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Combinatorial Properties, Invariants and Structures Associated with the Direct Product of Alternating and Cyclic Groups Acting on the Cartesian Product of Two Sets

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Abstract: In relation to group action, much research has focused on the properties of individual permutation groups acting on both ordered and unordered subsets of a set, particularly within the Alternating group and Cyclic group. However, the action of the direct product of Alternating group and Cyclic group on the Cartesian product of two sets remains largely unexplored, suggesting that some properties of this group action are still undiscovered. This research paper therefore, aims to determine the combinatorial properties - specifically transitivity and primitivity - as well as invariants which includes ranks and subdegrees of this group action. Lemmas, theorems and definitions were utilized to achieve the objectives of study with significant use of the Orbit-Stabilizer theorem and Cauchy-Frobeneus lemma. Therefore in this paper, the results shows that for any value of $n \geqslant 3$, the group action is transitive and imprimitive. Additionally, we found out that when $n = 3$, the rank is 9 and the corresponding subdegrees are ones repeated nine times that is, $1, 1, 1, 1, 1, 1, 1, 1, 1$. Also, for any value of $n \geqslant 4$, the rank is $2n$ with corresponding subdegrees comprising of n suborbits of size 1 and n suborbits of size $(n-1)$.

Keywords: Ranks, Subdegrees, Transitivity, Primitivity, Direct Product, Cartesian Product, Alternating Group, Cyclic Group

1. Introduction

Consider two groups, Alternating group denoted as A_n and Cyclic group denoted as C_n acting on disjoint sets X and Y respectively. Then the group action of $A_n \times C_n$ acting on $X \times Y$ is defined as $(g_1, g_2)(x, y) = (g_1x, g_2y)$ $\forall$ $g_1 \in A_n$, $g_2 \in C_n$, $x \in X$ and $y \in Y$.

This paper will determine the transitivity, primitivity, ranks and subdegrees of group action $A_n \times C_n$ acting on $X \times Y$ when $n = 3, 4$ and 5 before generalizing for $n \geq 3$.

2. Notation and Preliminary Results

Definition 2.1. [1] A set G together with a binary operation is said to be a *group* if it satisfies the axioms of closure, associativity, identity and inverse.

Definition 2.2. [2] Let X be a non-empty set, then a permutation is a bijective mapping from set X to itself. The group is said to be Symmetric if it consist of all permutations of a set X and is denoted as S_n. The subgroup of S_n that consist of all even permutations is referred to as the *Alternating* group and is denoted by A_n whose order is $|A_n| = \frac{n!}{2}$.

Definition 2.3. [3] A *Cyclic* group is a group that can be generated by a single element $g \in G$ such that $G = \langle g \rangle$. It is denoted as C_n and has an order of n.

Definition 2.4. [4] A group G is said to act on the left side of set X if the following axioms are satisfied:

1. There is a unique element $gx \in X$ for all $g \in G$ and $x \in X$.
2. $ex = x$ for all $x \in X$ where e is an identity element of G.

3. $g_1(g_2x) = (g_1g_2)x$ for all $g_1, g_2 \in G$ and $x \in X$.

Definition 2.5. [5] Suppose that G acts on X. Then a point is said to be *fixed* if the elements of set X is fixed by element of a group. It is denoted as $Fix_G(x)$ and given by : $Fix_G(x) = \{x \in X : gx = x\}$.

Definition 2.6. [6] Suppose that G acts on X. Then the *stabilizer* of point x in G is the set of elements of the group which leaves elements of a set unchanged under group action. It is denoted as $Stab_G(x)$ and given by: $Stab_G(x) = \{g \in G : gx = x\}$.

Definition 2.7. [7] Suppose G acts on X. Then the *orbits of point* x in G are the partitions under the group action that represents the equivalence classes of elements of a set when mapped by the elements of the group. It is denoted as $Orb_G(x)$ and given by $Orb_G(x) = \{gx : g \in G\}$.

Definition 2.8. [8] The action of G on X is said to be *transitive* if there is only one orbit. That is, for any two elements $x, y \in X$, there exist an element g in G such that $y = gx$. Therefore, the action is transitive if and only if $|orb_G(x)| = |X|$.

Theorem 2.1. (Orbit-Stabilizer Theorem)[9]. Suppose that G acts on a set X for all $x \in X$. Then the number of element in the orbit of x is the index of the stabilizer of x in G given by: $|Orb_G(x)| = |G : Stab_G(x)|$.

Lemma 2.1. (Cauchy Frobeneus Lemma) [10]. Suppose that G acts on set X. Then the number of suborbits of G is given as $\frac{1}{|G|} \sum\limits_{g \in G} |Fix_G(x)|$.

Theorem 2.2. [11] Consider an Alternating group A_n acting on a set X. For each $x \in X$, the stabilizer of A_n is isomorphic to A_{n-1}.

Theorem 2.3. [12] Suppose a Cyclic group C_n acts on a set X. For each $x \in X$, the stabilizer $\mathrm{Stab}_{C_n}(x)$ is $\{e\}$ which is the identity.

Definition 2.9. [13] Suppose that G_1 and G_2 are permutation groups acting on the set X and Y respectively. Then $G_1 \times G_2$ is said to be a *direct product* acting on the Cartesian product $X \times Y$ with the group operation defined by $(g_1\, g_2)(x, y) = (g_1x, g_2y)$.

Definition 2.10. [14] Suppose G acts transitively on X and let U be a subset of X. Then U is said to be a *block action* if $gU = U$ (g fixes U) or $gU \cap U = \emptyset$ (g moves U entirely). If a group acts transitively on X and the results are an identity, the whole set X (all elements of X) and $\{x\}$, $\forall\ x \in X$, then the group action said to have trivial blocks. Therefore, a group action with trivial blocks is said to be *primitive*. If not, it is said to be *imprimitive*.

Definition 2.11. [15] Suppose G acts on X transitively and $Stab_G(x)$ is the stabilizer of points x in G. Then the *suborbits* are the orbits of the stabilizers which arises when the elements of the stabilizer act on set X. They are denoted as Δ_0 and defined given by: $\Delta_0 = \{x\}, \Delta_1, \Delta_2, ..., \Delta_{r-1}$ where r represents the rank. The total number of suborbits is the *rank* and the size of these suborbits is the *subdegrees*.

3. Main Results

3.1. Transitivity of $A_3 \times C_3$ on $X \times Y$

Lemma 3.1. Let $G = A_3 \times C_3$.Then the action of G on $X \times Y$ is transitive.

Proof. Let $G = A_3 \times C_3$ such that
$A_3 = \{e_x, (x_1\, x_2\, x_3), (x_1\, x_3\, x_2)\}$ and
$C_3 = \{e_y, (y_1\, y_2\, y_3), (y_1\, y_3\, y_2)\}$.

Therefore, the elements of G generated by
$\langle\{(x_1\, x_2\, x_3), (y_1\, y_2\, y_3)\}\rangle$ are given as:
$\{(e_x, e_y), (e_x, (y_1\, y_2\, y_3)), (e_x, (y_1\, y_3\, y_2)),$
$((x_1\; x_2\; x_3), e_y), ((x_1\; x_2\; x_3), (y_1\; y_2\; y_3)), ((x_1\; x_3\; x_2), e_y),$
$((x_1\; x_2\; x_3), (y_1\; y_3\; y_2)), ((x_1\; x_3\; x_2), (y_1\; y_2\; y_3)),$
$((x_1\; x_3\; x_2), (y_1\; y_3\; y_2))\}. \Rightarrow |G| = 9.$

Also, let $V = X \times Y$ such that $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$.

Then the element of V are:
$\{(x_1, y_1), (x_2, y_1), (x_1, y_2), (x_3, y_1), (x_2, y_2), (x_1, y_3),$
$(x_3, y_2), (x_2, y_3), (x_3, y_3)\}$.

This shows that $|V| = 9$.

By definition 2.6, the $Stab_G(x_1, y_1) = \{(e_x\; e_y)\}$, the identity. Implies that $|Stab_G(x_1, y_1)| = 1$.

Applying theorem 2.1 (Orbit Stabilizer Theorem), it is suffices to show if the size of $|Orb_G(x_1, y_1)|$ is equals to $|X \times Y|$. Therefore,

$$\begin{aligned} |Orb_G(x_1, y_1)| &= |G| : |Stab_G(x_1, y_1)| \\ &= \frac{|G|}{|stab_G(x_1, y_1)|} \\ &= \frac{9}{1} = 9 \\ &= |X \times Y|. \end{aligned}$$

Hence, the group action is transitive.

3.2. Transitivity of $A_4 \times C_4$ on $X \times Y$

Lemma 3.2. Suppose that $G = A_4 \times C_4$, then G acts transitively on $X \times Y$.

Proof. Let $G = A_4 \times C_4$ such that the elements of $A_4 = \{e_x, (x_1\, x_2\, x_3), (x_1\, x_2\, x_4), (x_1\, x_3\, x_4), (x_1\, x_4\, x_3), (x_2\, x_4\, x_3),$
$(x_1x_3x_2), (x_1x_4x_2), (x_2x_3x_4), (x_1x_4)\ (x_2x_3), (x_1x_2)(x_3x_4),$
$(x_1\; x_3)(x_2\; x_3)\}$ and $C_4 = \{e_y, (y_1\; y_2\; y_3\; y_4), (y_1\; y_4\; y_3\; y_2),$
$(y_1\; y_3)(y_2\; y_4)\}$.

Then the size of elements of G is $|G| = 48$. Also, let $V = X \times Y$ such that $X = \{x_1, x_2, x_3, x_4\}$ and
$Y = \{\mathrm{y}_1, y_2, y_3, y_4\}$, then the of elements of V are: $\{(x_1, y_1),$
$(x_1, y_2), (x_1, y_3), (x_1, y_4), (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4),$
$(x_3, y_1), (x_3, y_2), (x_3, y_3), (x_3, y_4), (x_4, y_1), (x_4, y_2), (x_4, y_3),$
$(x_4, y_4)\}. \Rightarrow |V| = 16.$

From definition 2.6, the $Stab_G(x_1, y_1) = (e_x, e_y), ((x_2\, x_3\, x_4), e_y), ((x_2\, x_4\, x_3), e_y). \Rightarrow |Stab_G(x_1, y_1)| = 3.$

Applying theorem 2.1 (Orbit Stabilizer Theorem), it is suffices to show if the size of $|Orb_G(x_1, y_1)|$ is equals to $|X \times Y|$.

$$\begin{aligned}|Orb_G(x_1,y_1)| &= |G| : |Stab_G(x_1,y_1)| \\ &= \frac{|G|}{|stab_G(x_1,y_1)|} \\ &= \frac{48}{3} = 16 \\ &= |X \times Y|.\end{aligned}$$

Hence, the group action is transitive.

3.3. Transitivity of $A_5 \times C_5$ on $X \times Y$

Lemma 3.3. Suppose that $G = A_5 \times C_5$, then the action of G is transitive on $X \times Y$.

Proof. Let $G = A_5 \times C_5$ such that the elements of A_5 are:

$A_5 = \{e_x, (x_1\, x_5\, x_4), (x_1\, x_4\, x_5), (x_1\, x_2\, x_5), (x_1\, x_3\, x_5), (x_2\, x_5\, x_4), (x_1\, x_5)(x_2\, x_4), (x_1\, x_4\, x_2), (x_1\, x_2)(x_4\, x_5), (x_1\, x_3\; x_5\, x_4\, x_2), (x_2\, x_4\, x_5), (x_1\, x_5\, x_2), (x_1\, x_4)(x_2\, x_5), (x_1\, x_2\, x_4), (x_1\, x_3\, x_5\, x_2\, x_4), (x_2\, x_3\, x_5), (x_1\, x_5\, x_2\, x_3\, x_4), (x_1\, x_4\, x_5\, x_2\, x_3), (x_1\, x_2\, x_3), (x_1\, x_3)(x_2\, x_5), (x_3\, x_5\, x_4), (x_1\, x_5)(x_3\, x_4), (x_1\, x_4\, x_3), (x_1\, x_2\, x_5\, x_4\, x_3), (x_1\, x_3)(x_4\, x_5), (x_2\, x_5)(x_3\, x_4), (x_1\, x_5\, x_2\, x_4\, x_3), (x_2\, x_4\, x_3), (x_1\, x_4\, x_3\, x_5\, x_2), (x_1\, x_2)(x_3\, x_4), (x_1\, x_3\, x_4\, x_5\, x_2), (x_1\, x_5\, x_3), (x_1\, x_5\, x_4\, x_3\, x_2), (x_2\, x_3\, x_4), (x_1\, x_4\, x_3\, x_2\, x_5), (x_1\, x_2\, x_3\, x_4\, x_5), (x_1\, x_3\, x_2\, x_4\, x_5), (x_2\, x_3)(x_4\, x_5), (x_1\, x_5)(x_2\, x_3), (x_1\, x_4)(x_2\, x_3), (x_1\, x_2\, x_3\, x_5\, x_4), (x_1\, x_3\, x_2\, x_5\, x_4), (x_3 x_4\, x_5), (x_1\, x_3\, x_2), (x_1\, x_4)(x_3\, x_5), (x_1\, x_2\, x_5\, x_3\, x_4), (x_1\, x_3\, x_4), (x_2\, x_5\, x_3), (x_1\, x_5\, x_3\, x_2\, x_4), (x_1\, x_4\, x_5\, x_3\, x_2), (x_1\, x_2)(x_3\, x_5), (x_2\, x_4)(x_3\, x_5), (x_1\, x_5\, x_3\, x_4\, x_2), (x_1\, x_4\, x_2\, x_5\, x_3), (x_1\, x_2\, x_4\, x_5\, x_3), (x_1\, x_3)(x_2\, x_4), (x_1\, x_5\, x_4\, x_2\, x_3), (x_1\, x_4\, x_2\, x_3\, x_5), (x_1\, x_3\, x_4\, x_2\, x_5), (x_1\, x_2\, x_4\, x_3\, x_5)\}$.

The elements of C_5 are:

$C_5 = \{e_y, (y_1\, y_5\, y_4\, y_3\, y_2), (y_1\, y_4\, y_2\, y_5\, y_3), (y_1\, y_3\, y_5\, y_2\, y_4), (y_1\, y_2\, y_3\, y_4\, y_5)\}$.

Therefore, the size of elements of G generated by $\langle\{(x_1,x_2,x_3,x_4,x_5), (x_3,x_4,x_5), (y_1,y_2,y_3,y_4,y_5)\}\rangle$ are: $|G| = 300$. Also, let $V = X \times Y$ such that $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$.

Then the of elements of V are:

$\{(x_1,y_1), (x_2,y_1), (x_1,y_2), (x_3,y_1), (x_2,y_2), (x_1,y_3), (x_4,y_1), (x_3,y_5), (x_4,y_5), (x_3,y_2), (x_2,y_3), (x_1,y_4), (x_5,y_1), (x_4,y_2), (x_3,y_3), (x_2,y_4), (x_1,y_5), (x_5,y_2), (x_4,y_3), (x_3,y_4), (x_2,y_5), (x_5,y_3), (x_4,y_4), (x_5,y_4), (x_5,y_5)\}. \Rightarrow |V| = 25$.

By definition 2.6, the $Stab_G(x_1,y_1)$ of G are:

$Stab_G(x_1,y_1) = \{(e_x,e_y), ((x_2\, x_4\, x_5), e_y), ((x_2\, x_5\, x_4), e_y), ((x_2\, x_3\, x_5), e_y), ((x_3\, x_5\, x_4), e_y), ((x_2\, x_4\, x_3), e_y), ((x_3\, x_4\, x_5), e_y), ((x_2\, x_5\, x_3), e_y), ((x_2\, x_5)(x_3\, x_4), e_y), ((x_2\, x_3\, x_4), e_y), ((x_2\, x_3)(x_4\, x_5), e_y), ((x_2\, x_4)(x_3\, x_5), e_y)\}$.

Implying that $|Stab_G(x_1,y_1)| = 12$.

Applying theorem 2.1 (Orbit Stabilizer Theorem), it is suffices to show if the size of $|Orb_G(x_1,y_1)|$ is equals to $|X \times Y|$. Therefore,

$$\begin{aligned}|Orb_G(x_1,y_1)| &= |G| : |Stab_G(x_1,y_1)| \\ &= \frac{|G|}{|stab_G(x_1,y_1)|} \\ &= \frac{300}{12} = 25 \\ &= |X \times Y|\end{aligned}$$

Hence, the group action is transitive.

3.4. Transitivity of $A_n \times C_n$ on $X \times Y$

Theorem 3.1. For $n \geqslant 3$, the group action of $A_n \times C_n$ on $X \times Y$ is transitive where $X = \{x_1, x_2, x_3, ..., x_n\}$ and $Y = \{y_1, y_2, y_3, ..., y_n\}$.

Proof. For $n \geqslant 3$, it is suffices to show that the order of $Orb_G(x_1,y_1)$ is equivalent to the order of $X \times Y$. Let $G = A_n \times C_n$. Since $|A_n| = \frac{n!}{2}$ and $|C_n| = n$, then $|G| = |A_n \times C_n| = \frac{n \times n!}{2}$. Again, let $V = X \times Y$, then $|V| = n \times n = n^2$. By theorem 2.2 and theorem 2.3, the order of stabilizers of $|Stab_{A_n}(x_1)| = \frac{(n-1)!}{2}$ and $|Stab_{C_n}(y_1)| = \{e_y\}$ respectively such that $|Stab_G(x_1,y_1)| = |Stab_{A_n}(x_1)| \times |Stab_{C_n}(y_1)| = \frac{(n-1)!}{2} \times 1 = \frac{(n-1)!}{2}$. Applying theorem 2.1 (Orbit Stabilizer Theorem):

$$\begin{aligned}|Orb_G(x_1,y_1)| &= \frac{|G|}{|Stab_G(x_1,y_1)|} \\ &= \frac{\frac{n \times n!}{2}}{\frac{(n-1)!}{2} \times 1} = \frac{n \times n!}{(n-1)!} \quad \text{Since } n! = n \times (n-1)! \\ &= \frac{n \times n \times (n-1)!}{(n-1)!} \\ &= \frac{n^2 \times \cancel{(n-1)!}}{\cancel{(n-1)!}} \\ &= n^2 \\ &= |X \times Y|\end{aligned}$$

Hence, the group action of $A_n \times C_n$ on $X \times Y$ is transitive.

3.5. Primitivity of $A_3 \times C_3$ on $X \times Y$

Lemma 3.4. Given that $G = A_3 \times C_3$, then G acts imprimitively on $X \times Y$.

Proof. Consider $G = A_3 \times C_3$ and $V = X \times Y$ such that $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ where G acts transitively on V. From lemma 3.1, the elements of V are $\{(x_1, y_1), (x_2, y_1), (x_1, y_2), (x_3, y_1), (x_2, y_2), (x_1, y_3), (x_3, y_2), (x_2, y_3), (x_3, y_3)\}$. Implies $|V| = 9 = 3 \times 3 = 3^2$. Now, by definition 2.10, let U be a subset of V such that $|U|$ divides $|V|$ by any divisor of $|V|$. For this case, let $|U| = \{(x_1, y_1), (x_2, y_2), (x_3\ , y_3)\} = 3$ such that $\frac{|V|}{|U|} = \frac{3^2}{3}$. Then, there exist $g \in G$ which moves the elements of U to the elements that is not in U such that $gU \cap U = \emptyset$. For instance, from lemma 3.1 let $g = (e_x, (y_1\ y_2\ y_3))$ such that $gU = (e_x, (y_1\ y_2\ y_3))(x_1, y_1) = (x_1, y_2) \Rightarrow gU \cap U = (x_1, y_2) \cap (x_1, y_1) = \emptyset$. By this argument, it is shows that the U is a non-trivial block action and the conclusion follows that the action group is imprimitive

3.6. Primitivity of $A_n \times C_n$ on $X \times Y$

Theorem 3.2. Suppose that $G = A_n \times C_n$ on $X \times Y$, then G acts on $X \times Y$ and $V = X \times Y$ imprimitively.

Proof. Consider $G = A_n \times C_n$ acting on $X \times Y$ and $V = X \times Y$ such that $X = \{x_1, x_2, x_3, ..., x_n\}$ and $Y = \{y_1, y_2, y_3, ..., y_n\}$ where G acts on V transitively. By theorem 3.1, $|V| = n \times n = n^2$. From definition 2.10, let U be a subset of V such that $|U|$ divides $|V|$. For this case, let $|U| = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), ..., (x_n, y_n)\} = n$ such that $\frac{|V|}{|U|} = \frac{n^2}{n}$. There exist $g \in G$ which moves the elements of U to the elements not in U such that $gU \cap U = \emptyset$. By this argument, it is sufficient that U is a nontrivial block action and the conclusion follows that the action is imprimitive.

3.7. Ranks and Subdegrees of $A_3 \times C_3$ on $X \times Y$

Lemma 3.5. Suppose that the group action of $A_3 \times C_3$ acts on $X \times Y$ then the rank is 9 and the corresponding subdegrees are $1, 1, 1, 1, 1, 1, 1, 1, 1$.

Proof. Since $G = A_3 \times C_3$ and $V = X \times Y$. From lemma 3.1, the elements of V are $\{(x_1, y_1), (x_2, y_1), (x_1, y_2), (x_3, y_1), (x_2, y_2), (x_1, y_3), (x_3, y_2), (x_2, y_3), (x_3, y_3)\}$.

Also, the elements of the stabilizer are $Stab_G(x_1, y_1) = \{(e_x, e_y)\}$, which is an identity. Therefore, the number of elements of V fixed by each $g \in G$ is 9 since the identity fixes all the elements of V as shown in the table below.

Table 1. *Permutations in $Stab_G(x_1, y_1)$ and the Number of Fixed Points.*

Permutation types in $Stab_G(x_1, y_1)$	Number of permutations (H)	$Fix_G(g)$
(e_x, e_y)	1	9
TOTAL	1	

By lemma 2.1 (Cauchy Frobeneus Lemma), the number of suborbits of G acting on V is given as:

$\frac{1}{H} \sum\limits_{g \in G} |Fix_G(g)| = \frac{1}{1}[1 \times 9] = 9.$

This shows that there are 9 suborbits classified into four possible categories, as shown below.

Note: Let $K = (x_1, y_1)$ be a distinct point in set V. If an element (the identity) in G acts on K and the point remains fixed, then K forms trivial orbit under the group action. Therefore, these possible categories are:

a) Suborbit where both the first and the second components are identical to those of K.

$\Delta_0 = Orb_H(x_1, y_1) = \{(x_1, y_1)\} = 1$ (Trivial Orbit)

b) Suborbits where the first component is identical to that of K but the second components are different.

$\Delta_1 = Orb_H(x_1, y_2) = \{(x_1, y_2)\} = 1$

$\Delta_2 = Orb_H(x_1, y_3) = \{(x_1, y_3)\} = 1$

c)Suborbits where the second component is identical to that of K but the first components are different.

$\Delta_3 = Orb_H(x_2, y_1) = \{(x_2, y_1)\} = 1$

$\Delta_4 = Orb_H(x_3, y_1) = \{(x_3, y_1)\} = 1$

d) Suborbits where neither the first nor the second components are identical to those of K.

$\Delta_5 = Orb_H(x_2, y_2) = \{(x_2, y_2)\} = 1$

$\Delta_6 = Orb_H(x_2, y_3) = \{(x_2, y_3)\} = 1$

$\Delta_7 = Orb_H(x_3, y_2) = \{(x_3, y_2)\} = 1$

$\Delta_8 = Orb_H(x_3, y_3) = \{(x_3, y_3)\} = 1$

From the above sequence, it shows that the group action has a rank of 9 and its corresponding subdegrees being $1, 1, 1, 1, 1, 1, 1, 1, 1$.

3.8. Ranks and Subdegrees of $A_4 \times C_4$ on $X \times Y$

Lemma 3.6. Let $G = A_4 \times C_4$ acts on $X \times Y$. Then the rank is 8 and the subdegrees are $1, 1, 1, 1, 3, 3, 3, 3$.

Proof. Let $G = A_4 \times C_4$ and $V = X \times Y$. From lemma 3.2, the elements of V are $\{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4), (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4), (x_3, y_1), (x_3, y_2), (x_3, y_3), (x_3, y_4), (x_4, y_1), (x_4, y_2), (x_4, y_3), (x_4, y_4)\}$.

Also, the elements of the stabilizers are: $Stab_G(x_1, y_1) = \{(e_x, e_y), ((x_2\ x_3\ x_4), e_y), ((x_2\ x_4\ x_3), e_y)\}$. Therefore, the number of elements of V fixed by each $g \in G$ are as shown in the table below:

Table 2. *Permutations in $Stab_G(x_1, y_1)$ and the Number of Fixed Points.*

Type of permutation in $Stab_G(x_1, y_1)$	Number of permutations (H)	$Fix_G(g)$
(e_x, e_y)	1	16
$((a\ b\ c), e_y)$	2	4
TOTAL	3	

By lemma 2.1 (Cauchy Frobeneus Lemma), the number of suborbits of G acting on V is given as: $\frac{1}{H}\sum_{g\in G}|Fix_G(g)| = \frac{1}{3}[(1\times 16)+(2\times 4)] = \frac{24}{3} = 8$.

This shows that there are 8 suborbits classified into the following four possible categories:

a) Suborbit where both the first and the second components are identical to those of K.

$\Delta_0 = Orb_H(x_1, y_1) = \{(x_1, y_1)\} = 1$ (Trivial Orbit)

b) Suborbits where the first component is identical to that of K but the second components are different.

$\Delta_1 = Orb_H(x_1, y_2) = \{(x_1, y_2)\} = 1$

$\Delta_2 = Orb_H(x_1, y_3) = \{(x_1, y_3)\} = 1$

$\Delta_3 = Orb_H(x_1, y_4) = \{(x_1, y_4)\} = 1$

c) Suborbit where the second component is identical to that of K but the first components are different.

$\Delta_4 = Orb_H(x_2, y_1) = \{(x_2, y_1), (x_3, y_1), (x_4, y_1)\} = 3$

d) Suborbits where neither the first nor the second components are identical to those of K.

$\Delta_5 = Orb_H(x_2, y_2) = \{(x_2, y_2), (x_3, y_2), (x_4, y_2)\} = 3$

$\Delta_6 = Orb_H(x_2, y_3) = \{(x_2, y_3), (x_3, y_3), (x_4, y_3)\} = 3$

$\Delta_7 = Orb_H(x_2, y_4) = \{(x_2, y_4), (x_3, y_4), (x_4, y_4)\} = 3$

From above sequence, the rank is 8 and the corresponding subdegrees are $1, 1, 1, 1, 3, 3, 3, 3$.

3.9. Ranks and Subdegrees of $A_5 \times C_5$ on $X \times Y$

Lemma 3.7. Suppose that the group action of $A_5 \times C_5$ acts on $X \times Y$, then the rank is 10 and the corresponding subdegrees are $1, 1, 1, 1, 1, 4, 4, 4, 4, 4$.

Proof Let $G = A_5 \times C_5$ and $V = X \times Y$. From lemma 3.3, the elements of V are $\{(x_1,y_1), (x_2,y_1), (x_1,y_2), (x_3,y_1), (x_2,y_2), (x_1,y_3), (x_4,y_1), (x_3,y_5), (x_4,y_5), (x_3,y_2), (x_2,y_3), (x_1,y_4), (x_5,y_1), (x_4,y_2), (x_3,y_3), (x_2,y_4), (x_1,y_5), (x_5,y_2), (x_4,y_3), (x_3,y_4), (x_2,y_5), (x_5,y_3), (x_4,y_4), (x_5,y_4), (x_5,y_5)\}$.

Also, the elements of stabilizers are $Stab_G(x_1, y_1) = \{(e_x, e_y), ((x_2\, x_4\, x_5), e_y), ((x_2\, x_5\, x_4), e_y), ((x_2 x_3 x_5), e_y), ((x_3\, x_5\, x_4), e_y), ((x_2\, x_4\, x_3), e_y), ((x_3\, x_4\, x_5), e_y), ((x_2\, x_5\, x_3), e_y), ((x_2\, x_3\, x_4), e_y), ((x_2\, x_5)(x_3\, x_4), e_y), ((x_2\, x_3)(x_4\, x_5), e_y), ((x_2\, x_4)(x_3\, x_5), e_y)\}$.

Therefore, the number of elements of V fixed by each $g \in G$ are as shown in the table below:

Table 3. *Permutations in $Stab_G(x_1, y_1)$ and the Number of Fixed Points.*

Type of permutation in $Stab_G(x_1, y_1)$	Number of permutations (H)	$Fix_G(g)$
(e_x, e_y)	1	25
$((a\ b\ c), e_y)$	8	10
$((a\ b)(c\ d), e_y)$	3	5
TOTAL	12	

By lemma 2.1 (Cauchy Frobeneus Lemma), the number of suborbits of G acting on V is given as:
$\frac{1}{H}\sum_{g\in G}|Fix_G(g)| = \frac{1}{12}[(1\times 25)+(8\times 10)+(3\times 5)] = \frac{120}{12} = 10$.

This implies that there are 10 suborbits classified into the following four possible categories:

a) Suborbit where both the first and second components are identical to those of K.

$\Delta_0 = Orb_H(x_1, y_1) = \{(x_1, y_1)\} = 1$ (Trivial Orbit)

b) Suborbits where the first component is identical to that of K but the second components are different.

$\Delta_1 = Orb_H(x_1, y_2) = \{(x_1, y_2)\} = 1$

$\Delta_2 = Orb_H(x_1, y_3) = \{(x_1, y_3)\} = 1$

$\Delta_3 = Orb_H(x_1, y_4) = \{(x_1, y_4)\} = 1$

$\Delta_4 = Orb_H(x_1, y_4) = \{(x_1, y_5)\} = 1$

c) Suborbit where the second component is identical to that of K but the first components are different.

$\Delta_5 = Orb_H(x_2, y_1) = \{(x_2, y_1), (x_3, y_1), (x_4, y_1), (x_5, y_1)\} = 4$

d) Suborbits where neither the first nor the second components are identical to those of K.

$\Delta_6 = Orb_H(x_2, y_2) = \{(x_2, y_2), (x_3, y_2), (x_4, y_2), (x_5, y_2)\} = 4$

$\Delta_7 = Orb_H(x_2, y_3) = \{(x_2, y_3), (x_3, y_3), (x_4, y_3), (x_5, y_3)\} = 4$

$\Delta_8 = Orb_H(x_2, y_4) = \{(x_2, y_4), (x_3, y_4), (x_4, y_4), (x_5, y_4)\} = 4$

$\Delta_9 = Orb_H(x_2, y_5) = \{(x_2, y_5), (x_3, y_5), (x_4, y_5), (x_5, y_5)\} = 4$

From the above sequence, the group action has a rank of 10 and the corresponding subdegrees are $1, 1, 1, 1, 1, 4, 4, 4, 4, 4$.

3.10. Ranks and Subdegrees of $A_n \times C_n$ on $X \times Y$

Theorem 3.3. Let $G = A_n \times C_n$ acts on $X \times Y$, then the rank is $2n$ and subdegrees are $\underbrace{1, 1, ..., 1}_{n}, \underbrace{n-1, n-1, ..., n-1}_{n}$ for $n \geqslant 4$.

Proof. By generalizing the results obtained from lemma 3.5, lemma 3.6 and lemma 3.7, the number of suborbits are classified into the following four possible categories. They are:

a) Suborbit where both the first and the second components are identical to those of K.

$\Delta_0 = Orb_H(x_1, y_1) = (x_1, y_1) = 1$ (Trivial Orbit)

b) Suborbits where the first component is identical to that of K but the second components are different.

$\Delta_1 = Orb_H(x_1, y_2) = (x_1, y_2) = 1$

$\Delta_2 = Orb_H(x_1, y_3) = (x_1, y_3) = 1$

$\vdots$

$\Delta_{(n-1)} = Orb_H(x_1, y_n) = (x_1, y_n) = 1$

c) Suborbit where the second component is identical to that of K but the first components are different.

$\Delta_{(n-1)+1} = Orb_H(x_2, y_1) = (x_2, y_1), (x_3, y_1), (x_4, y_1), ..., (x_n, y_1) = n - 1$

d) Suborbits where neither the first nor the second components are identical to those of K.

$\Delta_{(n-1)+2} = Orb_H(x_2, y_2) = (x_2, y_2), (x_3, y_2), (x_4, y_2), ..., (x_n, y_2) = n - 1$

$\Delta_{(n-1)+3} = Orb_H(x_2, y_3) = (x_2, y_3), (x_3, y_3), (x_4, y_3), ..., (x_n, y_3) = n - 1$

$\Delta_{(n-1)+4} = Orb_H(x_2, y_4) = (x_2, y_4), (x_3, y_4), (x_4, y_4), ..., (x_n, y_4) = n - 1$

$\vdots$

$\Delta_{(n-1)+n} = Orb_H(x_2, y_n) = (x_2, y_n), (x_3, y_n), (x_4, y_n), ..., (x_n, y_n) = n - 1$

From the above sequence, it shows that there are n suborbits of size 1 and n suborbits of size $(n - 1)$. Therefore, the rank is $2n$ and the corresponding subdegrees are $\underbrace{1, 1, ..., 1}_{n}, \underbrace{n-1, n-1, ..., n-1}_{n}$.

To prove that $\Delta_0, \Delta_1, ..., \Delta_{(n-1)+n}$ are the only suborbits of the group action, we have to show that $\Delta_i = \{\Delta_0, \Delta_1, ..., \Delta_{(n-1)+n}\}$ is a partition of $X \times Y$.

$$
\begin{aligned}
\sum_{i=0}^{(n-1)+n} |\Delta_i| &= \sum_{i=0}^{n-1} |\Delta_i| + \sum_{i=(n-1)+1}^{(n-1)+n} |\Delta_i| \\
&= \sum_{i=0}^{n-1} |1, 1, ..., 1| + \sum_{i=(n-1)+1}^{(n-1)+n} |n-1, n-1, ..., n-1| \\
&\text{Since } S_n = \frac{n}{2}[a + l], then \\
&= \frac{n}{2}[1 + 1] + \frac{n}{2}[(n - 1) + (n - 1)] \\
&= \frac{2n}{2} + \frac{n}{2}[2n - 2] \\
&= n + n[n - 1] \\
&= n + n^2 - n \\
&= n^2 \\
&= |X \times Y|
\end{aligned}
$$

Therefore, it follows that $\bigcup_{i=1}^{n} \Delta_i = X \times Y$. Hence, Δ_i is a partition of $X \times Y$.

Corollary 4.1: Determine the rank and subdegrees of group action of $A_6 \times C_6$ acting on $X \times Y$.

Proof. From theorem 3.3, the rank is given as $2n$ and the corresponding subdegrees as $\underbrace{1, 1, ..., 1}_{n}$, $\underbrace{n-1, n-1, ..., n-1}_{n}$. Therefore, when $n = 6$, the rank is $2 \times 6 = 12$ and the corresponding subdegrees are: $1, 1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 5$.

Corollary 4.2: Determine the rank and the corresponding subdegrees of group action of $A_8 \times C_8$ which acts on $X \times Y$.

Proof. From theorem 3.3, the rank is given as $2n$ and the corresponding subdegrees as $\underbrace{1, 1, ..., 1}_{n}$, $\underbrace{n-1, n-1, ..., n-1}_{n}$. Therefore, when $n = 8$, the rank is $2 \times 8 = 16$ and the corresponding subdegrees are:

$1, 1, 1, 1, 1, 1, 1, 1, 7, 7, 7, 7, 7, 7, 7, 7$.

Corollary 4.3: Determine the rank and subdegrees of $A_9 \times C_9$ acting on $X \times Y$.

Proof. From theorem 3.3, the rank is given as $2n$ and the corresponding subdegrees as $\underbrace{1, 1, ..., 1}_{n}$, $\underbrace{n-1, n-1, ..., n-1}_{n}$. Therefore, when $n = 9$, the rank is $2 \times 9 = 18$ and the corresponding subdegrees are $1, 1, 1, 1, 1, 1, 1, 1, 1, 8, 8, 8, 8, 8, 8, 8, 8, 8$.

4. Conclusion

In this paper, properties of group action of $A_n \times C_n$ on $X \times Y$ for $n \geqslant 3$ were studied. Transitivity and primitivity were determined where it was proven that the group action is transitive and imprimitve. Also, the ranks and subdegrees were established where it was found that when $n = 3$, the rank is 9 and the corresponding subdegree being $1, 1, 1, 1, 1, 1, 1, 1, 1$. When $n \geqslant 4$, the rank was found to be $2n$ and the corresponding subdegrees are $\underbrace{1, 1, ..., 1}_{n}$, $\underbrace{n-1, n-1, ..., n-1}_{n}$.

Abbreviations

A_n	Alternating group of degree n whose order is $\frac{n!}{2}$
C_n	Cyclic group of order n
e_x	An identity element of Alternating group
e_y	An identity element of Cyclic group
$V = X \times Y$	Cartesian product of set X and set Y
$\text{Stab}_G(x)$	Stabilizer of a point x in G
$\text{Fix}_G(x)$	A point x fixed by the element in G
$\text{Orb}_G(x)$	The orbit of a point x in G
Δ	Suborbit of G on set V
$\emptyset$	An empty set
H	Number of permutations
$G = A_n \times C_n$	Direct product of Alternating group and Cyclic group

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Conflicts of Interest

The authors declare no conflicts of interest.

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