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Research Article

N-prime Ideal Graph of a Finite Commutative Ring

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Abstract

In this paper, we introduce and investigate the N-prime ideal graph of a finite commutative ring with unity. The concept of prime ideal graphs over rings has been studied by many authors. Let R be a finite commutative ring with unity and let J be a proper ideal of R. The N-prime ideal graph of R with respect to J, denoted by G_J, is the simple undirected graph whose vertex set consists of the elements of the ring R and the two distinct vertices are adjacent if and only if their product belongs to that ideal and at least one of them belongs to the sum of that ideal and the nilradical of the ring. The aim of this article is to study the structural properties of the N-prime ideal graph and establish the connection between the algebraic properties of ideals and their graph theoretic properties.In particular, the N-prime ideal graphs are characterized as star, pineapple, and complete multipartite graphs. The subgraphs induced by nilpotent and idempotent elements are investigated,along with their connections to zero-divisor and product maximal graphs. Furthermore, the properties of the N-prime ideal graph as a cograph and a threshold graph are studied and illustrated with examples. The study demonstrates how N-prime ideal properties can be effectively represented and analyzed using graph theoretic methods.

Keywords

N-prime Ideal Graph, Primary Ideal, Nilradical, Nilpotent and Idempotent Elements, Complete Multipartite Graph, Cograph, Split Graph, Threshold Graph

1. Introduction

Algebraic graph theory has developed into an active field of research. Graph-theoretic methods provide a useful approach to examining algebraic systems. The investigation of graph from ring is a vast research area. Its historical foundation can be traced to cayley's graphical representation of groups in 1878 [4]. This approach was later extended to semigroups by Bosák in 1964 [5]. A systematic treatment of the subject was later presented by Biggs in Algebraic Graph theory in 1974 [6]. Since then, graphs associated with groups, semigroups, rings, modules, and ideals have provided an effective method for translating algebraic properties into combinatorial form. Many authors developed this area such as intersection graphs arising from finite abelian groups and power graphs defined on finite groups of even order, On Idempotent graph of rings, Unit graph associated with rings, Maximal graph of a finite commutative ring, etc.

Zero-divisor graphs provide a graph-theoretic framework for studying the algebraic properties of rings. Beck initiated this approach for finite commutative rings in 1988 through the graph $\Gamma_0(R)$ [7]. In 1999, Anderson and Livingston formulated the version of the zero-divisor graph that is now widely used for commutative rings [8]. For a commutative ring R with unity, the zero-divisor graph $\Gamma(R)$ is the simple undirected graph on the nonzero zero-divisors of R, where two distinct vertices x and y are joined precisely when $xy = 0$. Several generalizations of $\Gamma(R)$ have subsequently been developed.

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In 2003, Redmond defined the ideal-based zero-divisor graph $\Gamma_J(R)$ [10]. Its vertex set is $\{x \in R \setminus J : xy \in J \text{ for some } y \in R \setminus J\}$, and distinct vertices x and y are adjacent when $xy \in J$. The annihilating-ideal graph $\mathbb{AG}(R)$ was defined by Behboodi and Rakeei [11]. The vertex set of this graph consists of all nonzero annihilating ideals of R, and the two distinct vertices J_1 and J_2 are joined by an edge if and only if $J_1J_2 = (0)$. Subsequently, Badawi introduced the annihilator graph $AG(R)$ in 2014 [12]. Its vertex set is $Z(R) \setminus \{0\}$. An edge connects distinct vertices x and y if and only if $\mathrm{Ann}_R(xy) \neq \mathrm{Ann}_R(x) \cup \mathrm{Ann}_R(y)$. These graphs provide useful tools for examining the annihilator structure of a commutative ring through graph-theoretic properties.

The role of nilpotent elements was investigated by Nikmehr and Khojasteh in 2013 through the nilpotent graph Γ_N [13]. In this graph, the two different vertices x and y are connected by an edge if and only if their product belongs to the nilradical N; that is, $xy \in N$. This construction extends the usual zero-product adjacency by incorporating the nilpotent elements of the ring. In 2021, Nikmehr, Azadi, and Nikandish introduced the weakly zero-divisor graph $W\Gamma(R)$ [14]. For $x, y \in Z(R)^*$ with $x \neq y$, their adjacency is determined by the existence of elements $r \in \mathrm{Ann}(x)$ and $s \in \mathrm{Ann}(y)$ satisfying $rs = 0$. The ordinary zero-divisor graph $\Gamma(R)$ is a spanning subgraph of $W\Gamma(R)$. Motivated by this weak adjacency condition, Patel introduced the weakly annihilating-ideal graph $WAG(R)$ in 2023 [15]. Its vertex set consists of all nonzero annihilating ideals of R. The two distinct vertices J_1 and J_2 are adjacent if and only if there exist nonzero ideals A and B of R satisfying $A \subseteq \mathrm{Ann}(J_1)$, $B \subseteq \mathrm{Ann}(J_2)$, $AB = (0)$. These weak graphs provide broader frameworks for studying the relationship between annihilator structures and graph-theoretic properties.

Graphs arising from maximal ideals and generalized prime ideals form another important class of graphs associated with rings. The product maximal graph $\Gamma_{pm}(R)$ associated with a finite commutative ring R was defined by Kalamani and Ramya [16]. Let $M_1, M_2, \ldots, M_r$ be the maximal ideals of R. The graph $\Gamma_{pm}(R)$ is defined on the vertex set R. For $x, y \in R$ with $x \neq y$, their adjacency is determined by the condition $xy \in M_i$ for some $i \in \{1, 2, \ldots, r\}$. The concept of an S-prime ideal was proposed by Hamed and Malek in 2020 [17]. Let S be a multiplicatively closed subset of R. A proper ideal J satisfying $J \cap S = \varnothing$ is called S-prime ideal if there is an element $s \in S$ such that, for every $a, b \in R$, the condition $ab \in J$ implies that $sa \in J$ or $sb \in J$. In 2021, Almahdi, Bouba, and Tamekkante introduced the notion of a weakly S-prime ideal [18]. Under the same assumptions on J and S, the ideal J is weakly S-prime if there exists a fixed element $s \in S$ such that $0 \neq ab \in J \; implies \; sa \in J$ or $sb \in J$, for all $a, b \in R$.

Based on the concept of an S-prime ideal, Kalamani and Mythily introduced the S-prime ideal graph $G_{S_d}(R)$ in 2023 [19]. Let R be a finite commutative ring with unity, let S be a multiplicatively closed subset of R, and let S_d be an S-prime ideal generated by a divisor d of $|R|$. The graph $G_{S_d}(R)$ has vertex set $V\left(G_{S_d}(R)\right) = R$ with $ab \in J$ and the two distinct vertices a and b are joined by an edge if and only if $sa \in S_d \; or \; sb \in S_d$, for some $s \in S$. Furthermore, they investigate several graph-theoretic properties of the S-prime ideal graph, including its diameter, girth, and chromatic number, as well as algebraic properties such as its rank and nullity.

In 2026, Kalamani and Vasuki introduced the weakly S-prime ideal graph $G_F(R)$, where F is a proper ideal of R disjoint from the multiplicatively closed subset S [20]. The vertex set consists of all elements $u \in R$ for which $0 \neq uw \in F$ and either $su \in F$ or $sw \in F$, for some $w \in R$ and $s \in S$. The two distinct vertices u and w are adjacent if and only if either $su \in F$ or $sw \in F$ for some $s \in S$. They also related the properties of this graph to those of the ring.

Esra Sengelen Sevim and Suat Koc introduced an intermediate class of ideals, called N-prime ideals in 2018 [9], which lies between prime ideals and quasi-primary ideals, and investigated several of their fundamental properties. Let R be a commutative ring. A proper ideal J of R is called a N-prime ideal if $ab \in J$, for each $a, b \in R$, then either $a \in J + N(R)$ or $b \in J + N(R)$. Moreover, they defined a topology on the set of all N-prime ideals and studied various topological properties of this space, including irreducibility, connectedness, and the separation axioms.

Our aim is to introduce a new graph called the N-prime ideal graph of a finite commutative ring and it is represented by G_J. Throughout this paper, R denotes the finite commutative ring of with unity. For every proper ideal J of R, we write $J' = J + N$. With this notation, the two distinct vertices $a, b \in R$ are adjacent in G_J if and only if $ab \in J$ and either $a \in J'$ or $b \in J'$. The sets of nilpotent and idempotent elements of R are denoted by N and I respectively. Furthermore, $G_J[N] \; and \; G_J[I]$ denote the subgraphs of G_J induced by the nilpotent and idempotent elements of R, respectively.

2. Preliminaries

In this section, some fundamental definitions are given. The algebraic terminology used in this section follows from [1] and [2], while the standard graph-theoretic terminology follows from [3].

Definition 2.1. A proper ideal H of a commutative ring R is prime whenever $uw \in H$ implies that $u \in H$ or $w \in H$, for all

$u, w \in R$ [1].

Definition 2.2. An element $u \in R$ is nilpotent if $u^m = 0$ for some positive integer m [1].

Definition 2.3. For an ideal J of a commutative ring R, the radical of J is $\sqrt{J} = \{u \in R : u^m \in J \text{ for some } m \geq 1\}$. In particular, the nilradical is $N = \sqrt{\langle 0 \rangle}$. The ring R is called reduced if $N = \{0\}$.

Definition 2.4. A proper ideal H of R is called primary if $uw \in H$ and $u \notin H$ imply $w^m \in H$ for some positive integer m; equivalently, $w \in \sqrt{H}$

Definition 2.5. A commutative ring R is called local if it has a unique maximal ideal. An element $f \in R$ is called idempotent if $f^2 = f$.

Definition 2.6. For an ideal J of R and $u \in R$, the colon ideal is $(J : u) = \{r \in R : ru \in J\}$. For ideals U and W, their product UW is the ideal generated by all products uw with $u \in U$ and $w \in W$. In particular, $N^2 = NN$.

Definition 2.7. Let R be a ring. A nonzero element $u \in R$ is a zero divisor if some nonzero element $w \in R$ satisfies $uw = 0$ or $wu = 0$ [1].

Definition 2.8. A simple graph is complete when every pair of distinct vertices is adjacent. A complete graph of order n is denoted by K_n [3].

Definition 2.9. The two graphs $G_1 = (V_1, Y_1)$ and $G_2 = (V_2, Y_2)$ are said to be isomorphic. If there exists a bijection $f : V_1 \to V_2$ such that $u, w \in V_1$ are adjacent if and only if $f(u), f(w) \in V_2$ are adjacent in G_2.
The two graphs are said to be isomorphic if they have same number of edges and vertices.

Definition 2.10. A graph $G = (V, E)$ is k-partite iff V can be partitioned into k-independent sets.

Definition 2.11. A k-partite graph is called a complete k-partite graph if any vertex of $v \in V$ is adjacent to all vertices not belonging to the same partition as v. it is denoted by $K_{n_1,n_2,\ldots,n_k}$. A complete k-partite graph, for some $k \geq 2$, is also called a complete multipartite graph.

Definition 2.12. A star graph $K_{1,s}$ is a tree on $s + 1$ vertices in which one vertex is adjacent to each of the remaining s vertices.

Definition 2.13. A triangular book graph is formed from n copies of the cycle C_3 having one edge in common. The shared edge is called the spine, or the base, of the book. This graph is denoted by $B(3, n)$ and is isomorphic to the complete tripartite graph $K_{1,1,n}$ [24].

Definition 2.14. The butterfly graph B_n is the planar graph formed by taking $2n$ copies of C_3 with one edge shared by all the copies. It has $2n + 2$ vertices and $4n + 1$ edges [30].

Definition 2.15. For $u \geq 3$ and $w \geq 1$, the pineapple graph K_u^w is formed by attaching w pendant edges to one fixed vertex of the complete graph K_u [21].

Definition 2.16. Let S be a subset of the vertex set of a graph G. The subgraph induced by S, denoted by $G[S]$, has vertex set S and includes exactly those edges of G whose two endpoints lie in S. A clique is a set of mutually adjacent vertices, whereas an independent set is a set in which no two vertices are adjacent [3].

Definition 2.17. A graph is a cograph when it has no induced P_4 [23]. It is a split graph when its vertex set admits a partition into a clique and an independent set [25, 26]. It is a threshold graph when none of P_4, C_4, and $2K_2$ occurs as an induced subgraph [22].

3. Characterization of N-prime Ideal Graphs

In this section, the definition of N-prime ideal graph of a finite commutative ring is given and we investigate the graph properties of the N-prime ideal graphs and characterize their graph structure in terms of the properties of the ring and its corresponding ideal.The results describe the cases in which G_J is a star, pineapple, complete multipartite, triangular book, or butterfly graph. The triangle-free case and the existence of an Eulerian trail are also considered. Appropriate examples are given for the main results.

Definition 3.1. Let R be a finite commutative ring with unity and let J be a proper ideal of R. The N-prime ideal graph of R with respect to J, denoted by G_J, is the simple undirected graph whose vertex set is R and the two distinct vertices $a, b \in R$ are adjacent if and only if $ab \in J$ and either $a \in J + N$ or $b \in J + N$, where N denotes the nilradical of R.

Theorem 3.2. *Let R be a ring of order p, where p is a prime. Then the N-prime ideal graph G_0 is a star graph.*

Proof. Let N be the nilradical of R, and $J = (0)$ be the only proper ideal of R.

Since R is of order p, we have $J = N$. Hence, $J = J'$.

Let a and b be two distinct vertices of G_J.

If $a \in J$ and $b \notin J$, then $a = 0$, since $J = (0)$.

Therefore, $ab = 0 \in J$, and hence there is an edge between a and b.

Now, suppose that $a \notin J$ and $b \notin J$. Then $a \neq 0$ and $b \neq 0$.

Since R has prime order, it is a field and therefore has no zero divisors. Hence,$ab \neq 0$, which implies that $ab \notin J$. Therefore, there is no edge between a and b.

Thus, the vertex 0 is adjacent to every other vertices of G_J, and there is no adjacency between any two nonzero vertices.

Hence, G_J is a star graph whose central vertex is 0, and all the other vertices are pendant vertices. □

Theorem 3.3. *Let R be a reduced ring. Then G_0 is a star graph.*

Proof. Let R be a reduced ring with unity and J=0.

The vertices of G_J are elements of R.

Since R is a reduced ring, $N = 0$. Therefore $J = J'$.

Let a and b be the two distinct elements of R such that $ab \in J$. Since $J = 0$, ab=0 then the following cases arise.

Case(i): Either $a \neq 0$ and $b = 0$

Suppose that $a \neq 0$ and $b = 0$ then b is an element of J. it follows that $b \in J'$.

Hence there is an edge between a and b.

Case(ii): Both a and b $\neq 0$.

Suppose that $a, b \in R \setminus \{0\}$ are distinct. Since $J' = J + N = \{0\}$, neither a nor b belongs to J'.

Hence there is no edge between a and b.

Thus, the vertex 0 is adjacent to all other non-zero vertices of G_0 and there is no edge between any two non-zero vertices.

Therefore,G_0 is a star graph whose central vertex is zero. □

Example 3.4. Let $R = \mathbb{Z}_{10}$ and $J = \langle 0 \rangle$. Since R is reduced, $N = \{0\}$ and hence $J' = J + N = \{0\}$. Thus 0 is the only vertex of G_0 that belongs to J'. For every nonzero element $a \in Z_{10}$, we have $0.a = 0 \in J$ and $0 \in J'$. therefore, 0 is adjacent to every nonzero vertex. No two nonzero vertices are adjacent because neither a nor b belongs to J'. Therefore, the nonzero elements of R form an independent set. Hence, G_0 is a star graph. Thus $G_0 \cong K_{1,9}$, with $|V(G_0)| = 10$ and $|E(G_0)| = 9$. The graph is illustrated in Figure 1.

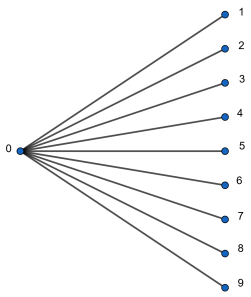


Figure 1. *The star graph $G_0 \cong K_{1,9}$*

Corollary 3.5. *If G_J is a star graph then the following conditions are equivalent.*

(i) G_J is a triangular free.

(ii)G_J is tree.

(iii)G_J is bipartite.

(iv) $G_J \cong K_{1,|R|-1}$.

Theorem 3.6. *Let $R = \mathbb{Z}_n$ be a ring of order p^2, $p \geq 3$. Then the N-prime ideal graph G_0 is a pineapple graph K_p^q, where $q = n - p$.*

Proof. Let R be a of order p^2 where $p \geq 3$ and $J = 0$.

If P is prime ideal of R, then $N = P$. Hence $J' = P$.

Choose a and b in R such that $a \neq b$ and $ab \in J =< 0 >$. Then either $a \neq 0$ or $b \neq 0$.

Consider $a \neq 0$. If $b = 0$ then b is an element of J'.

Hence there is an edge between a and b.

Thus the ideal element 0 is adjacent to all the other non-zero elements of G_J.

Suppose $a \neq 0$ and $b \neq 0$ then the elements a and b are nonzero zero-divisors of R. Since R is a local ring, $Z(R) = N = P$, $|P| = p$, and $P^2 = \langle 0 \rangle$ [27]. Therefore, both a and b belong to $J' = P$. Moreover, the product of any two distinct elements of P belongs to $P^2 = \langle 0 \rangle = J$. Hence, every two distinct elements of P are adjacent.

Thus, the set of zero-divisors of G_0 form a complete graph K_p in G_0. The remaining $(n - p)$ vertices form the pendent edges with ideal element 0.

Therefore, the N-prime ideal graph G_0 is a pineapple graph $K_p^{(n-p)}$, where $q = n - p$. □

Example 3.7. Let $R = \mathbb{Z}_{25}$ and $J = \langle 0 \rangle$. Since R is a local ring, its nilradical is $N = \langle 5 \rangle = \{0, 5, 10, 15, 20\}$. Hence,$J' = J + N = N$. The product of any two distinct elements of N is zero in $\mathbb{Z}_{25}$, and all these elements belong to J'. Therefore, N induces the complete graph K_5. The remaining twenty elements of R are units. Every unit is adjacent to 0, since its product with 0 belongs to J and $0 \in J'$. No unit is adjacent to a nonzero element of N, and no two units are adjacent. Hence, the twenty units are pendant vertices attached to 0. Thus, $G_0 \cong K_5^{20}$, where K_5^{20} is the pineapple graph obtained by attaching twenty pendant vertices to the vertex 0 of K_5. Moreover,

$$|V(G_0)| = 25 \qquad \text{and} \qquad |E(G_0)| = \binom{5}{2} + 20 = 30.$$

The resulting pineapple graph is shown in Figure 2.

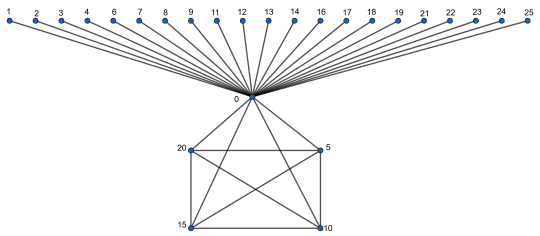


Figure 2. *The pineapple graph* $G_0 \cong K_5^{20}$.

Complete multipartite zero-divisor graphs have been studied in [28][Theorem 3.2]. The following theorem determines how the vertex set of G_J is partitioned into partite sets and specifies the exact number of these sets.

Theorem 3.8. *Let* $R = \mathbb{Z}_n$ *be a ring and* $J =< d >$ *where d is the divisor of n.Then the N-prime ideal graph* G_J *is a complete k-partite graph if d is either a prime or a product of distinct primes, where* $k = \frac{n}{d} + 1$.

Proof. Let R be a ring and $J =< d >$. Now consider $d = p_1.p_2...p_m$, where $p_1.p_2...p_m$ are distinct primes. Case(i): m = 1, then d = p

This implies $J =< p >$. As every prime ideal contains nilradical ideal, $J + N = J$.

Let a and b be the two distinct elements of R such that $ab \in J$. Since J is prime ideal of R, either $a \in J$ or $b \in J$. If a is an element of J then there is an edge between a and b.

Hence, the ideal element a is adjacent to all other elements of G_J. If $a \notin J$ and $b \notin J$ then $ab \notin J$. Hence, there is no edge between a and b.

Thus every ideal element is adjacent to all other elements of G_J and no adjacency between any two non-ideal elements of G_J.

Case(ii): If $m > 1$ then $d = p_1.p_2...p_r$, $2 \leq r \leq m$.

This implies that $J =< p_1.p_2...p_r >$ where $2 \leq r \leq m$.

Since nilradical ideal is the product of all distinct primes, $N \subseteq J$. It follows that,$J + N = J$.

Choose a and b in R such that $a \neq b$ and $ab \in J = p_1.p_2...p_r$ (i.e) $ab = h.p_1.p_2...p_r$, $h \in Z^+$.

If a = h, then $b = p_1.p_2...p_r$ is an element of J. Hence, there is an edge between a and b.

Thus, the ideal element $p_1.p_2...p_r$ is adjacent to all other elements of G_J.

Let us take $a = h.p_1.p_2...p_s$, $s < r$ and $b = p_{s+1}.p_2...p_r$. If h = b then $a = p_1.p_2...p_r$ is an element of J. Hence there is an edge between a and b.

If $h \neq b$ then a and b are not the elements of J. Hence there is no adjacency between them.

In either cases, Every element of J is adjacent to every other vertex of G_J. If $x, y \in R \setminus J$ are distinct, then neither x nor y belongs to $J' = J$. Hence, x and y are nonadjacent.

Therefore, the partite sets of G_J are the singleton sets $\{a\}$ for $a \in J$, together with the independent set $R \setminus J$. Since $|J| = n/d$, the number of partite sets is

$$k = |J| + 1 = \frac{n}{d} + 1.$$

Hence, G_J is a complete k-partite graph. □

Example 3.9. Let $R = \mathbb{Z}_{12}$ and $J = \langle 6 \rangle$. Since $N = J$, we have $J' = J + N = J$. The vertices 1 and 6 are adjacent because $1 \cdot 6 = 6 \in J$ and $6 \in J'$. For every $x \in R$, the products $0x$ and $6x$ lie in J, and $0, 6 \in J'$. Thus, 0 and 6 are adjacent to each other and to every other vertex of G_J.

The remaining ten vertices are not in J', so no two of them are adjacent. In particular, $2 \cdot 3 = 6 \in J$, but $2, 3 \notin J'$, and therefore 2 and 3 are nonadjacent. Also, $5 \cdot 7 \notin J$, so 5 and 7 are nonadjacent. Thus, no two distinct elements of $R \setminus J$ are adjacent in G_J. The vertex set is partitioned into $V_1 = \{0\}$, $V_2 = \{6\}$, and $V_3 = R \setminus J$. Each V_i is independent, and every two vertices from distinct parts are adjacent. Since $|J| = 2$, the number of partite sets is $k = |J| + 1 = 3$. Therefore, $G_J \cong K_{1,1,10}$, and G_J is a complete tripartite graph.

In particular, $|V(G_J)| = 12$ and $|E(G_J)| = 21$.

Corollary 3.10. *Let R be a ring and* $J =< d >$ *where n is even and* $d = \frac{n}{2}$. *If d is either a prime or a product of distinct primes then the following are equivalent.*

(i) The N-prime ideal graph G_J *is a triangular book graph* $B(3, n-2)$ *or* $K_{1,1,n-2}$.

(ii) G_J *is a butterfly graph* $B_{(n-2)/2}$.

(iii) G_J *has an Eulerian trail.*

Example 3.11. Let $R = \mathbb{Z}_{14}$ and $J = \langle 7 \rangle = \{0, 7\}$. Since R is reduced, its nilradical is $N = \{0\}$, and hence $J' = J + N = J$. Since J is an ideal, $0x \in J$ and $7x \in J$ for every $x \in R$. It follows that 0 and 7 are adjacent to each other and to every other vertex of G_J. The remaining twelve vertices are outside J', so there are no edges among them. Therefore, 0 and 7 form two singleton partite sets, while the other twelve vertices form the third partite set. Hence, G_J is the complete tripartite graph $G_J \cong K_{1,1,12}$.

For every $x \in R \setminus J$, the vertices 0, 7, and x form a triangle. Hence, the graph consists of twelve triangles sharing the common edge $\{0, 7\}$. Therefore, G_J is a triangular book graph with twelve pages, and $G_{\langle 7 \rangle} \cong B(3, 12)$.

Since the graph $B(3, 12)$ has an even number of triangular pages, these twelve pages can be grouped into six pairs and each pair forming a butterfly. Hence, $B(3, 12) \cong B_6$. Therefore,

$$G_{\langle 7 \rangle} \cong K_{1,1,12} = B(3, 12) \cong B_6.$$

Moreover, $|V(G_J)| = 14$, $|E(G_J)| = 25$, $\deg(0) = \deg(7) = 13$, and every other vertex has degree 2. The graph is connected and has exactly two vertices of odd degree. Therefore, by the Euler trail criterion [29], it admits an Eulerian trail but has no Eulerian circuit. The graph $G_{\langle 7\rangle}$ is shown in Figure 3.

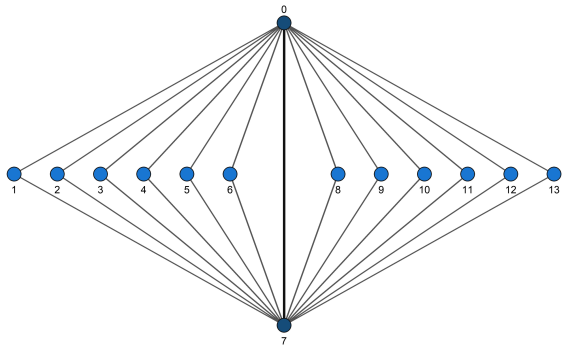


Figure 3. *The N-prime ideal graph* $G_{\langle 7\rangle}$

4. Relations and Operations on N-Prime Ideal Graphs

In this section, we examine the connection between N-prime ideal graphs and other graphs associated with commutative rings. We first compare G_J with the all-element zero-product graph $\Gamma_0(R)$. We then establish an isomorphism between certain N-prime ideal graphs associated with prime-power rings. We also investigate intersections of N-prime ideal graphs. Finally, we study the relationship between G_J and the product maximal graph associated with R.

Theorem 4.1. *Let R be a local ring. Then* $G_0 = \Gamma_0(R)$.

Proof. Since R is a local ring and $J = 0$, $J \subseteq N$.

The vertices of G_0 and $\Gamma_0(R)$ are the elements of R.

Choose a and b belongs to R with $a \neq b$ such that $ab \in J = 0$ then either $a = 0$ and $b \neq 0$ or $a \neq 0$ and $b \neq 0$.

If $a = 0$ then a is an element of J. Since $J \subseteq J'$, a is an element of J'.

Hence, there is an edge between a and b. If $a \neq 0$ and $b \neq 0$ then a and b are non-zero divisors of R. Let M be the unique maximal ideal of the finite local ring R. Since R is Artinian, M is nilpotent; hence $N = M = Z(R)$ [31, Chapter 8]. Since J is proper, $J \subseteq M$, and therefore $J' = N = Z(R)$.

As a and b are elements of J', there is an edge between a and b. If a and b are zero divisor pairs then they are adjacent.

Hence, the vertex set and edge set of G_0 is equal to $\Gamma_0(R)$.

Therefore, $G_0 = \Gamma_0(R)$. □

Example 4.2. Let $R = \mathbb{Z}_8$ and $J = \langle 0\rangle$. Then $N = \langle 2\rangle$ and $J' = J + N = N$. If $ab = 0$ then $\{a, b\} \cap N \neq \varnothing$. Therefore, two distinct elements $a, b \in R$ are adjacent in G_0 if and only if $ab = 0$. Thus, two distinct vertices are adjacent in G_0 of R exactly when they are adjacent in $\Gamma_0(\mathbb{Z}_8)$. Since both graphs have the same vertex set, it follows that $G_0 = \Gamma_0(\mathbb{Z}_8)$.

Graph isomorphisms induced by algebraic maps between commutative rings have been studied in Theorems 5.1, 5.4 and 5.8]AndersonLewis2016. The following result gives a related construction for N-prime ideal graphs over prime-power rings.

Theorem 4.3. *Let p be a prime, m* ≥ 2, *and* $1 \leq k \leq m-1$. *If* $R_1 = \mathbb{Z}_{p^m}$ *and* $R_2 = \mathbb{Z}_{p^{m+1}}$, *with* $J_1 = \langle p^{m-k}\rangle$ *and* $J_2 = \langle p^{m+2-k}\rangle$, *respectively, then*

$$G_{J_1} \cong G_{J_2}[N_2],$$

where N_2 *denotes the nilradical of* R_2.

Proof. Let G_{J_1} and G_{J_2} be the N-prime ideal graphs associated with the ideals J_1 of R_1 and J_2 of R_2, respectively. Also, let $G_{J_2}[N_2]$ denote the subgraph of G_{J_2} induced by the set N_2 of nilpotent elements of R_2. If N_1 is the nilradical of R_1, $N_1 = \langle p\rangle \subseteq R_1$ and $N_2 = \langle p\rangle \subseteq R_2$ [27].

Define a vertex mapping $\phi : V(G_{J_1}) \longrightarrow V\big(G_{J_2}[N_2]\big)$ by $\phi(a) = pa$. Suppose that $a = b$ in R_1. Then $a \equiv b \pmod{p^m}$, and hence $p^m \mid (a-b)$. Multiplying by p gives $p^{m+1} \mid p(a-b)$. Therefore, $pa \equiv pb \pmod{p^{m+1}}$, and hence $\phi(a) = \phi(b)$ in R_2. Moreover, $\phi(a) = pa \in \langle p\rangle = N_2$. Thus, ϕ is well-defined.

Let $a, b \in V(G_{J_1})$ and suppose that $\phi(a) = \phi(b)$. Then $pa \equiv pb \pmod{p^{m+1}}$, and hence $p^{m+1} \mid p(a-b)$.

Since $p^{m+1} = p \cdot p^m$, it follows that $p^m \mid (a-b)$. Therefore, $a \equiv b \pmod{p^m}$, and hence $a = b$ in R_1.

Thus, ϕ is injective.

Let $x \in N_2$. Since $N_2 = \langle p\rangle$, the element x can be written as $x = pa$ for some $a \in R_1$. Therefore, $x = \phi(a)$, and hence ϕ is surjective. Consequently, ϕ is bijective.

Let a and b be two distinct adjacent vertices in G_{J_1}. Since $1 \leq k \leq m-1$, it follows that $m - k \geq 1$, and hence $J_1 = \langle p^{m-k}\rangle \subseteq \langle p\rangle = N_1$. Therefore, $J_1 + N_1 = N_1$. Since a and b are adjacent in G_{J_1}, $ab \in J_1$, and at least one of a and b belongs to N_1. Moreover, there exists an integer $t \in \mathbb{Z}$ such that $ab \equiv tp^{m-k} \pmod{p^m}$. It follows that

$$\begin{aligned}\phi(a)\phi(b) &= (pa)(pb)\\ &= p^2ab\\ &\equiv tp^{m+2-k} \pmod{p^{m+1}}.\end{aligned}$$

Thus, $\phi(a)\phi(b) \in J_2$. Also, $\phi(a), \phi(b) \in N_2 \subseteq J_2 + N_2$. Therefore, both the product condition and the endpoint condition are satisfied, and hence $\phi(a)$ and $\phi(b)$ are adjacent in $G_{J_2}[N_2]$.

Conversely, let $\phi(a)$ and $\phi(b)$ be two distinct adjacent vertices in $G_{J_2}[N_2]$. Since this graph is induced by N_2, $\phi(a), \phi(b) \in N_2$. By the definition of adjacency, $\phi(a)\phi(b) \in J_2$. Since $\phi(a) = pa$ and $\phi(b) = pb$, $p^2ab \in \langle p^{m+2-k}\rangle$, which implies that $ab \in \langle p^{m-k}\rangle = J_1$. Since $m - k \geq 1$, the condition $ab \in J_1$ implies that $p \mid ab$. As p is prime, either $p \mid a$ or $p \mid b$. Hence, at least one of a and b belongs to $N_1 = J_1 + N_1$. Therefore, the product condition and the endpoint condition are satisfied, and so a and b are adjacent in G_{J_1}.

Therefore, two distinct vertices a and b are adjacent in G_{J_1} if and only if their images $\phi(a)$ and $\phi(b)$ are adjacent in $G_{J_2}[N_2]$. Thus, ϕ preserves both adjacency and non-adjacency. Since ϕ is bijective, it is a graph isomorphism. Therefore,

$$G_{J_1} \cong G_{J_2}[N_2].$$

□

Example 4.4. Let $R_1 = \mathbb{Z}_8$ and $R_2 = \mathbb{Z}_{16}$. Take $p = 2$ and $m = 3$. Then Then $1 \leq k \leq 2$. The nilradical of R_2 is $N_2 = \langle 2\rangle$. Consider the mapping $\phi : R_1 \longrightarrow N_2$ defined by $\phi(a) = 2a$. This mapping is a bijection between R_1 and N_2.

For $k = 1$, the corresponding ideals are $J_1 = \langle 2^{3-1}\rangle = \langle 4\rangle$ and $J_2 = \langle 2^{3+2-1}\rangle = \langle 16\rangle = \langle 0\rangle$. Under the correspondence determined by ϕ, two distinct vertices are adjacent in G_{J_1} precisely when their images are adjacent in $G_{J_2}[N_2]$. Hence,

$$G_{\langle 4\rangle} \cong G_{\langle 0\rangle}[N_2].$$

Both graphs have eight vertices and fourteen edges.

For $k = 2$, the corresponding ideals are $J_1 = \langle 2^{3-2}\rangle = \langle 2\rangle$ and $J_2 = \langle 2^{3+2-2}\rangle = \langle 8\rangle$. In this case, two distinct vertices a and b are adjacent in G_{J_1} if and only if $\phi(a)$ and $\phi(b)$ are adjacent in $G_{J_2}[N_2]$. Consequently,

$$G_{\langle 2\rangle} \cong G_{\langle 8\rangle}[N_2].$$

Both graphs have eight vertices and twenty-two edges.

Thus, the isomorphism holds for both possible values $k = 1$ and $k = 2$.

The intersection of N-prime ideal graphs is not always equal to the N-prime ideal graph associated with the intersection of the corresponding ideals. The following theorem gives the necessary and sufficient condition for the intersection of N-prime ideal graph.

Theorem 4.5. *Let J_1 and J_2 be any two proper ideals of R. Then $G_{J_1} \cap G_{J_2}$ is an N-prime ideal graph iff $G_{J_1} \cap G_{J_2} = G_{J_1 \cap J_2}$.*

Proof. Since $V(G_{J_1})$ and $V(G_{J_2})$ are the elements of R, $V(G_{J_1} \cap G_{J_2}) = R$. $V(G_{J_1} \cap G_{J_2}) = V(G_{J_1}) \cap V(G_{J_2}) = R$.

Assume that $G_{J_1} \cap G_{J_2}$ is an N-prime ideal graph of R.

Then $G_{J_1} \cap G_{J_2} = G_J$ for some ideal J. For every $x \in R$, the vertices 1 and x are adjacent in G_J if and only if $x \in J$. Since $G_J = G_{J_1} \cap G_{J_2}$, this occurs if and only if 1 and x are adjacent in both G_{J_1} and G_{J_2}, which is equivalent to $x \in J_1$ and $x \in J_2$. Thus, $x \in J$ if and only if $x \in J_1 \cap J_2$, and hence $J = J_1 \cap J_2$. Consequently, $G_{J_1} \cap G_{J_2} = G_J = G_{J_1 \cap J_2}$. Let a and b be the two distinct adjacent vertices of G_J.

Then a and b are adjacent in both G_{J_1} and G_{J_2}. We have $ab \in J_1$, this implies either $a \in J_1 + N$ or $b \in J_1 + N$ and $ab \in J_2$ implies either $a \in J_2 + N$ or $b \in J_2 + N$. By the distributive law for ideals, $(J_1 + N) \cap (J_2 + N) = (J_1 \cap J_2) + N$. Thus, if the same endpoint belongs to both $J_1 + N$ and $J_2 + N$, then it belongs to $(J_1 \cap J_2) + N$.

Since $J = J_1 \cap J_2$ and a and b are adjacent in G_J, we have $ab \in J_1 \cap J_2$ and either $a \in (J_1 \cap J_2) + N$ or $b \in (J_1 \cap J_2) + N$.

Hence, there is an edge between any two vertices in G_J implies the existence of an edge between their corresponding vertices in $G_{J_1 \cap J_2}$. Therefore $G_{J_1} \cap G_{J_2} \subseteq G_{J_1 \cap J_2}$.

Let a and b be the adjacent vertices in $G_{J_1 \cap J_2}$. Then $ab \in J_1 \cap J_2$ implies that either $a \in (J_1 \cap J_2) + N$ or $b \in (J_1 \cap J_2) + N$.

Since $ab \in J_1 \cap J_2$, $ab \in J_1$ and $ab \in J_2$. If $a \in (J_1 \cap J_2) + N$ then $a \in J_1 + N$ and $a \in J_2 + N$. If $b \in (J_1 \cap J_2) + N$ then $b \in J_1 + N$ and $b \in J_2 + N$. It follows that $ab \in J_1$ implies either $a \in J_1 + N$ or $b \in J_1 + N$ and $ab \in J_2$ implies either $a \in J_2 + N$ or $b \in J_2 + N$.

Hence, a and b are adjacent in both G_{J_1} and G_{J_2}. Thus a and b are adjacent in $G_{J_1} \cap G_{J_2}$.

Hence if two vertices are adjacent in $G_{J_1} \cap G_{J_2}$ then their corresponding vertices are also adjacent in G_J. i.e., $G_{J_1 \cap J_2} \subseteq G_{J_1} \cap G_{J_2}$. Therefore $G_{J_1} \cap G_{J_2} = G_{J_1 \cap J_2}$.

Conversely, assume that $G_{J_1} \cap G_{J_2} = G_{J_1 \cap J_2}$. Since $J_1 \cap J_2$ is a proper ideal of R, $G_{J_1 \cap J_2}$ is an N-prime ideal graph. Hence, $G_{J_1} \cap G_{J_2}$ is an N-prime ideal graph. □

Corollary 4.6. *Let $J_I, J_2, .., J_k$ be the proper ideals of R. If $G_{J_1} \cap G_{J_2} \cap ...G_{J_k}$ is an N-prime ideal graph then $G_{J_1} \cap G_{J_2} \cap ... \cap G_{J_k} = G_{J_1 \cap J_2 \cap ... \cap J_k}$. Thus, the equality holds for any finite collection pf proper ideals of R.*

Example 4.7. Let $R = \mathbb{Z}_{12}$, $J_1 = \langle 3\rangle$, and $J_2 = \langle 4\rangle$. Their intersection is $J_1 \cap J_2 = \langle 0\rangle$, and the nilradical of R is $N = \langle 6\rangle = \{0, 6\}$. In G_{J_1}, the vertices 3 and 4 are adjacent because $3 \cdot 4 = 0 \in J_1$ and $3 \in J_1 + N$. Similarly, they are adjacent in G_{J_2} because $3 \cdot 4 = 0 \in J_2$ and $4 \in J_2 + N$. Hence, $\{3, 4\}$ is an edge of $G_{J_1} \cap G_{J_2}$.

On the other hand, $G_{J_1 \cap J_2} = G_0$ and $(J_1 \cap J_2) + N = N$. Since neither 3 nor 4 belongs to N, they are not adjacent in G_0. Therefore,

$$G_{J_1} \cap G_{J_2} \neq G_{J_1 \cap J_2}.$$

Moreover, both graphs have 12 vertices, whereas $G_{J_1} \cap G_{J_2}$ has 19 edges and $G_{J_1 \cap J_2}$ has 15 edges. Thus, the intersection

of two N-prime ideal graphs is not, in general, equal to the N-prime ideal graph associated with the intersection of the corresponding ideals.

The product maximal graph $\Gamma_{pm}(R)$ was introduced in [16]. The following theorem expresses this graph as a union of N-prime ideal graphs associated with the maximal ideals of R.

Theorem 4.8. *Let R be a ring and $M_1, M_2, ..., M_k$ be the maximal ideals of R. Then $\bigcup_{i=1}^{k} G_{M_i} = \Gamma_{pm}(R)$.*

Proof. Let $\Gamma_{pm}(R)$ be the product maximal graph of R and $\bigcup_{i=1}^{k} G_{M_i}(R)$ be the N-prime ideal graph for the maximal ideals $M_1, M_2, ..., M_k$.

The elements of R are the vertices of both product maximal graph $\Gamma_{pm}(R)$ and union of N-prime ideal graph for the maximal ideals $M_1, M_2, ..., M_k$ $\bigcup_{i=1}^{k} G_{M_i}(R)$. Let a and b be the two distinct adjacent vertices of $\Gamma_{pm}(R)$. Then $ab \in M_i$, for some $i = 1, 2, .., k$. Since M_i is a prime ideal, either a or $b \in M_i$. As $M_i + N = M_i$, there is an edge between a and b in $G_{M_i}(R)$. Hence a and b are adjacent in $\bigcup_{i=1}^{k} G_{M_i}(R)$.

Let a and b be the two distinct adjacent vertices of $\bigcup_{i=1}^{k} G_{M_i}(R)$.

Then a and b are adjacent in G_{M_i} for some $i = 1, 2, .., k$. This implies $ab \in M_i$ and either a or $b \in M_i + N = M_i$ for some $i = 1, 2, .., k$. [16]

Hence there is an edge between a and b in $\Gamma_{pm}(R)$.

Thus, the two vertices a and b are adjacent in $\bigcup_{i=1}^{k} G_{M_i}(R)$ if and only if they are adjacent in $\Gamma_{pm}(R)$.

Therefore $\bigcup_{i=1}^{k} G_{M_i} = \Gamma_{pm}(R)$.

□

Theorem 4.9. *Let R be a reduced ring and $P_1, P_2, ...P_k$ be all the prime ideals of R. Then $\bigcap_{i=1}^{k} G_{P_i}$ is $\Gamma_0(R)$.*

Proof. Let ab is an edge belongs to the edge set of $\bigcap_{i=1}^{k} G_{P_i}$.

Then a and b are adjacent in G_{P_i} for every $i = 1, 2, .., k$.

This implies $ab \in P_i$, for every $i = 1, 2, ..., k$ and either a or $b \in P_i + N = P_i$. Hence $ab \in \bigcap_{i=1}^{k} P_i$.

Since $P_1, \ldots, P_k$ are all the prime ideals of R, the nilradical

$$N = \bigcap_{i=1}^{k} P_i$$

[31, Proposition 1.8].

As R is reduced, N =0 and $ab \in \{0\}$.

Now, by the definition of $\Gamma_0(R)$, $ab = 0$ implies that a and b are adjacent in $\Gamma_0(R)$.

Thus, every edge in $\bigcap_{i=1}^{k} G_{P_i}$ is also an edge in $\Gamma_0(R)$.

Conversely, let a and b be adjacent in $\Gamma_0(R)$. Then $ab = 0$, and hence $ab \in P_i$ for every i. Since each P_i is prime, either $a \in P_i$ or $b \in P_i$. Moreover, $N = \{0\}$ because R is reduced, and therefore $P_i + N = P_i$. Hence, a and b are adjacent in G_{P_i} for every i. Thus,$\Gamma_0(R) \subseteq \bigcap_{i=1}^{k} G_{P_i}$.

Therefore $\bigcap_{i=1}^{k} G_{P_i}$ is a zero divisor graph $\Gamma_0(R)$.

□

5. Subgraphs Induced by Idempotent and Nilpotent Elements

In this section, we consider $G_J[I]$ and $G_J[N]$, the subgraphs induced by the idempotent and nilpotent elements of R, respectively. The results determine the conditions under which the idempotent-induced subgraph is complete multipartite and the nilpotent-induced subgraph is complete.

Graphs defined an orthogonal idempotent elements of $\mathbb{Z}_n$ have been studied in [32]. The following results instead examine the subgraphs of G_J induced by the set I of idempotent elements.

Theorem 5.1. *Let J be the primary ideal of R. Then $G_J[I]$ is a complete $(2^{s-1} + 1)$ partite graph where s is the number of distinct primes of R.*

Proof. Let I be the set of all idempotent elements of R.

Let $n = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}$, where $p_1, p_2, \ldots, p_s$ are distinct primes. Then $R = \mathbb{Z}_n$ has exactly 2^s idempotent elements [32, Theorem 2]. where s is the number of distinct primes.

For every $e \in I$, $e(1-e) = 0 \in J$. Since J is primary, $e \in J$ or $(1-e)^m \in J$ for some $m \geq 1$.

Suppose $e \notin J$. Then $(1-e)^m \in J$. As $(1-e)$ is idempotent, $(1-e)^m = 1-e$ and $(1-e) \in J$. Hence $e \in J$ or $(1-e) \in J$.

Suppose both e and $(1-e)$ belongs to J. Then $e + 1 - e = 1 \in J$.

This implies that $J = R$. Since J is a proper ideal, which contains any one of the idempotent element in every complementary pair $(e, 1-e)$. As R has 2^s idempotent elements, there are 2^{s-1} complementary pairs. Therefore $|I \cap J| = 2^{s-1}$ and $|I \setminus J| = 2^s - 2^{s-1} = 2^{s-1}$.

Now let $G_J[I]$ be the graph induced by the set of all idempotent elements of G_J.

Choose the arbitrary idempotent elemnts $e, f \in I \cap J$ with $e \neq f$.

Then $ef \in J$ implies that $ef \in J'$. since J' is a prime ideal, either $e \in J'$ or $f \in J'$. Hence there is an edge between e and f in $G_J[I]$.

Thus any two idempotent elements of J are adjacent in $G_J[I]$.

Let us take two distinct elements $e, f \in I$ which is not contained in J.

Suppose that $ef \in J$. Since J is primary, $ef \in J$ primary, $ef \in J$ implies $e \in J$ or $f^m \in J$ for some $m \geq 1$.

As f is idempotent, $f^m = f$. It follows that $e \in J$ or $f \in J$ which is a contradiction.

Thus $ef \notin J$ and e and f are not adjacent in $G_J[I]$.

Hence no two distinct elements of $I \setminus J$ are adjacent in $G_J[I]$.

If $e \in I \cap J$ and $f \in I \setminus J$ then $ef \in J$ this implies $ef \in J'$.

Since J' is a prime ideal, e or $f \in J'$. Hence e and f are adjacent.

Thus every idempotent element of J is adjacent to every element of $I \setminus J$. In the above cases, every element of $I \cap J$ is adjacent to every other elements of I.

Hence each element of $I \cap J$ is a singleton independent set.

As $I \cap J$ contains 2^{s-1} elements. there are 2^{s-1} such independent sets and also $I \setminus J$ is an independent set.

Therefore, $G_J[I]$ is a complete $(2^{s-1}+1)$ partite graph. □

Example 5.2. Let $R = \mathbb{Z}_{18}$ and $J = \langle 2 \rangle$.The ideal J is prime and hence primary. As R has $s = 2$ distinct prime divisors, The set of idempotent elements of R is $I = \{0, 1, 9, 10\}$. Also, $I \cap J = \{0, 10\}$ contains $2^{s-1} = 2$ elements. These elements form two singleton partite sets, while $I \setminus J = \{1, 9\}$ forms one independent set. Hence, the total number of partite sets is $2^{s-1} + 1 = 2^{2-1} + 1 = 3$. Therefore,

$$G_{\langle 2 \rangle}(\mathbb{Z}_{18})[I] \cong K_{1,1,2}.$$

Thus, the induced graph is a complete 3-partite graph with four vertices and five edges.

Corollary 5.3. *Let J_1 and J_2 be the primary ideals of R_1 and R_2 respectively. If R_1 and R_2 have the same number of distinct primes then $G_{J_1}[I_1] \cong G_{J_2}[I_2]$.*

Proof. By Theorem 5.1, both induced graphs are complete $(2^{s-1}+1)$-partite graphs. There exists a bijection between their vertex sets preserves adjacency and non-adjacency. Hence, the two induced graphs are isomorphic. □

Corollary 5.4. *Let J_1 and J_2 be the distinct primary ideals of R. Then $G_{J_1}[I] \cong G_{J_2}[I]$.*

Theorem 5.5. *Let J be a primary ideal of R with $I \cong Z_2^s$. Then $G_J[I] \cong G_{P_i}$ for every prime ideal P_i of Z_2^s.*

Proof. By Theorem 5.1,

$$G_J[I] \cong K_{\underbrace{1,\ldots,1}_{2^{s-1}\text{ times}},\, 2^{s-1}}.$$

Thus, $G_J[I]$ is a complete $(2^{s-1}+1)$-partite graph whose vertex partition consists of 2^{s-1} singleton sets and one independent set of cardinality 2^{s-1}.

Let P be an arbitrary prime ideal of $\mathbb{Z}_2^s$, and let G_P denote the N-prime ideal graph associated with P. Then, for some $i \in \{1, 2, \ldots, s\}$,$P = \{(x_1, x_2, \ldots, x_s) \in \mathbb{Z}_2^s : x_i = 0\}$. Hence, $|P| = 2^{s-1}$.

Every element of P is adjacent to every other vertex of G_P, whereas the vertices in $\mathbb{Z}_2^s \setminus P$ form an independent set.

Thus, G_P has the complete $(2^{s-1}+1)$-partite form as $G_J[I]$.

Therefore, $G_J[I] \cong G_P$. for every prime ideal P of $\mathbb{Z}_2^s$. □

Example 5.6. Let $R = \mathbb{Z}_{30}$ and $J = \langle 2 \rangle$. Since $R/J \cong \mathbb{Z}_2$, the ideal J is prime and hence primary. Here $s = 3$, and the idempotent elements of R are $I = \{0, 1, 6, 10, 15, 16, 21, 25\} \cong \mathbb{Z}_2^3$. The set $C = I \cap J = \{0, 6, 10, 16\}$ forms a clique and is adjacent to every vertex of $F = I \setminus J = \{1, 15, 21, 25\}$, while F is independent. Hence, $G_{\langle 2 \rangle}(\mathbb{Z}_{30})[I] \cong K_{1,1,1,1,4}$. Let P be an arbitrary prime ideal of $\mathbb{Z}_2^3$, and let G_P denote the N-prime ideal graph associated with P. For some $i \in \{1, 2, 3\}$, $P = \{(x_1, x_2, x_3) \in \mathbb{Z}_2^3 : x_i = 0\}$, and both P and $\mathbb{Z}_2^3 \setminus P$ contain four elements. If $x \in P$, then $xy \in P$ for every $y \in \mathbb{Z}_2^3$, so x is adjacent to every other vertex of G_P. If $x, y \notin P$, then $x_i = y_i = 1$ and $xy \notin P$, so x and y are nonadjacent. Thus, $G_P \cong K_{1,1,1,1,4}$ and hence $G_J[I] \cong G_P$. The graph has eight vertices and 22 edges.

Corollary 5.7. *Let $s \geq 1$. If J is the prime ideal of $\mathbb{Z}_{2^s}$ and P is any prime ideal of $\mathbb{Z}_2^s$, then $G_J \cong G_P$.*

Example 5.8. Consider $s = 3$. The prime ideal of $\mathbb{Z}_8$ is $J = \langle 2 \rangle$, and let P be any prime ideal of $\mathbb{Z}_2^3$. By corollary 5.5, $G_J \cong G_P$. In fact, both graphs are isomorphic to $K_{1,1,1,1,4}$.

Graphs defined through nilpotent elements of rings have been studied in [13]. The following theorem instead concerns the subgraph of G_J induced by the nilradical N.

Theorem 5.9. *Let J be a proper ideal of R. If $N^2 \subseteq J$ then $G_J[N] \cong K_{|N|}$.*

Proof. Let a,b ∈ N be distinct.

Since $N^2 \subseteq J$, ab∈ J. Also $a, b \in N \subseteq J + N$.

Hence there is an edge between a and b in G_J. Thus every two distinct elements of N are adjacent.

Hence N induces a complete subgraph of G_J.

Therefore, $G_J[N] \cong K_{|N|}$. □

Example 5.10. Let $R = \mathbb{Z}_{81}$ and $J = \langle 9 \rangle$. The nilradical of R is $N = \langle 3 \rangle$, with $|N| = 27$. Moreover,

$$N^2 = \langle 9 \rangle = J.$$

Thus, for any two distinct elements $a, b \in N$, we have $ab \in J$ and $a, b \in N \subseteq J + N$. Hence,

$$G_J[N] \cong K_{27}.$$

Therefore, the induced graph has twenty-seven vertices and $|E| = \binom{27}{2} = 351$ edges.

6. Special Classes of N-Prime Ideal Graphs

In this section, three well-known graph classes are considered in the context of N-prime ideal graphs, namely cographs, split graphs, and threshold graphs. Throughout this section, R denotes the ring of integers modulo n. The conditions on the ring R and the ideal J are given under which G_J belongs to each of these classes. These graph classes admit classical forbidden induced subgraph characterizations [22, 23, 25], which are applied to the results presented below.

Theorem 6.1. *Let J be a proper primary ideal of R. Then the N-prime ideal graph G_J is a cograph.*

Proof. Suppose that G_J is not a cograph.

Then G_J contains an induced path P_4 with four distinct vertices $a, b, c, d \in R$.

Since J is primary, $\sqrt{J} =< p >$. it follows that $J + N = \sqrt{J} =< p >$. Hence, whenever $ab \in J$ implies $a \in J'$ or $b \in J'$.

Thus every pair whose product belongs to J satisfies the N-prime ideal graph adjacency condition. Since the path is induced, ac and bd are non-edges.This implies that $ac \notin J$ and $bd \notin J$.

Now consider the colon ideals $(J : b) = \{x \in R : xb \in J\}$ and $(J : c) = \{x \in R : xc \in J\}$. Since $J = \langle p^\alpha \rangle$, the colon ideals $(J : b)$ and $(J : c)$ are generated by powers of p. Hence,$(J : b) \subseteq (J : c)$ or $(J : c) \subseteq (J : b)$. If $(J : b) \subseteq (J : c)$ then $ab \in J$ implies $a \in (J : b) \subseteq (J : c)$, hence $ac \in J$ and if $(J : c) \subseteq (J : b)$ then $cd \in J$ implies $d \in (J : c) \subseteq (J : b)$, hence $bd \in J$. which is the contradiction.

Thus, G_J contains no induced P_4. By the standard P_4-free characterization of cographs [23], G_J is a cograph.

□

Example 6.2. Let $R = \mathbb{Z}_{32}$ and $J = \langle 8 \rangle$. Since J is a primary ideal, $\sqrt{J} = \langle 2 \rangle = N$. Partition the vertex set as

$$A = \langle 8 \rangle, \quad B = \langle 4 \rangle \setminus \langle 8 \rangle, \quad C = \langle 2 \rangle \setminus \langle 4 \rangle, \quad D = R \setminus \langle 2 \rangle.$$

Here, $|A| = 4$, $|B| = 4$, $|C| = 8$, and $|D| = 16$. The vertices of A form a clique and are adjacent to every other vertex. The vertices of B form a clique and are adjacent to every vertex of C, but not to the vertices of D. Both C and D are independent sets, and there are no edges between them. Consequently,

$$G_{\langle 8 \rangle}(\mathbb{Z}_{32}) \cong K_4 \vee \left((K_4 \vee \overline{K_8}) \cup \overline{K_{16}}\right),$$

where $\vee$ denotes graph join and $\cup$ denotes disjoint union. Since cographs are closed under graph join and disjoint union, $G_{\langle 8 \rangle}(\mathbb{Z}_{32})$ is a cograph. Moreover, it has 32 vertices and

$$|E| = \binom{4}{2} + 4(28) + \binom{4}{2} + 4(8) = 156$$

edges.

Remark

If J is a proper ideal of R such that every admissible path a–b–c–d satisfies $ac \in J$ or $ad \in J$ or $bd \in J$, with the corresponding N-prime ideal graph adjacency condition, then G_J is P_4-free and hence a cograph.

Theorem 6.3. *Let $J = \langle d \rangle$ be a proper ideal of R. Then the N-prime ideal graph G_J is a split graph iff d is either a prime power or cube free.*

Proof. Let $d = p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}...p_s^{\alpha_s}$ be the prime factorization of d.

Suppose that d is cube free then $d = p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}...p_s^{\alpha_s}$ where each $\alpha_i \leq 2$

Let $r = p_1.p_2.p_3...p_s$. Since $J' = J + N, J' = \langle r \rangle = \langle rad(d) \rangle$.

Let $x, y \in J'$ be distinct. Since $J' = \langle r \rangle$, there exist $a, b \in R$ such that $x = ra$ and $y = rb$. Since d is cube-free and $r = \text{rad}(d)$, every prime divisor of d occurs in d with exponent at most two. Hence, $d \mid r^2$, and therefore $xy = r^2ab \in \langle d \rangle = J$. Moreover, $x, y \in J'$. Thus, x and y are adjacent in G_J, and hence J' induces a clique.

Now let $x, y \in R \setminus J'$ be distinct. Neither x nor y belongs to J'. Therefore, x and y are nonadjacent. Hence, $R \setminus J'$ is an independent set. Finally, suppose that $x \in J'$ and $y \notin J'$. then $x = ra$ for some $a \in R$. Since $r \mid d$, $d = rq$, where $q = d/r$. If $q \mid ay$, then $ay = qk$ for some $k \in R$, and hence $xy = (ra)y = rqk = dk \in J$. Since $x \in J'$, the vertices x and y are adjacent in G_J. If $q \nmid ay$, then $xy \notin J$, and therefore x and y are nonadjacent. Thus, J' induces a clique and $R \setminus J'$ induces an independent set. Hence, G_J is a split graph. Now suppose that $d = p^k$ for some prime p. Let $t = \lceil k/2 \rceil$ and $H = \langle p^t \rangle$.

Take distinct two elements $x, y \in H$. Then $x = p^t a$, $y = p^t b$ for some a,b belongs to R. Hence $xy = p^{2t}ab$.

Since $2t \geq k$, the product xy belongs to $J = \langle p^k \rangle$. Also $H \subseteq J'$. Hence every two distinct elements of H are adjacent, So H is a clique.

Now take $x, y \notin H$. Each of x and y contains fewer than t factors of p.

Therefore their product contains fewer than k factor of p, and hence $xy \notin J$.

Thus x and y are not adjacent.

Hence $R \setminus H$ is independent.

Therefore The vertex set has been partitioned into a clique and an independent set. Hence, G_J is a split graph [25].

Conversely , suppose that G_J is split graph and that d is neither a prime power nor cube free then d contains atleast two distinct prime factors, and one them occurs atleast three times (i.e) $d = p^k m$ where $k \geq 3$, $m > 1$ and p does not occurs as a factors of m.

Choose $u > 1$ such that p is not a factor of u, and consider a=p^{k-1}, b=pm, c=p^k, z=upm are four distinct vertices of R. Then the products ab,bc,cz,az belongs to J.Moreover $b, z \in J'$.

Hence a and b, b and c, c and z, z and a are adjacent.

However ac= p^{2k-1} contains no factor from m, so $ac \notin J$.

Also $bz = up^2m^2$ contains only two factors of p, whereas J requires atleast $k \geq 3$. Hence $bd \notin J$.

Therefore a and c, b and z are not adjacent.

Thus the four vertices a,b,c,z induce a C_4, which is impossible in a split graph.

Hence d must be either a prime power or cube free.

Therefore G_J is a split graph iff d is either a prime power or cube free. □

Example 6.4 (Prime-power case). Let $R = \mathbb{Z}_{20}$ and $J = \langle 4 \rangle$. The nilradical of R is $N = \langle 10 \rangle$, and hence $J' = \langle 2 \rangle$. Let $C = J'$, the set of ten even elements of R, and let $H = R \setminus J'$, the set of ten odd elements. If $x, y \in C$ are distinct, then $x = 2a$ and $y = 2b$ for some $a, b \in R$. Therefore,$xy = 4ab \in \langle 4 \rangle = J$. Since $x, y \in J'$, the vertices x and y are adjacent. Thus, C induces a clique. If $x, y \in H$ are distinct, then neither x nor y belongs to J', and hence they are nonadjacent. Therefore, H induces an independent set. Consequently, (C, H) is a split partition of the vertex set, and G_J is a split graph.

Example 6.5 (Cube-free case). Let $R = \mathbb{Z}_{60}$ and $J = \langle 12 \rangle$, We have $N = \langle 30 \rangle$ and $J + N = \langle 6 \rangle$. The set $C = \langle 6 \rangle$ contains ten vertices and forms a clique, since the product of any two multiples of 6 belongs to $\langle 12 \rangle$. The remaining set $H = R \setminus \langle 6 \rangle$ contains fifty vertices and is independent. Therefore, (C, H) is a split partition, and $G_{\langle 12 \rangle}$ is a split graph.

Corollary 6.6. *Let J be a primary ideal of R. Then the N-prime ideal graph G_J is a threshold graph.*

Proof. By Theorem 6.1, G_J is P_4-free, and by Theorem 6.2, it is C_4-free and $2K_2$-free. Therefore, the forbidden-subgraph characterization of threshold graphs [22] shows that G_J is a threshold graph. □

Example 6.7. Let $R = \mathbb{Z}_{27}$ and $J = \langle 9 \rangle$. Since $J = \langle 3^2 \rangle$, it is a primary ideal with $\sqrt{J} = \langle 3 \rangle = N$. Put $A = \langle 9 \rangle$, $B = \langle 3 \rangle \setminus \langle 9 \rangle$, and $C = R \setminus \langle 3 \rangle$. Here, $|A| = 3$, $|B| = 6$, and $|C| = 18$. The set A consists of universal vertices, B is a clique, C is independent, and there are no edges between B and C.

Since $A \cup B = \langle 3 \rangle$ is a clique and C is independent, $(A \cup B, C)$ is a split partition. Moreover, an induced subgraph containing a vertex of A cannot be P_4, C_4, or $2K_2$, because every vertex of A is universal. Such a subgraph also cannot occur in $B \cup C$, because B is a clique, C is independent, and there are no edges between them. Therefore, the graph is $\{P_4, C_4, 2K_2\}$-free and hence is a threshold graph. Furthermore,

$$|V| = 27 \qquad \text{and} \qquad |E| = \binom{6}{2} + 3(24) + \binom{3}{2} = 90.$$

7. Conclusion

In this paper, we introduced and investigated the N-prime ideal graph G_J associated with a proper ideal J of a finite commutative ring R. The study shows how the algebraic structure of the ring and its ideals influence the resulting graph. For reduced rings, including rings of prime order, the graph corresponding to the zero ideal is a star graph. For $R = \mathbb{Z}_{p^2}$ with $p \geq 3$, it has a pineapple structure. When $R = \mathbb{Z}_n$ and $J = \langle d \rangle$, where d is either a prime or a product of distinct prime divisors satisfying the stated conditions, the graph is complete multipartite. We also obtained isomorphisms between certain prime-power graphs, studied the subgraphs induced by nilpotent and idempotent elements, and established connections with the zero-product graph and the product maximal graph. Under the appropriate hypotheses on primary ideals and their generators, the associated graphs exhibit cograph, split, and threshold properties.

These results illustrate the close relationship between the algebraic properties of ideals and the structural properties of their associated graphs. At the same time, graph operations do not always behave exactly like the corresponding ideal operations; in particular, intersections of N-prime ideal graphs may require additional hypotheses. Further research may extend this study to other classes of finite local and product rings, determine when graph intersections preserve the N-prime ideal graph structure, and investigate additional parameters such as domination, spectrum, connectivity, and coloring. Computational analysis of larger finite rings may also provide new examples and support broader generalizations.

Symbols

R	Finite Commutative Ring with Unity
P	Prime Ideal of R
J	Proper Ideal of R
J'	Sum of the Ideal J and the Nilradical N

G_J	N-prime Ideal Graph Associated with J
N	Nilradical of R
I	Set of All Idempotent Elements of R
$G_J[I]$	Subgraph of G_J Induced by the Idempotent Elements of R
$G_J[N]$	Subgraph of G_J Induced by the Nilpotent Elements of R
$Z(R)$	Set of All Zero Divisors of R
$\mathbb{Z}_n$	Ring of Integers Modulo n
$\Gamma_0(R)$	All-element Zero-product Graph of R
$\Gamma_{pm}(R)$	Product Maximal Graph Associated with R

Author Contributions

Kalamani Duraisamy: Conceptualization, Methodology, Supervision, Validation, Writing – review & editing

Hariharan Murugan: Conceptualization, Investigation, Methodology, Visualization, Writing – original draft

Conflicts of Interest

The authors declare no conflicts of interest.

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