, d)$ in the form

$$\left.\begin{matrix} bn \\ abn + r_1 \\ \cdots \\ abn + r_N \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} dn \\ cdn + s_1 \\ \cdots \\ cdn + s_N \end{matrix}\right.$$

If the residues are in non-increasing order we call such f a simple generalization of $P(a, b, c, d)$. In this paper we initially consider the class of permutations consisting of $P(a, b, c, d)$ and their simple generalizations. This class will be extended later.

The first example in the subsection Examples i.e. the Collatz permutation is $P(2, 2, 1, 3)$. The second example is $P(2, 4, 3, 3)$.

$$\left.\begin{matrix} 4n \\ 8n+1 \\ 8n+2 \\ 8n+3 \\ 8n+5 \\ 8n+6 \\ 8n+7 \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} 3n \\ 9n+1 \\ 9n+2 \\ 9n+4 \\ 9n+5 \\ 9n+7 \\ 9n+8 \end{matrix}\right.$$

For both examples we will calculate bounds for cycles and divergent trajectories. Note that the third example in the subsection Examples cannot be written in the format $P(a, b, c, d)$. The fourth example can be written in the format $P(a, b, c, d)$. For both examples we prove the existence of divergent trajectories.

1.3. Main Result

For the original Collatz function $g(2n) = n, \ g(2n+1) = 3n + 2$ (which is not a permutation), a trajectory with K odd and L even elements, m local minima, m local maxima, and with the property that for all elements x_i of the trajectory $x_{i+K+L} = x_i$ is called an m-cycle by Simons and de Weger [19]. Steiner and also Davidson call a 1-cycle a circuit [6, 17]. Brox considers cycles of odd numbers only and calls a number a descendent if the next number is smaller [3]. A descendent is the (odd) predecessor of a local maximum in an m-cycle. For the Collatz function Steiner proved that 1-cycles, except $(1, 2)$, cannot exist [17]. Simons and de Weger generalized his proof to m-cycles up to $m = 75$. Their approach is not simply applicable to permutation sequences. We generalize the approach of Simons and de Weger in a nontrivial way. For permutation sequences we modify the definition of an m-cycle to : a trajectory with K elements $\not\equiv 0$ (mod b) and L elements $\equiv 0$ (mod b), m local minima and m local maxima and with the property that for all elements x_i of the trajectory $x_{i+K+L} = x_i$. Our main result is

Theorem 2 [Main Theorem] Let $P(a, b, c, d)$ and $PS(f)$ be defined as above.

1. For all a, b, c, d, if $f = P(a, b, c, d)$ then $PS(f)$ has for each m a finite number of m-cycles. These cycles can be computed.
2. For all a, b, c, d, the set of sequences associated with a simple generalization of $P(a, b, c, d)$ has for each m a finite number of m-cycles. These cycles can be computed.
3. The set of Collatz permutation sequences associated with $P(1, 3, 2, 2)$ has for $m \le 5$ no other cycles than those listed in Corollary 11.
4. The set of permutation sequences associated with the simple generalization of $P(1, 3, 2, 2)$ has for $m \le 5$ no other cycles than those listed in the subsection Simple generalizations.
5. If $f = P(2, 4, 3, 3)$ then $PS(f)$ has for $m \le 5$ no other cycles than those listed in Corollary 13.
6. There exist permutations $f = P(a, b, c, d)$ such that the set of associated permutation sequences $PS(f)$ contains an infinite number of divergent trajectories.

2. Conditions for the Existence of m-cycles

Consider $P(a, b, c, d)$ with $a > 0, \ c > 0, \ b > 1, \ d > 1$. We choose $b > d$ which implies $c > a, \ ab < cd$. Then an m-

cycle consists of m pairs of an increasing subsequence of k_i elements $\not\equiv 0$ (mod b) followed by a decreasing subsequence of ℓ_i elements $\equiv 0$ (mod b).

Suppose an m-cycle exists for $P(a,b,c,d)$ with elements $x_i > X_0 = \frac{1+\epsilon}{\epsilon}(ab-a-1) \gg ab-a-1$ where X_0 is a numerical lower bound. By convention this m-cycle has m pairs of an increasing subsequence followed by a decreasing subsequence. The elements in the m-cycle are denoted by x_i. We call the local minima $\overline{x}_0 \ldots \overline{x}_{m-1}$, the local maxima $\overline{y}_0 \cdots \overline{y}_{m-1}$ and the length of the subsequences $k_0 \ldots k_{m-1}$ and $\ell_0 \ldots \ell_{m-1}$ respectively. We put $K = \sum_{j=0}^{m-1} k_j$ and $L = \sum_{j=0}^{m-1} \ell_j$, hence $x_{K+L} = x_0$. For m-cycles we define $\Lambda = K\log(cd) - K\log(ab) + L\log(d) - L\log(b)$.

2.1. An Upper Bound for Λ in Terms of the Local Minima $\overline{x}_j$

For the first pair of subsequences (neglecting the indices of r and s) we have

$$\frac{x_1}{x_0} = \frac{cd.n_0+s}{ab.n_0+r}, \cdots, \frac{x_{k_0}}{x_{k_0-1}} = \frac{cd.n_{k_0-1}+s}{ab.n_{k_0-1}+r}, \frac{x_{k_0+1}}{x_{k_0}} = \cdots = \frac{x_{k_0+\ell_0}}{x_{k_0+\ell_0-1}} = \frac{d}{b}$$

Multiplication of all chaining relations an using $x_{k_0+\ell_0} = \overline{x}_1$ yields

$$\frac{\overline{x}_1}{\overline{x}_0} = \frac{x_{k_0+\ell_0}}{x_0} = \frac{x_1}{x_0}\frac{x_2}{x_1}\cdots\frac{x_{k_0+\ell_0}}{x_{k+\ell_0-1}} = \left(\frac{cd}{ab}\right)^{k_0}\left(\frac{d}{b}\right)^{\ell_0}\prod_{i=0}^{k_0-1}\frac{1+\frac{s}{cdn_i}}{1+\frac{r}{abn_i}}$$

Applying this to each pair of subsequences we find for $j = 0\ldots m-1$

$$\frac{\overline{x}_{j+1}}{\overline{x}_j} = \left(\frac{cd}{ab}\right)^{k_j}\left(\frac{d}{b}\right)^{\ell_j} Pr(j)$$

where $Pr(j) = \prod_{i=0}^{k_j-1}\frac{1+\frac{s}{cdn_i}}{1+\frac{r}{abn_i}}$.

Multiplication of all the fractions of local minima and using $\overline{x}_m = \overline{x}_0$ leads to

$$F(K,L) = \left(\frac{cd}{ab}\right)^{K}\left(\frac{d}{b}\right)^{L} = \prod_{j=0}^{m-1}(Pr(j))^{-1}$$

Suppose $F(K,L) > 1$. Then we obtain

$$1 < F(K,L) < \prod_{j=0}^{m-1}\prod_{i=0}^{k_j-1}[1+\frac{r}{abn_i}] < \prod_{j=0}^{m-1}\left(1+\frac{ab-a-1}{\overline{x}_j-ab+a+1}\right)^{k_j}$$

which implies after taking logs

$$0 < \Lambda = \log F(K,L) < \sum_{j=0}^{m-1} k_j \log\left(1+\frac{ab-a-1}{\overline{x}_j-ab+a+1}\right) < \sum_{j=0}^{m-1} k_j\frac{ab-a-1}{\overline{x}_j-ab+a+1} \tag{1}$$

Suppose $F(K,L) < 1$. Then we find similarly

$$0 < -\Lambda = \log(F(K,L))^{-1} < \sum_{j=0}^{m-1} k_j \log\left(1+\frac{cd-c-1}{\overline{x}_j-cd+c+1}\right) < \sum_{j=0}^{m-1} k_j\frac{cd-c-1}{\overline{x}_j-cd+c+1} \tag{2}$$

From equations 1 and 2 and using the lower bound $\overline{x}_j > X_0$ we find

Corollary 3 If $F(K,L) > 1$ then $0 < \Lambda < K\frac{ab-a-1}{X_0-ab+a+1} = \epsilon K$
If $F(K,L) < 1$ then $0 < -\Lambda < K\frac{ab-a-1}{X_0-ab+a+1} = \epsilon K$

2.2. An Upper Bound for Λ in Terms of K, L and m

For local minima we have by convention $\overline{x}_j = e_j.d^{\ell_j} \geq d^{\ell_j}$. For the maximal local minimum we find $\max(\overline{x}_j)^m > \prod_{j=0}^{m-1}\overline{x}_j > d^L$ and consequently

$$\max(\overline{x}_j) > d^{\frac{L}{m}} \tag{3}$$

We now derive a chaining relation between the magnitudes of successive local minima $\overline{x}_j$. Under the assumption $a <$

$c, b > d, ab < cd$, consider the trajectory from $\overline{x}_j$ up to $\overline{y}_j$ and down to $\overline{x}_{j+1}$. Then $\overline{x}_j$ must be less than or equal to the largest possible predecessor of $\overline{y}_j$. The largest possible predecessor occurs if $abn + ab - a - 1 \leftrightarrow cdn + 1$ applies to that predecessor. Let $\overline{y}_j = e_j b^{\ell_j}$, $\delta_1 = \log_d b + 1 (> 1)$, $\alpha_1 = -\frac{ab}{cd} + ab - a - 1 (> 0)$. Then we find the asserted relation

$$\begin{aligned}
\overline{x}_j &\leq abn + ab - a - 1 = \frac{ab}{cd}(cdn+1) - \frac{ab}{cd} + ab - a - 1 = \frac{ab}{cd}\overline{y}_j + \alpha_1 \\
&= \frac{ab}{cd} e_j b^{\ell_j} + \alpha_1 = \frac{ab}{cd}\left(\frac{b}{d}\right)^{\ell_j} e_j d^{\ell_j} + \alpha_1 = \frac{ab}{cd}(d^{\delta_1})^{\ell_j} e_j d^{\ell_j} + \alpha_1 \\
&= \frac{ab}{cd}\frac{(e_j d^{\ell_j})^{\delta_1+1}}{e_j^{\delta_1}} + \alpha_1 < \frac{ab}{cd}\overline{x}_{j+1}^{\delta_1+1} + \alpha_1
\end{aligned}$$

Let $\gamma_1 = \frac{ab}{cd} + \frac{\alpha_1}{X_0^{\delta_1+1}}$. Then $\alpha_1 = (\gamma_1 - \frac{ab}{cd})X_0^{\delta_1+1} < (\gamma_1 - \frac{ab}{cd})\overline{x}_{j+1}^{\delta_1+1}$ which implies $\overline{x}_j < \frac{ab}{cd}\overline{x}_{j+1}^{\delta_1+1} + (\gamma_1 - \frac{ab}{cd})\overline{x}_{j+1}^{\delta_1+1} = \gamma_1 \overline{x}_{j+1}^{\delta_1+1}$. We find (taking into account that the worst case applies if $\overline{x}_j > \overline{x}_{j+1}$ for $j = 0 \cdots m-2$)

$$\max(\overline{x}_j) < \gamma_1^{1+(\delta_1+1)+\cdots+(\delta_1+1)^{m-2}} (\min(\overline{x}_j))^{(\delta_1+1)^{m-1}} \tag{4}$$

Let $\beta_1 = \gamma_1^{1+(\delta_1+1)+\cdots+(\delta_1+1)^{m-2}} = \gamma_1^{\frac{(\delta_1+1)^{m-1}-1}{\delta_1}}$.
Inserting equations 3 and 4 into equation 1 we obtain an upper bound for Λ (if $\Lambda > 0$) in terms of K, L and m.

$$\begin{aligned}
0 < \Lambda = \log F(K,L) &< \sum_{j=0}^{m-1} k_j \frac{ab-a-1}{\overline{x}_j - ab + a + 1} < K\frac{ab-a-1}{\min(\overline{x}_j) - ab + a + 1} \\
&= K\frac{(1+\epsilon)(ab-a-1)}{\min(\overline{x}_j)}\frac{\min(\overline{x}_j)}{\min(\overline{x}_j) - ab + a + 1}\frac{1}{1+\epsilon} \\
&< K\frac{(1+\epsilon)(ab-a-1)}{\min(\overline{x}_j)}\frac{X_0}{X_0 - ab + a + 1}\frac{1}{1+\epsilon} \\
&= K\frac{(1+\epsilon)(ab-a-1)}{\min(\overline{x}_j)} \\
&< K\frac{(1+\epsilon)(ab-a-1)}{\left(\frac{\max(\overline{x}_j)}{\beta_1}\right)^{\frac{1}{(\delta_1+1)^{m-1}}}} < K\frac{(1+\epsilon)(ab-a-1)}{\left(\frac{d^{\frac{L}{m}}}{\beta_1}\right)^{\frac{1}{(\delta_1+1)^{m-1}}}}
\end{aligned}$$

A similar expression can be derived if $\Lambda < 0$. So we have

Corollary 4. If an m-cycle exists for $P(a,b,c,d)$ with $0 < a < c,\ b > d > 0$ and $\epsilon = \frac{ab-a-1}{X_0-(ab-a-1)}$ then $|\Lambda| < K\frac{(1+\epsilon)(ab-a-1)}{\left(\frac{d^{\frac{L}{m}}}{\beta_1}\right)^{\frac{1}{(\delta_1+1)^{m-1}}}}$.

2.3. Conditions on K, L and m from Continued Fractions

A consequence of Corollary 3 is the availability of a sharp lower and upper bound for $\frac{K}{L}$. Suppose an m-cycle exists for $P(a,b,c,d)$ with $a < c,\ b > d$. Recall that $\alpha = \log cd - \log ab > 0$ and $\beta = \log b - \log d > 0$. From Corollary 3 we find

$$If\ F(K,L) > 1\ then\ 0 < \alpha K - \beta L < \epsilon K.\ \ If\ F(K,L) < 1\ then\ -\epsilon K < \alpha K - \beta L < 0.$$

This means that $\frac{K}{L}$ must be a good approximation to $\rho = \frac{\beta}{\alpha} > 0$. Then Corollary 3 implies

Corollary 5 If $F(K,L) > 1$ then $0 < \Lambda < \epsilon\frac{\beta}{\alpha-\epsilon}L$
If $F(K,L) < 1$ then $0 < -\Lambda < \epsilon\frac{\beta}{\alpha}L < \epsilon\frac{\beta}{\alpha-\epsilon}L$ and Corollary 4 implies

Corollary 6 If an m-cycle exists for $P(a,b,c,d)$ then $|\Lambda| < \frac{\beta}{\alpha-\epsilon}L\dfrac{(1+\epsilon)(ab-a-1)}{\left(\frac{d^{\frac{L}{m}}}{\beta_1}\right)^{\frac{1}{(\delta_1+1)^{m-1}}}}$.

The bound for Λ in Corollary 6 is for fixed m a negative exponential function of L. For large L this bound is smaller than $L^{-(2+\delta)}$ for some $\delta > 0$. So from Roth's lemma it follows that $P(a,b,c,d)$ has for fixed m a finite number of m-cycles [9]. This proves the existence part of theorem 2 (1).

We denote by $\frac{p_n}{q_n}$ the nth convergent to $\rho \in \mathbb{R}$ with partial quotient a_n. We have the following results [8, Chapter 10]).

Lemma 7 (a) If $\dfrac{p}{q}$ is a rational approximation to ρ satisfying $|p - q\rho| < \frac{1}{2q}$, then $\frac{p}{q}$ is a convergent.
(b) $|p_n - q_n\rho| > \frac{1}{q_n+q_{n+1}} > \frac{1}{(a_{n+1}+2)q_n}$.
(c) If $\frac{p}{q}$ is a rational approximation to ρ, and if $q \leq q_n$, then $|p - q\rho| \geq |p_n - q_n\rho|$.
(d) If n is odd then $p_n - q_n\rho > 0$; if n is even then $p_n - q_n\rho < 0$.

Using Corollary 5 we derive like Crandall a lower bound for L [5].

Lemma 8 If $q_n + q_{n+1} \leq \left(\frac{\alpha(\alpha-\epsilon)}{\beta}\right)\frac{X_0-ab+a+1}{ab-a-1}\frac{1}{L}$, then $L > q_n$.

Proof Assume $L \leq q_n$. By lemma 7 (c) and (b)

$$|\Lambda| = |\alpha K - \beta L| = \alpha|K - \rho L| \geq \alpha|p_n - \rho q_n| > \frac{\alpha}{q_n + q_{n+1}} \geq \frac{\beta}{\alpha-\epsilon}\frac{ab-a-1}{X_0 - ab + a + 1}L = \epsilon\frac{\beta}{\alpha-\epsilon}L$$

Which contradicts Corollary 5.

Lemma 8 implies

Lemma 9 If $q_n + q_{n+1} \leq \left(\frac{\alpha(\alpha-\epsilon)}{\beta}\right)\frac{X_0-ab+a+1}{ab-a-1}\frac{1}{q_n}$ then $L > q_n$

Proof Assume that $L \leq q_n$. Then $\frac{1}{q_n} \leq \frac{1}{L}$ and $q_n + q_{n+1} \leq \left(\frac{\alpha(\alpha-\epsilon)}{\beta}\right)\frac{X_0-ab+a+1}{ab-a-1}\frac{1}{q_n} \leq \left(\frac{\alpha(\alpha-\epsilon)}{\beta}\right)\frac{X_0-ab+a+1}{ab-a-1}\frac{1}{L}$. Then from lemma 8 it follows that $L > q_n$ which contradicts the assumption $L \leq q_n$.

Applying lemma 9 we find

Table 1. *Lower bound for L as a function of X_0.*

$\log_{10} X_0$	$L(1,3,2,2) >$	$L(2,4,3,3) >$
3	5	2
4	17	2
5	22	9
6	127	52
7	276	52
8	276	113
9	6475	113
10	13226	2651

2.4. Application of Transcendental Number Theory

Transcendental number theory shows that linear forms in logarithms of integers cannot be too small in terms of their coefficients. The original paper of Baker gives a lower bound for a linear form in n logarithms [1]. Laurent a.o. give a bound for two logarithms and Rhin derived a sharper lower bound for the specific case $x\log 2 + y\log 3$[12], [16]. The result of Baker supplies a lower bound for Λ of $P(a,b,c,d)$. Corollary 6 supplies an upper bound for Λ as a function of L and m. For fixed m the upper bound is a negative exponential function of L which for large L is smaller than the lower bound of Baker and this latter bound is polynomial in L. So for each m the upper bound for L can be computed. This proves the computability part of theorem 2 (1).

3. Cycle Existence

For the functions $P(2,2,1,3)$ and $P(2,4,3,3)$ Rhin's lower bound is applicable. We calculated theoretical and numerical bounds for cycles up to an arbitrary chosen upper bound for m. Recall that $\alpha = \log(cd) - \log(ab)$, $\beta = \log(b) - \log(d)$.

3.1. Numerical Results for $P(2,2,1,3)$

Because of the constraint $\alpha, \beta > 0$ we analyze $P(1,3,2,2)$ which has the same trajectories. Let $X_0 = 10^6$. From Rhin we derive the following estimate with $\epsilon = \frac{1}{X_0-1}$ [16].

Lemma 10

$$|\Lambda| > e^{-13.3(1.34+\log(L))}.$$

Proof We apply Rhin's proposition on p. 160 with $u_0 = 0,\ u_1 = 2K + L,\ u_2 = -(K + L)$. Then $H = u_1 = 2K + L$ and Rhin's estimate leads to $|\Lambda| \geq [2K + L]^{-13.3}$. From Corollary 3 we have: if $\Lambda > 0$ then $0 < \alpha K - \beta L < \epsilon K$ which implies $K < \frac{\beta}{\alpha-\epsilon}L$ and if $\Lambda < 0$ then $0 < -\alpha K + \beta L < \epsilon K$ which implies $K < \frac{\beta}{\alpha}L$. So in both cases we have $K < \frac{\beta}{\alpha-\epsilon}L$ and we find $|\Lambda| \geq [2K + L]^{-13.3} > \left(\frac{\alpha+2\beta-\epsilon}{\alpha-\epsilon}L\right)^{-13.3} > e^{-13.3(1.34+\log(L))}$.

Let $x = x_3(m)$ be the solution of $e^{-13.3(1.34+\log(x))} = \frac{\beta x}{\alpha-\epsilon}\frac{(1+\epsilon)(ab-a-1)}{\left(\frac{\frac{x}{m}d}{\beta}\right)^{\frac{1}{(\delta+1)^{m-1}}}}$. Then $L \leq x_3(m)$.

Table 2. *Upper bound for L as a function of m for $P(2, 2, 1, 3)$.*

m	$L \leq$
1	126
2	1241
3	8171
4	45588
5	2.3201×10^5
10	4.2643×10^8
20	4.9668×10^{14}
50	1.1449×10^{32}
100	2.2665×10^{60}

We now apply an upper bound reduction technique. Let $x = x_1(m)$ be the solution of $e^{-13.3(1.34+\log(x))} = \frac{\alpha}{2x}$. Lemma 7 implies that if $L \geq L_1 = x_1(m)$ then $\frac{K}{L}$ must be a convergent to $\frac{\beta}{\alpha}$. The first convergents are (3, 2), (7, 5), (24, 17), (31, 22), (179, 127), (389, 276), (9126, 6475), (18641, 13226), (46408, 32927) $\cdots$. For the L-interval up to $x_3(m)$ for $m \leq 20$ we found that the maximum partial quotient is 55. Let $x = x_2(m)$ be the solution of $e^{-13.3(1.34+\log(x))} = \frac{\alpha}{57x}$. Then lemma 7 implies that $L \leq L_2 = x_2(m)$. Applying these bounds as a function of m gives an interval for L being a convergent solution of a potential m-cycle.

Table 3. *L bounds for (convergent related) length of m-cycles for $P(2, 2, 1, 3)$.*

m	$L \geq L_1$	$L \leq L_2$
1	10	16
2	122	162
3	875	1085
4	5120	6103
5	26893	31240
10	5.3270×10^7	5.8249×10^7
20	6.5093×10^{13}	6.8249×10^{13}

For $m = 1$ the lower bound 127 from lemma 8 is larger than the upper bound L_2 in this table. For $m = 2, 3, 4, 5$ there are no convergents in the interval (L_1, L_2). We checked for $L < 31240$ if corollary 5 is satisfied. This is true for (K, L) is $(389, 276), (778, 552), (957, 679), (1167, 828)$ and many more pairs with $L > 828$. We computed all trajectories for starting values $< X_0 = 10^6$ until either a cycle appeared or an apparent divergent trajectory exceeded 10^8 with $m > 10$. From these calculations we find

Corollary 11 The Collatz permutation

(a) has for $m \leq 2$ no other cycles than $(1),\ (2, 3),\ (4, 6, 9, 7, 5)$.

(b) has for $m = 3$ the cycle (44, 66, 99, 74, 111, 83, 70, 105, 79, 59) and if $x_i \geq 10^6$ could have cycles with (K, L) = (389, 276), (778,552), (957,679), (1167,828).

(c) has for $m = 4$ no cycles with $x_i < 10^6$ and if $x_i \geq 10^6$ could have cycles with (K, L) = (389, 276), (778, 552), (957, 679), (1167, 828) and 828 ¡ L ¡ 5120.

(d) has for $m = 5$ no cycles with $x_i < 10^6$ and if $x_i \geq 10^6$ could have cycles with (K, L) = (389, 276), (778, 552), (957, 679), (1167, 828) and 828 ¡ L ¡ 26893.

This proves theorem 2 (3)

3.2. Numerical Results for $P(2, 4, 3, 3)$

Let $a = 2, b = 4, c = 3, d = 3, X_0 = 10^6$. From Rhin we derive [16]

Lemma 12
$$|\Lambda| > e^{-13.3(1.77+\log(L))}.$$

Proof We apply Rhin's proposition on p. 160 with $u_0 = 0,\ u_1 = 2K + L,\ u_2 = -(3K + 2L)$. Then $H = u_2 = 3K + 2L$ and Rhin's estimate leads to $|\Lambda| \geq [3K + 2L]^{-13.3}$. From Corollary 3 we have: if $\Lambda > 0$ then $0 < \alpha K - \beta L < \epsilon K$ which implies $K < \frac{\beta}{\alpha-\epsilon}L$ and if $\Lambda < 0$ then $0 < -\alpha K + \beta L < \epsilon K$ which implies $K < \frac{\beta}{\alpha}L$. So in both cases we have $K < \frac{\beta}{\alpha-\epsilon}L$ and we find $|\Lambda| \geq [3K + 2L]^{-13.3} > \left(\frac{2\alpha+3\beta-2\epsilon}{\alpha-\epsilon}L\right)^{-13.3} > e^{-13.3(1.77+\log(L))}$.

Let $x = x_3(m)$ be the solution of $e^{-13.3(1.77+\log(x))} = \frac{\beta x}{\alpha-\epsilon}\frac{(1+\epsilon)(ab-a-1)}{\left(\frac{\frac{x}{m}d}{\beta}\right)^{\frac{1}{(\delta+1)^{m-1}}}}$. Then $L \leq x_3(m)$. We calculated the cross-over point $L = x_3(m)$ to find

Table 4. *Upper bound for L of potential m-cycle for $P(2, 4, 3, 3)$.*

m	$L \leq$
1	88
2	754
3	4422
4	22142
5	1.0150×10^5
10	1.1314×10^8
20	5.0142×10^{13}
50	6.6686×10^{29}
100	1.1640×10^{56}

We now apply an upper bound reduction technique. Let $x = x_1(m)$ be the solution of $e^{-13.3(1.77+\log(x))} = \frac{\alpha}{2x}$. Lemma 7 implies that if $L \geq L_1 = x_1(m)$ then $\frac{K}{L}$ must be a convergent to $\frac{\beta}{\alpha}$. The first convergents are (5, 2), (17, 7), (22, 9), (127, 52), (276, 113), (6475, 2651), (13226, 5415), (32927, 13481), $\cdots$. For the L-interval up to $x_3(m)$ for $m \leq 20$ we found that the maximum partial quotient is 55. Let $x = L_2 = x_2(m)$ be the solution of $e^{-13.3(1.77+\log(x))} = \frac{\alpha}{57x}$. Then lemma 7 implies that $L \leq x_2(m)$. We took as a numerical lower bound for cycles the value $X_0 = 10^6$. Applying these bounds as a function of m gives an interval for L being a convergent solution of a potential m-cycle.

Table 5. *L bounds for (convergent related) length of m-cycles for* $P(2,4,3,3)$.

m	$L \geq L_1$	$L \leq L_2$
1	9	12
2	84	107
3	517	625
4	2661	3124
5	12437	14307
10	1.4561×10^7	1.5903×10^7
20	6.676×10^{12}	7.034×10^{13}

For $m = 1$ the lower bound 52 from lemma 8 is larger than the upper bound L_2 in this table. For $m = 2, 3, 4$ there are no convergents in the interval (L_1, L_2). For $m = 5$ there is one convergent ($K = 32927, L = 13481$) in the interval (L_1, L_2). We computed all trajectories for starting values $< X_0 = 10^6$ until either a cycle appeared or an apparent divergent trajectory exceeded 10^8 with $m > 20$. From these calculations we find

Corollary 13 The permutation $P(2,4,3,3)$

(a) has for $m = 1, 2$ no other cycles than listed in the table below.

(b) has for $m = 3$ the 3-cycle listed in the table below and if $x_i \geq 10^6$ could have cycles with $55 < L < 517$.

(c) has for $m = 4$ no cycles with $x_i < 10^6$ and if $x_i \geq 10^6$ could have cycles with $55 < L < 2661$.

(d) has for $m = 5$ no cycles with $x_i < 10^6$ and if $x_i \geq 10^6$ could have cycles with $55 < L < 12437$ and with $K = 32927, L = 13481$.

Table 6. *Numerically found cycles for* $P(2,4,3,3)$.

nr	x_{min}	x_{max}	$length$	m
1	1	1	1	0
2	2	2	1	0
3	3	4	2	1
4	5	5	1	0
5	6	8	3	1
6	9	16	7	1
7	15	32	14	3
8	27	176	51	10
9	33	52	7	2
10	90	1972	93	19
11	213	700	31	7
12	645	1612	31	8

This result proves theorem 2 (5).

4. Generalizations of $P(a,b,c,d)$

4.1. Simple Generalizations

The definition of $P(a,b,c,d)$ guarantees that only elements $\equiv 0 \pmod b$ occur in decreasing subsequences in between a local maximum and a local minimum. If the residues are in arbitrary order and we consider elements with $n > \frac{ab-2}{cd-ab}$, then the worst case $abn + ab - 1 \to cdn + 1$ still results in an increasing subsequence. So there exist $(ab-a)! - 1$ functions, each being a simple generalization of $P(a,b,c,d)$ for which the developed theory and calculation of m-cycles is applicable. This proves theorem 2 (2).

The Collatz permutation has $2! - 1 = 1$ simple generalization. We computed all trajectories for starting values $< X_0 = 10^6$ until either a cycle appeared or an apparent divergent trajectory exceeded 10^8 with $m > 20$. We found the following results:

Table 7. *Cycles of the simple generalization of the Collatz permutation* $P(2,2,1,3)$.

nr	x_{min}	x_{max}	$length$	m
1	1	3	3	1
2	4	27	11	2
3	5	5	1	0
4	10	15	2	1
5	14	21	3	1
6	16	261	34	8
7	20	45	5	1
8	220	555	12	4

This result cannot be obtained through the SdW approach. This result and the potential cycles in Corollary 11 prove theorem 2 (4).

For the second example $P(2,4,3,3)$ there are $6! - 1 = 719$ simple generalizations. We computed for the first (ordered by permutation number) 150 simple generalizations all trajectories for starting values $< X_0 = 10^5$ until either a cycle appeared or an apparent divergent trajectory exceeded 10^8 with $m > 20$. See the section Closing remarks 3. There is a "large" variety for the parameters of the cycle structure per function: number of cycles from 9 to 23, maximal cycle length from 11 to 1291, maximal element from 124 to 30 010 496. Potential cycles in Corollary 13 could exist for all these simple generalizations too.

4.2. Extended Generalizations

If also the first residue class changes, there is no fixed ratio $\frac{cd}{ab}$ for the increasing subsequences. Consequently the developed theory for cycle existence cannot simply be applied. See the section Extending the class of permutations.

For the Collatz permutation, there are 4 extended generalizations. We computed trajectories with a starting value

$< X_0 = 10^6$ until a cycle appeared or an apparent divergent trajectory exceeded 10^8. We found the following results. The Collatz permutation and its simple generalization are included in the table below for comparison.

Table 8. *Cycles of the extended generalizations of the Collatz permutation* $P(2, 2, 1, 3)$.

$f\ nr$	$\#\ of\ cycles$	$(x_{min}, x_{max}, length, m)$
1	4	(1, 1, 1, 0), (2, 3, 2, 1), (4, 9, 5, 2), (44, 111, 12, 4)
2	8	(1, 3, 3, 1), (4, 27, 11, 2), (5, 5, 1, 0), (10, 15, 2, 1), (14, 21, 3, 1), (16, 261, 34, 8), (20, 45, 5, 1), (220, 555, 12, 4)
3	3	(0, 1, 2, 1), (2, 7, 5, 1), (42, 109, 12, 4)
4	2	(0, 5, 5, 1), (40, 107, 12, 4)
5	6	(0, 7, 6, 1), (5, 5, 1, 0), (12, 19, 2, 1), (26, 61, 5, 1), (140, 5215, 94, 26), (306, 775, 12, 4)
6	7	(1, 1, 1, 0), (0, 23, 11, 2), (6, 11, 1, 1), (10, 17, 3, 1), (12, 257, 34, 8), (16, 41, 5, 1), (216, 551, 12, 4)

For the second example $P(2, 4, 3, 3)$ there are 4320 extended generalizations. We computed trajectories with a starting value $< X_0 = 10^5$ until a cycle appeared or an apparent divergent trajectory exceeded 10^8 for the first 150 extended generalizations. We found a "large" variety in the parameters for the cycle structure per function: number of cycles from 3 to 10, maximal cycle length from 2 to 118, maximal element from 14 to 214 649.

5. Divergent Trajectories

The conjecture for the Collatz sequence is that all trajectories end in the cycle $(1, 2)$. For the Collatz permutation sequences the analogous conjecture is that all trajectories are either one of the 4 known cycles or divergent. Numerical evidence and transcendental number theory lead to the common conjecture of a finite number of m-cycles for given m. For the Collatz sequence this implies that divergent trajectories can exist, for the Collatz permutation sequences this implies that divergent trajectories must exist. Indeed we found (no proof) that the number of apparent divergent trajectories increases linear with the range of starting values X_0. For $P(1, 3, 2, 2)$ the ratio is $.05X_0$ and for $P(2, 4, 3, 3)$ the ratio is $.12X_0$ approximately.

5.1. Distribution of Elements in Divergent Trajectories of $P(a, b, c, d)$

For the numbers in a divergent trajectory of the Collatz permutation sequence Guy mentions an intriguing paradox [7]. Assuming that numbers in such a trajectory are uniformly distributed (mod 2) and (mod 3), then from left to right the average multiplication factor is $\sqrt{\frac{3}{2}\frac{3}{4}} \approx 1.06066$ and from right to left the average multiplication factor is $\sqrt[3]{\frac{2}{3}(\frac{4}{3})^2} \approx 1.05827$. These factors should be reciprocal.

A divergent trajectory has two branches: a left branch downwards to the minimal element and a right branch from the minimal element upwards. Numerical evidence suggests that for the Collatz permutation sequence the distribution of elements in a branch of a divergent trajectory is uniform either (mod b) or (mod d). The next lemma proves that for $P(a, b, c, d)$ a uniform distribution of elements (mod b) and (mod d) in a branch of a divergent trajectory cannot exist.

Lemma 14 Assume that $P(a, b, c, d)$ has divergent trajectories. Then in a branch at least one of the distributions of elements (mod b) and (mod d) is non-uniform.

Proof From left to right in the decreasing part, let $P(x \equiv 0 \pmod{b}) = \alpha$ and let $P(x \equiv 0 \pmod{d}) = \beta$. From left to right we have in the decreasing part of the trajectory a multiplication factor $f_1 = (\frac{d}{b})^\alpha(\frac{cd}{ab})^{1-\alpha} = (\frac{c}{a})^{1-\alpha}\frac{d}{b}$. From right to left we have in the increasing part a multiplication factor $f_2 = (\frac{b}{d})^\beta(\frac{ab}{cd})^{1-\beta} = (\frac{a}{c})^{1-\beta}\frac{b}{d}$. By definition $f_1.f_2 = 1$ from which follows $\alpha = \beta$. A uniform distribution (mod b) respectively (mod d) implies $\alpha = \frac{1}{b}$ and $\beta = \frac{1}{d}$. So at least one of the distributions is non-uniform. A similar reasoning holds for the from left to right increasing part of the divergent trajectory.

5.2. Theoretical Conditions for Divergent Trajectories

Let $d > 2$. Let $x_i,\ r_0, \ldots, r_{d-1} \in \mathbb{Z}, m_0, \ldots, m_{d-1} \in \mathbb{N}$ with $r_i \equiv im_i \pmod{d}$. Matthews introduced the generalized Collatz function $T:\ \mathbb{Z} \to \mathbb{Z}$ [13].

$$x_{i+1} = T(x_i) = \frac{m_i x_i - r_i}{d} \text{ if } x_i \equiv i \pmod{d}$$

Matthews conjectures (i) if $\prod_{i=0}^{d-1} m_i < d^d$ then all trajectories end in a cycle (ii) if $\prod_{i=0}^{d-1} m_i > d^d$ then almost all trajectories are divergent. Levy (his theorem 18) proves that if $\sum_{i=1}^{k} \frac{1}{m_i} \le 1$ then $\prod_{i=0}^{d-1} m_i \ge d^d$ [11]. The equality holds if and only if $m_0 = \cdots = m_{d-1} = d$.

Let C be a set of $k \ge 2$ ordered pairs $\{(a_i, b_i), i = 1, \cdots, k\}$ with $a_i \in \mathbb{N}$ and $b_i \in \mathbb{N}_0$. We call C a complete coverage set (CC set) if for every x $\in \mathbb{N}$ there exists exactly one i for which $x \equiv b_i \pmod{a_i}$. Examples are $\{(3, 0), (3, 1), (3, 2)\}$, $\{(2, 0), (4, 1), (8, 3), (16, 7), (16, 15)\}$. See the section Closing remarks 4.

Lemma 15 Let $\{(a_i, b_i), i = 1, \cdots, k\}$ be a CC-set. Then $\sum_{i=1}^{k} \frac{1}{a_i} = 1$. *Proof* Let $A = 1cma_i$ and let $\gamma_i = \frac{A}{a_i}$.

The set $\{(A, b_1), (A, b_1 + a_1), \ldots, (A, b_1 + (\gamma_1 - 1)a_1), \ldots, (A, b_k), (A, b_k + a_k), \ldots, (A, b_k + (\gamma_k - 1)a_k)\}$ is a CC-set. Suppose that $x \equiv b_i \pmod{a_i}$. Then $x = b_i + m.a_i$. Let $m = r + s\gamma_i$. Then $0 \le r < \gamma_i$ and $x = b_i + ra_i + s\gamma_i a_i$. Hence $x \equiv b_i + ra_i \pmod{\gamma_i a_i} \equiv b_i + ra_i \pmod{A}$. Now suppose that $x \equiv b_i + r_i a_i \pmod{\gamma_i a_i}$ and $x \equiv b_j + r_j a_j \pmod{\gamma_j a_j}$. Then $x \equiv b_i \pmod{a_i}$ and $x \equiv b_j \pmod{a_j}$ which contradicts that $\{(a_i, b_i), i = 1, \cdots, k\}$ is a CC-set. Hence the set $\{(A, b_1), (A, b_1 + a_1), \ldots, (A, b_1 + (\gamma_1 - 1)a_1), \ldots, (A, b_k), (A, b_k + a_k), \ldots, (A, b_k + (\gamma_k - 1)a_k)\}$ is a CC-set.

This CC-set has $\sum_{i=1}^{k} \gamma_i$ elements. Let (a_j^*, b_j^*) be an element of this CC-set. Because of the constant $a_j^* = A$ the set $\{b_j^*\}$ is a complete residue system (mod A). Hence the set $\{b_j^*\}$ has A elements. We find that $\sum_{i=1}^{k} \gamma_i = A$ from which we conclude $\sum_{i=1}^{k} \frac{1}{a_i} = 1$.

A permutation can be written as a generalized Collatz function. Permutations are exactly those generalized Collatz functions for which Matthews does not conjecture convergence or divergence of trajectories.

5.3. Permutations with $d \equiv 0$ (Mod b)

If $b|d$ then a special structure of trajectories appears. Let $b = v.d$ with $v > 1$ and let $\gcd(b-1, d-1) = w \ge 1$. Then $a(b-1) = a.w.(b^*-1) = c(d-1) = c.w.(d^*-1)$. Consequently $a = u(d^*-1)$ and $c = u(b^*-1)$ and we find the permutation

$$\left.\begin{array}{c} v.d.n \\ u(d^*-1)v.dn + s_i \ (s_i \not\equiv 0 \pmod{v.d}) \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} d.n \\ u(b^*-1)d.n + r_i \ (r_i \not\equiv 0 \pmod{d}) \end{array}\right.$$

Recall that $b > d$ implies $ab < cd$. Let $x_i \ge v.d$. If $x_i \not\equiv 0 \pmod{v.d}$ then $x_{i+1} > x_i$. Hence $x_{i+1} \not\equiv 0 \pmod{d}$ and $x_{i+1} \not\equiv 0 \pmod{v.d}$. So x_i is the start of a divergent trajectory. If $x_i \equiv 0 \pmod{v.d}$ then $x_{i+1} < x_i$ and as long as $x_{i+1} \equiv 0 \pmod{v.d}$ the decreasing trajectory continues. Apart from trivial 1-cycles for $0 \le x_i < v.d$ this permutation has infinitely many divergent trajectories. Consider as an example $P(2, 6, 5, 3)$ with $u = 1$

$$\left.\begin{array}{c} 6n \\ 12n + 1, 2, 3, 4, 5, 7, 8, 9, 10, 11 \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} 3n \\ 15n + 1, 2, 4, 5, 7, 8, 10, 11, 13, 14 \end{array}\right.$$

For this permutation there are three trivial 1-cycles (0), (1), (2) and an infinite number of divergent trajectories with minima $3, 9, 15, \cdots$. This proves theorem 2 (6). For simple generalizations the same analysis is applicable, however for $n \le \frac{ab-2}{cd-ab}$ there can also exist m-cycles with $m > 1$. For extended generalizations the existence of divergent trajectories cannot be proved. This example shows the limitations of our approach that can only prove that for each m a finite number of m-cycles exists.

6. Extending the Class of Permutations

The class of permutations in lemma 1 with $a(b-1) = c(d-1)$ can be extended, such that our method of finding cycles is partly applicable.

Lemma 16 Let $a, b, c, d \in \mathbb{N}_0$, satisfying $a > 0,\ c > 0,\ b > 1,\ d > 1$, $\gcd(a, c) = \gcd(b, d) = 1$. Let $1 \le f_a | a, 1 \le f_c | c$ and let $N = a(b-1) + f_a = c(d-1) + f_c$. Let $R = \{r_i, i = 1 \cdots N\}$ be the set of different residues (mod ab) with $r_i \not\equiv 0 \pmod{b}$ and let $S = \{s_i, i = 1 \cdots N\}$ be the set of different residues (mod cd) with $s_i \not\equiv 0 \pmod{d}$. If $1 \le f_a \le a,\ 1 \le f_c \le c$ then the function f defined by

$$\left.\begin{array}{c} f_a.bn \\ f_a.bn + b \\ \cdots \\ f_a.bn + (f_a - 1)b \\ abn + r_i \ \ r_i \not\equiv 0 \pmod{b} \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} f_c.dn \\ f_c.dn + d \\ \cdots \\ f_c.dn + (f_c - 1)d \\ cdn + s_i \ \ s_i \not\equiv 0 \pmod{d} \end{array}\right.$$

is a permutation.

Proof We distinguish five cases

1. The case $f_a = f_c = 1$. Then a, b, c, d satisfy the conditions of lemma 1.
2. The case $f_a = 1, f_c = c > 1$. Now $ab - a + 1 = cd$. Since $ab \neq cd$ it follows that $a \neq 1$ and we find the function

$$\left.\begin{array}{c} bn \\ ab.n + r_i \;\; r_i \not\equiv 0 \pmod b \end{array}\right\} \leftrightarrow cd.n + r_j \;\; r_j \in \{0, 1, \cdots, cd-1\}$$

This function is invariant under the substitution $c \to c* = 1,\ d \to d* = cd,\ f_c \to f_{c*} = 1$. $P(a, b, c*, d*)$ satisfies lemma 1.

3. The case $f_a = a > 1, f_c = 1$. After the transformation $a \to c,\ b \to d$ this case is identical to the case $f_a = 1, f_c = c > 1$.
4. The case $f_a = a, f_c = c$. Now $ab = cd$ and f becomes the trivial function $n \leftrightarrow n$.
5. The case $1 < f_a < a, 1 < f_c < c$. Now the conditions of lemma 1 are not satisfied. For $a = 10, b = 8, f_a = 5, c = 9, d = 9, f_c = 3$ we find the permutation

$$\left.\begin{array}{c} 40n, 40n+8, 40n+16, 40n+24, 40n+32 \\ 80n+1\cdots 7, \cdots, 80n+73\cdots 79 \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} 27n, 27n+9, 27n+18 \\ 81n+1\cdots 8, \ldots, 81n+73\cdots 80 \end{array}\right.$$

For permutations of lemma 16 our method to find cycles and divergent trajectories must be adapted in a nontrivial way, since increasing subsequences do not have a fixed ratio $\frac{cd}{ab}$ anymore. In general a linear form in three or more logarithms appears. Then Baker's result to find an upper bound for the cycle length is still applicable, however reducing the upper bound becomes nontrivial.

Let $a_i, b_i, c_i, d_i | i = 1, \cdots N \in \mathbb{N}_0$. Consider the function $f : \mathbb{N}_0 \to \mathbb{N}_0$ defined by

$$\left.\begin{array}{c} a_1 n + b_1 \\ a_2 n + b_2 \\ \cdots \\ a_N n + b_N \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} c_1 n + d_1 \\ c_2 n + d_2 \\ \cdots \\ c_N n + d_N \end{array}\right.$$

A necessary and sufficient condition for f being a permutation is that the sets $\{(a_i, b_i), i = 1 \cdots N\}$ and $\{(c_i, d_i), i = 1 \ldots N\}$ are CC-sets. Venturini chooses $a_i = pq, b_i = i$ and proves a similar result for permutations from $\mathbb{Z}$ onto $\mathbb{Z}$ [20].

7. Closing Remarks

1. There exist permutation functions outside the class of this paper. Let $P(n)$ be the n^{th} prime number and let $C(n)$ be the n^{th} composite number. Then the function

$$\begin{array}{rcl} 1 & \leftrightarrow & 1 \\ 2n & \leftrightarrow & P(n) \\ 2n+1 & \leftrightarrow & C(n) \end{array}$$

   is a permutation with (at least) the cycles (1), (2), (3, 4), (5, 6), (7, 8), (9), (10, 11), (12, 13), (14, 17, 15), a cycle $(18, \cdots)$ with length 22, a cycle $(62, \cdots)$ with length 3, a cycle $(84, \cdots)$ with length 3, and a cycle $(92, \cdots)$ with length 6. There exist numerically "divergent", however theoretically unknown trajectories.
2. The Collatz function $g(2n) = n,\ g(2n+1) = 3n+2$ is well known [10]. The Collatz function is not a permutation. The Collatz conjecture states that for all $n > 0$ finally the cycle $(1, 2)$ appears. Collatz also introduced the Collatz permutation and used the slightly different notation $f(2n) = 3n, f(4n+1) = 3n+1, f(4n-1) = 3n-1$. Known cycles and conjectured divergent trajectories are the same, however now if $n \in \mathbb{N}_0$ also the cycle (-1) appears. The Collatz permutation has different names in the literature. Lagarias and Tavares call it *Collatz's original problem* and Guy calls it *probably the inverse of Collatz's original problem* [10, 18], [7].
3. Cycle computation time with Mathematica can be quite large on a standard notebook with dual processor Intel(R) Celeron(R) CPU 1007U, 1.50GHz. Computation of the results for $P(1, 3, 2, 2)$ took 20 hours and computation of the results for $P(2, 4, 3, 3)$ took 24 hours (See subsection Extended generalizations). When a larger range for starting values is used then lemma 9 supplies a larger lower bound for the cycle length.
4. A CC set is a concept similar (not identical) to a covering system of congruences of Erdös : a finite set of k residue classes $b_i \pmod{a_i}$ with $2 \le a_i \le a_{i+1} \le a_k$ with the property that $\forall x \in \mathbb{N}$ there exists at least one i for which $x \equiv b_i \pmod{a_i}$ [4]. The difference is the uniqueness of the residue class for which $x \equiv b_i \pmod{a_i}$. It is also similar (not identical) to an exact covering system (See problem F14 in [7]). The difference is the existence of infinite CC sets, e.g., $\{(3, 0), (3, 1), (3.2^{k+1}, 3.2^k - 1) | k \ge 0\}$. Covering systems are extensively analyzed in the literature [2, 4, 14, 15].

Acknowledgements

The author expresses his gratitude to Aad Dijksma and to the anonymous reviewers for many useful remarks on the manuscript.

References

[1] A. Baker, *Linear forms in the logarithms of algebraic numbers IV, Mathematika*, 15 [1968], pp. 204-216.

[2] M. A. Berger, A. Felzenbaum, A. S. Fraenkel, *On infinite and finite covering systems*, Amer. Math. Monthly 98 (1991), pp. 739-742.

[3] T. Brox, *Collatz cycles with few descents*, Acta Arithmetica 92 (2000), pp. 181-88.

[4] S. L. G. Choi, *Covering the set of integers by congruence classes of distinct moduli, Math. Comp.*, 25 [1971], pp. 885-895.

[5] R. E. Crandall, *On the "$3x+1$" problem, Math. Comp.* 32 [1978], pp. 1281-1292.

[6] J. L. Davidson, *Some comments on an Iteration problem*, Proceedings of the Sixty Manitoba Conference on Numerical Mathematics (Univ. Manitoba, Winnipeg, Manitoba, 1976 pp. 155-159, Congressus Numerantium XVII, Utilitas Math., Winnipeg, MN, 1977.

[7] R. K. Guy, *Unsolved Problems in Number Theory, second edition* (1994) p. 219.

[8] G. H. Hardy and E. M. Wright, "An Introduction to the Theory of Numbers", 5th ed., Oxford University Press, 1981.

[9] M. Hindry and J. H. Silverman, *Diophantine Geometry, An Introduction*, GTM 201, Springer Verlag, Berlin, 2000.

[10] J. C. Lagarias, *The 3x + 1 problem and its generalizations*, Amer. Math. Monthly 85 (1985).

[11] D. Levy, *Injectivity and surjectivity of Collatz functions, Discrete Mathematics* 285 [2004] pp. 191-199.

[12] Michel Laurent, Maurice Mignotte and Yuri Nesterenko, "Formes linéaires en deux logarithmes et déterminants d'interpolation", J. Number Th. 55 [1995], pp. 285-321.

[13] K. $\dot{R}$. Matthews, *Generalized $3x+1$ mappings: Markov chains and ergodic theory*, The Ultimate Challenge: The 3x + 1 Problem, (J. C. Lagarias, Ed.), pp. 79-104, Amer. Math. Society, Providence RI 2010.

[14] S. Porubsky and J. Schönheim, *The covering systems of Paul Erdös: Past, present and future*, Paul Erdös and his mathematics, Vol. I, Janos Bolyai and his mathematics, vol. I, Janos Bolyai Math. Soc., Budapest 2002, pp. 581-627.

[15] S. Porubsky, *Results and problems on covering systems of residue classes*, Mitteilungen aus dem Math. Sem. Giessen, Heft 150, Univ. Giessen 1981.

[16] G. Rhin, "Approximants de Padé et mesures effectives d'irrationalité", *Progress in Mathematics* 71 [1987], pp. 155-164.

[17] R. P. Steiner, "A Theorem on the Syracuse Problem", Proc. *7th Manitoba Conference on Numerical Mathematics 1977*, Winnipeg 1978, pp. 553-559.

[18] M. Tavares, "A remark about the density of the orbits of the Collatz permutation", *Discrete Mathematics 313 (2013)*, pp. 2139-2145.

[19] J. L. Simons and B. M. M. de Weger, *Theoretical and computational bounds for m-cycles of the $3n+1$ problem*. Acta arithmetica 117.1 [2005] pp. 51-70, see for latest results http://www.win.tue.nl/bdeweger/research.html.

[20] G. Venturini, *On a generalization of the 3x + 1 problem*, Adv. Appl. Math. 19 [1997], pp. 295-305.