International Journal of Theoretical and Applied Mathematics
2023; 9(2): 10-13
http://www.sciencepublishinggroup.com/j/ijtam
doi: 10.11648/j.ijtam.20230902.11
ISSN: 2575-5072 (Print); ISSN: 2575-5080 (Online)

Some Fixed Point Theorems for Countably Condensing

Abdul-Majeed Al-izeri[1,*], Ahmed Al-Haysah[2]

[1]Departement Mathematics (CNRS, LMBP), Clermont Auvergne University, Clermont-Ferrand, France
[2]Departement Mathematics, Albaydha University, Albaydha, Yemen

Email address:
Abdul_Majeed.Al_izeri@uca.fr (Abdul-Majeed Al-izeri), alhaysah2010@gmail.com (Ahmed Al-Haysah)
[*]Corresponding author

To cite this article:
Abdul-Majeed Al-izeri, Ahmed Al-Haysah. Some Fixed Point Theorems for Countably Condensing. *International Journal of Theoretical and Applied Mathematics*. Vol. 9, No. 2, 2023, pp. 10-13. doi: 10.11648/j.ijtam.20230902.11

Received: July 25, 2023; **Accepted:** September 11, 2023; **Published:** September 24, 2023

Abstract: Our aim in this article is to establish the principles results of a fixed point theorems for multivalued mappings of Krasnoselskii type setting in general classes Mönch's type. We seek to do that, we introduce and recall some theorems to aid our study. The beginning of this work has been introduced some properties of the measure of weak noncompactness under the weak topology and the definitions of countably condensing operators. We have shown that the operator $H(S)$ is relatively weakly compact by using some properties of weak topology. We investigate that all hypotheses guarantee that the operator $(B+H)(S)$ is relatively weakly compact and than simply to apply Himmelberg's theorem in Banach spaces. We extended two fixed point theorems for weakly sequentially upper semicontinuous mappings subjected the perturbation map satisfies the Mönch's type and we obtain our results in the second theorem with a less restrictive hypothesis. Using abstract measures of weak noncompactness, these results are applied to derive some fixed point theorems for a weakly sequentially upper semicontinuous countably μ-condensing multivalued mappins.

Keywords: Fixed Point Theorems, Weakly Sequentially Continuous Multivalued Maps, Measure of Noncompactness, Countably μ-condensing Perturbation

1. Introduction

In the last years, fixed point theory for single and multivalued mappings, under the weak topology, has known many developement. In particular, under various conditions, several works were dedicated to derive theorems of Schauder's type (Himmelberg's theorem [12]), Sadovskii's type, Krasnosel'skii's type (see, for example, [1–5, 7, 8, 10–18, 20, 21] and the references therein).

Our objectif in this work is to establish some fixed point results of multivalued version of the Himmelberg's theorem in Banach spaces for weakly sequentially continuous maps. This work is motivated by some results obtained in [2] essentially, Theorem 2.2.

2. Preliminaries

Let X be a Banach space. we denote by
$\mathsf{K}(X)=\{S\subset X : S \text{ is nonempty}\}$,
$\mathsf{K}_{bd}(X)=\{S\subset X : S \text{ is nonempty and bounded}\}$,
$\mathsf{K}_{cv}(X)=\{S\subset X : S \text{ is nonempty and convex}\}$,
$\mathsf{K}_{cl,cv}(X)=\{S\subset X : S \text{ is nonempty, convex and closed}\}$
and
$\mathsf{W}(X)=\{S\subset X : S \text{ is nonempty weakly compact}\}$.

We introduce the notions in Banach space for the measure of weak noncompactness.

Definition 2.1. Let $\mu : \mathsf{K}_{bd}(X) \to [0,+\infty[$ be said the measure of weak noncompactness on X if μ satisfies the following assertions:

(1) The family $\ker\mu := \{S\in \mathsf{K}_{bd}(X) : \mu(S)=0\}$ is nonempty and elso $\ker\mu$ is contained in all set of relatively weakly compact subsets of X, ß
(2) The monotonicIty: $S_1\subset S_2 \Rightarrow \mu(S_1)\le\mu(S_2)$,
(3) Invariance for the closed convex hull: $\mu(\overline{co}(S))=\mu(S)$ where $\overline{co}$ denotes the closed convex hull of S,
(4) The homogeneity: $\mu(\lambda S)=\lambda\mu(S)\ \ \forall\lambda\in\mathbb{R}^+$.

The family $\ker\mu$ is given on first assertion and it is called the kernel of the measure μ. It should be noticed that the inclusions $S \subseteq \overline{S^w} \subseteq \overline{co}(S)$ together with the item (3) of Definition 2.1 imply

(5) $\mu(\overline{S^w}) = \mu(S)$.

Note that if $\mu(\cdot)$ is the full measure of weak noncompactness to having the maximum property, then it is non-singular, that is:

(6) $\mu(S \cup \{x\}) = \mu(S)$, for all $S \in \mathsf{K}_{bd}(X)$ and $x \in X$.

Before going further we recall the following definitions.

Definition 2.2 Let (X, d) and (Y, d) be two metric spaces and let $G : X \to \mathsf{K}_{cl,cv}(Y)$ be a multivalued map. ß • G is called weakly upper semicontinuous (w.u.s.c. for short) if G is upper semi-continuous with respect to the weak topologies of X and Y.ß • F is called weakly sequentially upper semicontinuous (w.s.u.s.c. for short) if for each weakly closed set F of Y, $G^{-1}(F)$ is weakly sequentially closed. □

Definition 2.3 Let $\mu(\cdot)$ a measure of weak noncompactness on X and $B : S \to \mathsf{K}(X)$ a multivalued map and S is a closed, convex, nonempty subset of a Hausdorff locally convex linear topological space X. We say that B is condensing with respect to the measure $\mu(\cdot)$ if

(a) $B(M)$ is bounded, ß

(b) $\mu(B(J)) < \mu(J)$, for all bounded subset J of S with $\mu(S) > 0$. □

Now, we recall the following results.

Theorem 2.1 [16] Let S be a weakly compact subset on Banach space X. If $B : \Omega \to \mathsf{K}_{cl,cv}(S)$ is w.s.u.s.c. multivalued map, then B is w.u.s.c. multivalued map.

Theorem 2.2 [2] Let S be a convex nonempty, closed subset on Banach space X. Let $B : S \to \mathsf{K}_{cl,cv}(S)$ be a w.s.u.s.c. multivalued map and $B(S)$ is relatively weakly compact. Then there exists $z \in S$ such that $z \in B(z)$.

3. Main Results

The goal of the following results is to improve some fixed point results of Krasnosel'skii theorems for a w.s.u.s.c. multivalued map subjected the perturbation H satisfies the Mönch's type.

Theorem 3.1 Suppose that $B : S \to \mathsf{K}_{cl,cv}(S)$ and $H : S \to \mathsf{K}_{cl,cv}(S)$ be two w.s.u.s.c. multivalued maps. Assume that

(a) $B(S)$ is relatively weakly compact;

(b) There exists $z_0 \in S$ satisfies

$$\begin{cases} M \subseteq S,\ \overline{M}^w \subseteq \overline{co}\big(\{z_0\} \cup H(\overline{M}^w)\big) \text{ so, there exists} \\ \text{a countable subset } F \text{ of } M \text{ with } \overline{M}^w = \overline{F}^w \\ \Longrightarrow \overline{M}^w \text{ is weakly compact;} \end{cases}$$

(c) H maps weakly compact sets into itself;

(d) For all $z \in S$, $B(z) \cap H(z) \neq \emptyset$,

(e) $H(S) + B(S) \subset S$.

Thus, there exists $x \in S$ such that $x \in H(z) + B(z)$. *Proof* Let P_m be defined by

$$P_0 = \{z_0\},\ P_m = co\big(\{z_0\} \cup H(P_{m-1})\big) \text{ for} m = 1, 2, \cdots, \text{ and } P = \bigcup_{m=0}^{\infty} P_m.$$

Assume that P_n be a relatively weakly compact for each $n \in \{1, 2, \cdots\}$. We note that according to the Theorem 2.1, $B : \overline{P_n^w} \to \mathsf{K}_{cl,cv}(S)$ is weakly semicontinuous, and so [6, p.464] ensures that $B\big(\overline{P_n^w}\big)$ is weakly compact. Theorem of Krein-Smulian [9, p.82] ensures that P_{n+1} is relatively weakly compact. by (b) there exists a sequence of countable sets $\{M_n\}_0^\infty$ with $\overline{C_m^w} = \overline{P_m^w}$ for $m = 0, 1, \cdots$. Let

$$P = \bigcup_{m=0}^{\infty} P_m \quad \text{and} \quad M = \bigcup_{m=0}^{\infty} M_m.$$

Thus, P is also convex, since $P_{m-1} \subseteq P_m$ for $m = 1, 2, \cdots$. We deduce that

$$P = co\big(\{z_0\} \cup H(P)\big). \tag{1}$$

Thus, [19, p.66] and hypothesis (b) guarantee that

$$\overline{P}(= \overline{P^w}) = \overline{co}\big(\{z_0\} \cup H(P)\big) \quad \text{and} \quad \overline{P^w} = \overline{M^w} \tag{2}$$

Then, Eqs. (1) and (2) guarantees that

$$\overline{M^w} = \overline{P^w} = \overline{co}\big(\{z_0\} \cup H(P)\big) \subseteq \overline{co}\big(\{z_0\} \cup H(\overline{P^w})\big) = \overline{co}\big(\{z_0\} \cup H(\overline{M^w})\big).$$

Now, hypothesis (b) satisfying that $\overline{M}^w$ (and so $\overline{P}^w$) is also weakly compact. This yields that $H(\overline{P}^w) \subset \overline{P}^w$ and therefore $H(\overline{P}^w)$ is relatively weakly compact.

Next, set $\mathcal{R} = F(M) \cap B(\overline{D^w})$. It is clear that $\mathcal{R}$ is nonempty set (using hypothesis (d)). Since $H(\overline{P^w})$ is relatively weakly compact, we deduce that $\mathcal{R}$ is relatively weakly compact.

On the other hand, because $\overline{\mathcal{R}} \subset S$, we have $B(\overline{\mathcal{R}}) \subset B(S)$ and by using hypothesis (a) shows $B(\overline{\mathcal{R}})$ is relatively weakly compact.

According to the steps in the above, $(B + H)(\overline{\mathcal{R}})$ is also relatively weakly compact. Applying Theorem 2.2, we conclude there exists $x \in \overline{\mathcal{R}}$ (and $x \in S$) such that $x \in (B + H)(x)$ which concludes the proof of Theorem.

Let S be a subset of the space X and let $H : S \to \mathsf{K}_{cl,cv}(S)$ be a w.s.u.s.c. multivalued map, in the following theorem, we replace the hypothesis (b) of Theorem 3.1 with a less restrictive hypothesis; a countable set $M \subseteq Q$, there exist $z_0 \in S$ such that $\overline{M}^w = \overline{co}\big(\{z_0\} \cup H(M)\big)$ implies that M is also relatively weakly compact.

Theorem 3.2 Suppose that $B : S \to \mathsf{K}_{cl,cv}(S)$ and $H : M \to \mathsf{K}_{cl,cv}(S)$ be two w.s.u.s.c. multivalued maps satisfying the conditions (a), (c), (d) and (e) of Theorem 3.1. In addition assume that.

(A) There exists $z_0 \in S$ such that

$$\begin{cases} M \subseteq S \text{ is countable and also } \overline{M}^w = \overline{co}\big(\{z_0\} \cup H(M)\big) \\ \text{satisfying } M \text{ is relatively weakly compact,} \end{cases}$$

(B) $H(\overline{F^w}) \subseteq \overline{H(F)^w}$ for each subset F of S are satisfied. Then there exists $x \in S$ such that

$$x \in H(x) + B(x).$$

Proof Let P_m defines as the same steps in Theorem 3.1, we have that P_m is relatively weakly compact for all $m = 0, 1, \cdots$. Assume that M_m, P and M be as the last theorem, we obtain

$$\overline{P}(=\overline{P^w}) = \overline{co}\big(\{z_0\} \cup H(P)\big) \quad \text{and} \quad \overline{P^w} = \overline{M^w}. \tag{3}$$

Now, condition (B) implies that

$$\{z_0\} \cup H(P) \subseteq \{z_0\} \cup H(\overline{P^w}) \subseteq \overline{\{z_0\} \cup H(P)^w} \subseteq \overline{co}\big(\{z_0\} \cup H(P)\big).$$

thus,

$$\overline{co}\big(\{z_0\} \cup H(P)\big) = \overline{co}\big(\{z_0\} \cup H(\overline{P^w})\big). \tag{4}$$

Then, Eqs. (3) and (4) imply that

$$\overline{M^w} = \overline{P^w} = \overline{co}\big(\{z_0\} \cup H(P)\big) = \overline{co}\big(\{z_0\} \cup H(\overline{P^w})\big) = \overline{co}\big(\{z_0\} \cup H(\overline{M^w})\big) = \overline{co}\big(\{z_0\} \cup H(M)\big).$$

Now, hypothesis (A) implies that $\overline{M}^w$ $(\overline{P}^w)$ is weakly compact. This yields that $H(\overline{M}^w) \subset \overline{M}^w$ and further $H(\overline{M}^w)$ is relatively weakly compact.

Next, put $\mathcal{R} = B(S) \cap H(\overline{M}^w)$. Note that, applying hypothesis (d), we get $\mathcal{R}$ is nonempty and $(B + H)(\overline{\mathcal{R}})$ is relatively weakly compact. Now arguing the same steps in the end of the proof of Theorem 3.1, we deduce that there exists $x \in \overline{\mathcal{R}}$ (and so $x \in S$) such that $x \in (B + H)(x)$ which conclude the proof.

Remark 3.1 The conditions (b), (A) and (B) in Theorems 3.1 and 3.2 introduced by Donal O'Regan [16], to prove a fixed set results for w.s.u.s.c. multivalued maps, our objective of this work , we use these conditions to show that the perturbation $B(M)$ is also relatively weakly compact in Banach space X with $S \subset X$.

Let S be a subset of the space X and let μ be the measure of weak noncompactness on X. Assume that $B : S \to \mathsf{K}_{cl,cv}(S)$ is a w.s.u.s.c. multivalued map. F is said to be countably μ-condensing if $\mu\big(B(S)\big) < \infty$ and $\mu(B(M)) < \mu(M)$ for all countable bounded subset M of S with $\mu(M) > 0$.

Definition 3.1 Assume S be a subset convex of X and let $B : S \to \mathsf{K}(X)$ be a given map. We say that $F \subset S$ is a Mönch-set for B if there exists $z_0 \in S$ such that $F^w = co\big(\{z_0\} \cup B(F)\big)$ and there exists a countable set $M^w \subset F^w$ with $\overline{F^w} = \overline{M^w}$.

Corollary 3.1 Assume hat S is a closed convex, nonempty subset of X, $z_0 \in S$ and $B : S \to \mathsf{K}_{cl,cv}(S)$. Let $H : S \to \mathsf{K}_{cl,cv}(S)$ be two w.s.u.s.c. multivalued maps satisfying the assumptions of Theorem 3.2. If H is countably μ-condensing, then there exists $z \in S$ such that $z \in H(z) + B(z)$.

Proof We start to prove that H satisfies assumption (A) of Theorem 3.2. We fix y_0 in S and we assume that M^w be a Mönch-set for the map H contained also in S, that is, $\overline{M^w} = \overline{co}\big(\{y_0\} \cup H(M)\big)$, we have

$$\overline{M^w} = \overline{F^w} \tag{5}$$

where F^w is also a countable subset of M^w. Since $F^w \subset \overline{co}(\{y_0\} \cup H(M))$, for each point of F^w, we can be written as a finite combination of points belonging to the M is relatively weakly compact, we set $\{z_0\} \cup H(M)$. Further, there exists a countable set $O \subset M^w$ such that

$$F^w \subset \overline{co}(\{z_0\} \cup H(O)). \tag{6}$$

We claim that the measure of F is equal zero, that is $\mu(F^w) = 0$. in fact, the use of (6), H is countably μ-condensing and also the properties of μ, we have

$$\mu(F^w) \leq \mu(co(\{z_0\} \cup H(O))) = \mu(\{z_0\} \cup H(O)) = \mu(H(O)) < \mu(O). \tag{7}$$

If $\mu(O) = 0$, then the proof of the corollary is finished. Otherwise, combining (5) and (7) we get

$$\mu(F^w) < \mu(O) \leq \mu(M^w) = \mu(\overline{M^w}) = \mu(\overline{F^w}) = \mu(F^w) \tag{8}$$

which is a contradiction. Thus, $\mu(F^w) = 0$ which gives the prove of our claim.

Consequently, the use of (5) improve that $\overline{M^w}$ is weakly compact. Keeping in mind that $\overline{M^w} = \overline{co}(\{z_0\} \cup H(M))$, we infer that $H(\overline{M^w}) \subset \overline{M^w}$ and further $H(\overline{M^w})$ is also relatively weakly compact.

Put $\mathcal{R} = B(S) \cap H(\overline{M^w})$. By the assumption (d), we have that $\mathcal{R}$ is nonempty and so $(B + H)(\overline{\mathcal{R}})$ is relatively weakly compact. Now arguing as the same steps in the end of the proof of Theorem 3.1, we deduce that there exists $z \in \overline{\mathcal{R}}$ (and also then $z \in S$) such that $z \in (B + H)(z)$.

4. Conclusion

We conclude that we have in this paper some contrubtion for the new results of fixed points theory setting in general classes Mönch's type for multivalued mappings by using the generation of Himmelberg theorem that is given in [2] and we develop these results by using the measure of weak noncompactness on X when the perturbation is countably μ-condensing.

Acknowledgments

The author expresses his gratitude to Latrach Khalid and to the anonymous reviewers for many useful remarks on the manuscript.

References

[1] MY, Abdallah, A. Al-Izeri & K. Latrach (2021). *Some remarks on fixed sets for perturbed multivalued mappings.* J. Fixed Point Theory Appl. 23 (3), Paper No. 38, 14 pp.

[2] A. Al-Izeri & K. Latrach. (2023). *A note on fixed point theory for multivalued mappings.* Fixed Point Theory 24 (1): 233-239

[3] A. Al-Izeri & K. Latrach. *Krasnosel'skii-type fixed point results for weakly sequentially upper semicontinuous multivalued mappings*. Journal of Mathematics and Applications (to appear).

[4] R. P. Agarwal, W. Hussain & M. A. Taoudi. (2012). *Fixed point theorems inAB ordered Banach spaces and applications to nonlinear integral equations. Abstr. Appl. Anal. Art.* ID 245872, 15 pp.

[5] R. P. Agarwal, D. O'Regan & M. A. Taoudi. (2010). *Browder-Krasnoleskii-Type Fixed point theorems in Banach spaces. Fixed Point Theory Appl.* 243716, 20 pp.

[6] CD. Aliprantis & K. C. Border. (1985). Infinite dimensional analysis. *Springer-Verlag, Berlin Heidelberg*. New York.

[7] J. Banaś & J. Rivero. (1988). *On measures of weak noncompactness. Ann. Mat. Pura Appl.* 151 (4): 213-224.

[8] G. Cai & S. Bu. (2013). *Krasnosel'skii-type fixed point theorems with applications to Hammerstein integral equations in L^1-spaces. Math. Nachr.* 14-15: 1452-1465.

[9] K. Floret. (1980). Weakly compact sets. *Springer-Verlag, Berlin, Heidelberg, New York.*

[10] J. Garcia-Falset. (1971). *Existence of fixed points and measure of weak noncompactness*. *Nonlinear Anal.* 71: 2625-2633.

[11] JR. Graef, J. Henderson & A. Ouahab. (2017). *Multivalued versions of a Krasnosel'skii-type fixed point theorem*. J. Fixed Point Theory Appl. 19: 1059-1082.

[12] Himmelberg CJ. (1972). *Fixed points of compact multifunctions*. *J. Math. Anal. Appl.* 38: 205-207.

[13] Y. Liu & Z. Li. (2006). *Schaefer type theorem and periodic solutions of evolution equations*. *J. Math. Anal. Appl.* 316: 237-255.

[14] K. Latrach, MA. Taoudi & A. Zeghal. (2006) *Some fixed point theorems of the Schauder and the Krasnosel'skii type and application to nonlinear transport equations.* J. Differential Equations. 221 (1): 256-271.

[15] EA. Ok. (2009). Fixed set theorems of Krasnoselskii type. Proc. Amer. Math. Soc. 137: 511-518.

[16] D. O'Regan. (2000). *Fixed point theory of Mönch type for weakly sequentially upper semicontinuous maps. Bull. Austral. Math. Soc.* 61 (3): 439-449.

[17] D. O'Regan. (1998). *Fixed point theory for weakly contractive maps with applications to operator inclusions in Banach spaces relative to the weak topology.* Z. Anal. Anwend. 17 (2): 281-296.

[18] D. O'Regan. (2000). *Fixed point theorems for weakly sequentially closed maps.* Arch. Math. (Brno). 36 (1): 61-70.

[19] W. Rudin. (1973). *Functional analysis McGraw Hill.* New York.

[20] P. Somyot & T. Thammathiwat. (2013). Fixed point theorems of Krasnosel'skii type for the sum of two multivalued mappings in Banach spaces. J. Nonlinear Convex Anal. 14: 183-191.

[21] F. Wang. (2013). *Fixed point theorems for the sum of two operators under ω-condensing*. *Fixed point theory and applications*. 102: 1-13.