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The Existence and Uniqueness Results for a Nonlocal Bounbary Value Problem of Caputo-type Hadamard Hybrid Fractional Integro-differential Equations

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Abstract: This article is dedicated to study the existence and uniqueness of solutions for a non local bounbary value problem of Caputo-type Hadamard hybrid fractional integro-differential equations in Banach space, the recent researches considered the study of differential equations of Caputo-type Hadamard hybrid fractional integro-differential equations with classical order and the study of existence and uniqueness of solutions using approched numerical methodes, the objective of this paper is the study of the existence and uniqueness of fractional order of integro-differential equations involving the Caputo-type Hadamard derivative using fixed point theory. This work have two important results, the first result was the discussion of a new results owing to the fixed point theorem. Before the prove of results the problem was trandformed to Hadamard type problem. The first result based on Dhage fixed point theorem, after transforming our nonlocal boundary value problem into integral equation we defined operator equation, then we applied the fixed point theorem to get the existence resutl. The second result was the existence and uniqueness of solution for our nonlocal boundary value problem, we get this result using the Banach fixed point theorem. We illustrate our results by example to ending our theorical study.

Keywords: Caputo-type Hadamard Derivative, Existence Results, Existence and Uniqueness Result, Non Local Boundary Conditions, Fixed Point Theorem

1. Introduction

Hybrid differential equations have been considered more important and served as special cases of dynamical systems. Dhage and Lakshmikantham were the first to study ordinary hybrid differential equation and studied the existence of solutions for this boundary value problem [1]. In recent years, with the wide study of fractional differential equations, the theory of hybrid fractional differential equations were also studied by several researchers, see [3-5] and the references there in.

As we know, Hadamard fractional derivative is also a famous fractional derivative given by Hadamard [2] in 1892, and we can find this kind of derivative in the literature. The key of this definition involves a logarithmic function of arbitrary exponent. In the past decades, there were more studies on Hadamard fractional differential equations under different boundary conditions, see [6-13].

Zidane Baitiche et al. considered the following boundary value problem of nonlinear fractional hybrid differential equations involving Caputo's derivative [14]

$$
{}^{C}D^{\alpha}_{0^+}\left(\frac{x(t)}{f(t,x(\mu(t)))}\right) = g(t,x(\mu(t))), t \in I = [0,1],
$$

$$
a\left[\frac{x(t)}{f(t,x(\mu(t)))}\right]|_{t=0} + b\left[\frac{x(t)}{f(t,x(\mu(t)))}\right]_{t=1} = c,
$$

where $0 < \alpha < 1$, ${}^{C}D^{\alpha}_{0^+}$ is the Caputo fractional derivative.

$f \in C(I \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(I \times \mathbb{R}, \mathbb{R})$.

As we all known, the hadamard fractional differential equations are also popular in the literature, see [15-16], so some authors began to study the theory of fractional hybrid differential equation of hadamard type.

Zidane Baitiche et al. studied the existence of solutions for fractional hybrid differential equation of hadamard type with dirichlet boundary conditions [17]

$$_{H}D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad 1<t<e, \quad 1<\alpha<2$$
$$x(1)=0, \quad x(e)=0$$

where $1<\alpha<2$, $_{H}D^{\alpha}$ is the Hadamard fractional derivative, $f \in C([1,e] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C([1,e] \times \mathbb{R}, \mathbb{R})$.

In this paper, we study the existence results for a non local bounbary value problem of Caputo-type Hadamard hybrid fractional integro-differential equations, consider the follwing problem

$$_{c}^{H}D^{\alpha}\left(\frac{u(t)-\sum_{i=1}^{m} I^{\beta_i} f_i(t,u(t))}{g(t,u(t))}\right) = h(t,u(t)) \quad (1)$$
$$u(1)=0, u(e)=\mu(u)$$

where $_{c}^{H}D^{\alpha}$ is the Caputo-type Hadamard fractional differential derivative of order $0<\alpha<1$.

The rest of the paper is organized as follows. In Section 2, we recall some useful preliminaries. Section 3 contains the main result which is obtained by means of a hybrid fixed point theorem for three operators in a Banach algebra due to Dhage [1]. An example is also discussed for illustration of the main result.

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [19-21] and present preliminary results needed in our proofs later.

Definition 2.1. The Hadamard fractional integral of order α for a function f is defined as

$$I_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds, \alpha > 0.$$

Provided the integral exists.

Definition 2.2. The Hadamard fractional derivative of fractional order α for a function $f : [a,+\infty) \mapsto \mathbb{R}$ is defined as

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \frac{f(s)}{s} ds,$$

where $n-1<\alpha<n, n=[\alpha]-1$.

Definition 2.3. The Caputo-type Hadamard fractional derivative of fractional order α for a function $f : [a,+\infty) \mapsto \mathbb{R}$ is defined as

$$_{c}^{H}D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(t\frac{d}{dt}\right)^n \frac{f(s)}{s} ds,$$

where $n-1<\alpha<n, n=[\alpha]-1$.

Lemma 2.1. If f is a function such that $_{c}^{H}D_a^{\alpha} f(t)$ and $^{H}D^{\alpha} f(t)$ exists, then

$$_{c}^{H}D^{\alpha} f(t) = ^{H}D^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{\left(t\frac{d}{dt}\right)^n f(t_0)}{\Gamma(k-\alpha+1)} \left(\log \frac{t}{t_0}\right)^{k-\alpha},$$

and when $0<\alpha<1$, then

$$_{c}^{H}D^{\alpha} f(t) = ^{H}D^{\alpha} f(t) - \frac{f(t_0)}{\Gamma(1-\alpha)} \left(\log \frac{t}{t_0}\right)^{-\alpha}.$$

Lemma 2.2.

1. The equality $D_a^{\alpha} x(t) = 0$ is valid if, and only if,
$$x(t) = \sum_{j=1}^{n} c_j \left(\log \frac{t}{s}\right)^{\alpha-j},$$
where $c_j \in \mathbb{R}$, $j=1,2,..,n$, are arbitrary constants, and $x \in C(J,\mathbb{R})$.
2. Let $x \in C(J,\mathbb{R})$
$$I_a^{\alpha}\left(I_a^{\beta} x\right) = I_a^{\alpha+\beta} x,$$
3. Let $x, y \in C(J,\mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$
$$I_a^{\alpha}(c_1 x(t) + c_2 y(t)) = c_1 I_a^{\alpha} x(t) + c_2 I_a^{\alpha} y(t),$$
4. Let $x \in C_{\delta}^{n}(J,\mathbb{R})$ and $c_j \in \mathbb{R}$, $(j=1,2,...,n)$
$$I_a^{\alpha}(D_a^{\alpha} x(t)) = x(t) - \sum_{j=1}^{n} c_j (\log \frac{t}{a})^{\alpha-j}.$$

Lemma 2.3. Let S be a nonempty, convex, closed, and bounded subset of a Banach algebra E and let $A, C : E \mapsto E$ and $B : S \mapsto E$ be three operators such that

1. (H 1) A and C are Lipschitzian with Lipschitz constants λ and ρ, respectively;
2. (H 2) B is completely continuous;
3. (H 3) $u = AuBv + Cu \Rightarrow u \in S, \forall v \in S$, and
4. (H 4) $\lambda M + \rho < 1$, where $M = \|B(S)\|$.

Then the operator equation $u = AuBu + Cu$ has a solution.

Lemma 2.4 (Ascoli-Arzela theorem). A be a subset of $C(J,E)$, A is relatively compact in $C(J,E)$ if and only if the following conditions are checked:

(i) The unit A is limited.
$\exists k > 0$ such that $\|f(x)\|_E \leq k$ for $x \in J$ and $f \in A$.
(ii) Unit A is equicontinuous.
$\forall \varepsilon > 0, \exists \delta > 0$ and for evry $t_1, t_2 \in J$ we have $|t_1, t_2| < \delta \Rightarrow \|f(t_1) - f(t_2)\|_E < \varepsilon$.
(iii) For any $x \in J$ the unit $\{f(x), f \in A\} \subset E$ is relatively compact.

Lemma 2.5 (Banach fixed point theorem). Let X be a nonempty complete metric space, and $T : X \mapsto X$ is a contraction mapping. Then, there exists a unique point $x \in X$ such that $Tx = x$.

Let $E = C(J, \mathbb{R})$ be a Banach space equipped with the norm

$$\|u\| = \sup_{t\in J} |u(t)| \quad and \quad uv(t) = u(t)v(t), \forall t \in J$$

Then E is a Banach algebra with the above norm and multiplication.

3. Existence Results

This section consider the study of the existence of solutions to the non local boundary value problem (1). By Lemma 3.1, the boundary value problem (1) be transformed into a fixed point problem.

Suppose that $\alpha, \beta_i,\ i = 1, 2, ..., m$, and fuctions f, g, h satisfy the problem (1). Then the unique solution of (1) is given by

Lemma 3.1.

$$u(t) = g(t, u(t))I^{\alpha}h(t, u(t)) + g(t, u(t)) \left[\frac{\mu(u) - \sum_{i=1}^{m} I^{\beta_i} f_i(e, \mu(u))}{g(e, \mu(u))} - I^{\alpha}h(e, \mu(u))\right] + \sum_{i=1}^{m} I^{\beta_i} f_i(t, u(t)). \tag{2}$$

Proof The problem (1)-(2) is equivalent to the following problem. Using Lemma 2.1 we find

$${}^{H}_{c}D^{\alpha}\left(\frac{u(t) - \sum_{i=1}^{m} I^{\beta_i} f_i(t, u(t))}{g(t, u(t))}\right) = {}^{H}D^{\alpha}\left(\frac{u(t) - \sum_{i=1}^{m} I^{\beta_i} f_i(t, u(t))}{g(t, u(t))}\right) - \frac{\frac{u(1)-\sum_{i=1}^{m} I^{\beta_i} f_i(1,u(1))}{g(1,u(1))}}{\Gamma(1-\alpha)}(\log t)^{-\alpha}.$$

Using the boundary condition $u(1) = 0$ we have the following problem

$${}^{H}D^{\alpha}\left(\frac{u(t) - \sum_{i=1}^{m} I^{\beta_i} f_i(t, u(t))}{g(t, u(t))}\right) = h(t, u(t)) \tag{3}$$

$$u(1) = 0, u(e) = \mu(u). \tag{4}$$

now, we apply the Hadamard fractional integral I^{α} to the both sides of (4) and apply Lemma 2.2, we have

$$\frac{u(t) - \sum_{i=1}^{m} I^{\beta_i} f_i(t, u(t))}{g(t, u(t))} - c_1(\log t)^{\alpha-1} = I^{\alpha}h(t, u(t))$$

then

$$u(t) = g(t, u(t)) \left[I^{\alpha}h(t, u(t)) + c_1(\log t)^{\alpha-1}\right] + \sum_{i=1}^{m} I^{\beta_i} f_i(t, u(t)), \tag{5}$$

using the boundary value condition $u(e) = \mu(u)$, we find

$$\mu(u) = g(e, \mu(u)) \left[I^{\alpha}h(e, \mu(u)) + c_1\right] + \sum_{i=1}^{m} I^{\beta_i} f_i(e, \mu(u)),$$

thus

$$c_1 = \frac{\mu(u) - \sum_{i=1}^{m} I^{\beta_i} f_i(e, \mu(u))}{g(e, \mu(u))} - I^{\alpha}h(e, \mu(u)).$$

Taking the value of c_1 into (6) , we find

$$u(t) = g(t, u(t))I^{\alpha}h(t, u(t)) + g(t, u(t)) \left[\frac{\mu(u) - \sum_{i=1}^{m} I^{\beta_i} f_i(e, \mu(u))}{g(e, \mu(u))} - I^{\alpha}h(e, \mu(u))\right] + \sum_{i=1}^{m} I^{\beta_i} f_i(t, u(t)).$$

The proof completed.

In the sequel, we need the following assumptons.

$(H1)$ The functions $f_i : J \times \mathbb{R} \mapsto \mathbb{R}$, $i = 1, 2, ..., m$ and $g : J \times \mathbb{R} \mapsto \mathbb{R} \setminus \{0\}$ are continuous and there exists positive functions ϕ_i, ψ, $i = 1, 2, ..., m$, with bounds $\|\phi_i\|$ and $\|\psi\|$, respectively, such that

$$|f_i(t, u) - f_i(t, v)| \leqslant \phi_i(t) |u - v| \qquad i = 1, 1, ..., m$$

and

$$|g(t, u) - g(t, v)| \leqslant \psi(t) |u - v| ,$$

for $t \in J$ and $u, v \in \mathbb{R}$.

(H_2) There exists a function $p \in C(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\varphi : [1, \infty) \mapsto (0, \infty)$ such that

$$h(t,u) \leqslant p(t)\varphi(u), \qquad (t,u) \in J \times \mathbb{R}.$$

(H_3) There exists a constant $M_0 > 0$ such that

$$|f_i(t,u)| \leqslant M_0, \quad (t,u) \in J \times \mathbb{R}, \quad i = 1,2,...,m.$$

(H_4) There exists a constant $M_1 > 0$ such that

$$\frac{\mu(u)}{g(e,\mu(u))} \leqslant M_1, \qquad \forall u \in C(J,\mathbb{R}).$$

(H_5) There exists a number $r > 0$ such that

$$r \geqslant \frac{G_0\left(\frac{2\|p\|\varphi(r)}{\Gamma(\alpha+1)} + M_1 + \frac{M_0 \sum_{i=1}^m \frac{1}{\Gamma(\beta_i+1)}}{G}\right) + \sum_{i=1}^m \frac{F_i}{\Gamma(\beta_i+1)}}{1 - \|\psi\|\left(\frac{2\|p\|\varphi(r)}{\Gamma(\alpha+1)} + M_1 + \frac{M_0 \sum_{i=1}^m \frac{1}{\Gamma(\beta_i+1)}}{G}\right) - \sum_{i=1}^m \frac{\|\varphi_i\|}{\Gamma(\beta_i+1)}}, \tag{6}$$

where $G_0 = \sup_{t\in J} |g(t,0)|$, $F_i = \sup_{t\in J} |f_i(t,0)|$, $i = 1,2,...,m$ and

$$\|\psi\|\left(\frac{2\,\|p\|\,\varphi(r)}{\Gamma(\alpha+1)} + M_1 + \frac{M_0 \sum_{i=1}^m \frac{1}{\Gamma(\beta_i+1)}}{G}\right) + \sum_{i=1}^m \frac{\|\varphi_i\|}{\Gamma(\beta_i+1)} < 1 \tag{7}$$

Theorem 3.1. Assume that conditions $(H_1) - (H_5)$ hold. Then there exists at least one solution for the non local problem (1).

Proof We consider a subse S of E given by

$$S = \{u \in E, \|u\| \leqslant r\}.$$

Notice that S is closed, convex and bounded subset of the Banach space E.

In view of Lemma 3.1 , we define an operator

$$Tu(t) = g(t,u(t))I^\alpha h(t,u(t)) + g(t,u(t))\left[\frac{\mu(u) - \sum_{i=1}^m I^{\beta_i} f_i(e,\mu(u))}{g(e,\mu(u))} - I^\alpha h(e,\mu(u))\right] + \sum_{i=1}^m I^{\beta_i} f_i(t,u(t)).$$

Notice that the fixed point problem $Tu = u$ is equivalent to problem (1)-(2). Next we introduce three operators $A : E \mapsto E$, $B : S \mapsto E$ and $C : E \mapsto E$ as follow

$$Au(t) = g(t,u(t)), t \in J,$$

and

$$\begin{aligned} Bu(t) &= I^\alpha h(t,u(t)) + \frac{\mu(u) - \sum_{i=1}^m I^{\beta_i} f_i(e,\mu(u))}{g(e,\mu(u))} - I^\alpha h(e,\mu(u)) \\ &= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{h(s,u(s))}{s}ds + \frac{\mu(u) - \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)}\int_1^e \left(\log\frac{e}{s}\right)^{\beta_i-1} \frac{f_i(e,\mu(u))}{s}ds}{g(e,\mu(u))} \\ &\quad -\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1} \frac{h(e,\mu(u))}{s}ds, \end{aligned}$$

and

$$Cu(t) = \sum_{i=1}^m I^{\beta_i} f_i(t,u(t)) = \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta_i-1} \frac{f_i(s,u(s))}{s}ds.$$

Observe that $Tu = AuBu + Cu$. Now let show that the operators A,B and C satisfy all conditions of Lemma 2.1 in series of steps

Step 1. First, show that A and C are Lipschitzian on E

Let $u, v \in E$. Then by (H_1), for $t \in J$, we have

$$|Au(t) - Av(t)| = |g(t, u(t)) - g(t, v(t))| \leqslant \psi(t)\,|u - v| \leqslant \|\psi\|\,\|u - v\|\,,$$

which implies $\|Au - Av\| \leqslant \|\psi\|\,\|u - v\|$ for all $u, v \in E$. Therefore, A is a lipschitzian on E with Lipschitz constant $\|\psi\|$. Analogously, for any $u, v \in E$, we have

$$\begin{aligned}
|Cu(t) - Cv(t)| &= \left|\sum_{i=1}^{m} I^{\beta_i} f_i(t, u(t)) - \sum_{i=1}^{m} I^{\beta_i} f_i(t, v(t))\right| \\
&\leqslant \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta_i - 1} \frac{|f_i(s, u(s)) - f_i(s, v(s))|}{s} ds \\
&\leqslant \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta_i - 1} \frac{\phi_i(s)\,|u(s) - v(s)|}{s} ds \\
&\leqslant \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i + 1)} \|\phi_i\|\,\|u - v\|\,,
\end{aligned}$$

which implies that

$$\|Cu - Cv\| \leqslant \sum_{i=1}^{m} \frac{\|\phi_i\|}{\Gamma(\beta_i + 1)} \|u - v\|\,.$$

Thus, C is an Lipschitzian on E with Lipschitz constant $\sum_{i=1}^{m} \frac{\|\phi_i\|}{\Gamma(\beta_i+1)}$.

Step 2. The operator B is completely continuous on S

We first show that the operator B is continuous on E. Let u_n be a sequence in S converging to a point $u \in S$. Then by Lebesgue dominated convergence theorem, for all $t \in J$, we obtain

$$\begin{aligned}
&\lim_{n \mapsto \infty} Bu_n(t) \\
= &\lim_{n \mapsto \infty} \{\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} \frac{h(s, u_n(s))}{s} ds + \frac{\mu(u_n) - \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^e \left(\log \frac{e}{s}\right)^{\beta_i - 1} \frac{f_i(e, \mu(u_n))}{s} ds}{g(e, \mu(u_n))} \\
&- \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha - 1} \frac{h(e, \mu(u_n))}{s} ds\} \\
= &\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} \lim_{n \mapsto \infty} \frac{h(s, u_n(s))}{s} ds + \frac{\lim_{n \mapsto \infty} \mu(u_n) - \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^e \left(\log \frac{e}{s}\right)^{\beta_i - 1} \lim_{n \mapsto \infty} \frac{f_i(e, \mu(u_n))}{s} ds}{\lim_{n \mapsto \infty} g(e, \mu(u_n))} \\
&- \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha - 1} \lim_{n \mapsto \infty} \frac{h(e, \mu(u_n))}{s} ds \\
= &\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} \frac{h(s, u(s))}{s} ds + \frac{\mu(u) - \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^e \left(\log \frac{e}{s}\right)^{\beta_i - 1} \frac{f_i(e, \mu(u))}{s} ds}{g(e, \mu(u))} \\
&- \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha - 1} \frac{h(e, \mu(u))}{s} ds, \\
= &Bu(t).
\end{aligned}$$

Thus implies that $Bu_n \mapsto Bu$ point-wise on J. Further it can be shown that Bu_n is equicontinuous sequence of functions. So $Bu_n \mapsto Bu$ uniformly and the operator B is continuous on S.

Next we will prove that the set $B(S)$ is uniformly bounded in S. For any $u \in S$, we have

$$
\begin{aligned}
|Bu(t)| &= \Big|\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{h(s,u(s))}{s}ds+\frac{\mu(u)-\sum_{i=1}^m\frac{1}{\Gamma(\beta_i)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta_i-1}\frac{f_i(e,\mu(u))}{s}ds}{g(e,\mu(u))}\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{h(e,\mu(u))}{s}ds\Big|\\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|h(s,u(s))|}{s}ds+\left|\frac{\mu(u)-\sum_{i=1}^m\frac{1}{\Gamma(\beta_i)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta_i-1}\frac{f_i(e,\mu(u))}{s}ds}{g(e,\mu(u))}\right|\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|h(e,\mu(u))|}{s}ds,\\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|p(s)|\,|\varphi(u)|}{s}ds+\left|\frac{\mu(u)}{g(e,\mu(u))}\right|+\left|\frac{\sum_{i=1}^m\frac{1}{\Gamma(\beta_i)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta_i-1}\frac{f_i(e,\mu(u))}{s}ds}{g(e,\mu(u))}\right|\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|p(s)|\,|\varphi(\mu(u))|}{s}ds,\\
&\leqslant \frac{2}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{p(s)\varphi(r)}{s}ds+M_1+\frac{\sum_{i=1}^m\frac{1}{\Gamma(\beta_i)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta_i-1}\frac{M_0}{s}ds}{G}\\
&\leqslant \frac{2\,\|p\|\,\varphi(r)}{\Gamma(\alpha+1)}+M_1+\frac{M_0\sum_{i=1}^m\frac{1}{\Gamma(\beta_i+1)}}{G}=K_1.
\end{aligned}
$$

For all $t\in J$. Therefore, $\|B\|\leqslant K_1$ which shows that B is uniformly bounded on S.
Now, we will show that $B(S)$ ia an equicontinuous set in E. Let $\tau_1,\tau_2\in J$ with $\tau_1<\tau_2$ and $u\in S$. Then we have

$$
\begin{aligned}
&|Bu(\tau_2)-Bu(\tau_1)|\\
\leqslant\ &\Big|\frac{1}{\Gamma(\alpha)}\int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{h(s,u(s))}{s}ds+\frac{\mu(u)-\sum_{i=1}^m\frac{1}{\Gamma(\beta_i)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta_i-1}\frac{f_i(e,\mu(u))}{s}ds}{g(e,\mu(u))}\\
&-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{h(e,\mu(u))}{s}ds\\
&-\frac{1}{\Gamma(\alpha)}\int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\frac{h(s,u(s))}{s}ds-\frac{\mu(u)-\sum_{i=1}^m\frac{1}{\Gamma(\beta_i)}\int_1^e\left(\log\frac{e}{s}\right)^{\beta_i-1}\frac{f_i(e,\mu(u))}{s}ds}{g(e,\mu(u))}\\
&+\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{h(e,\mu(u))}{s}ds\Big|\\
\leqslant\ &\left|\frac{1}{\Gamma(\alpha)}\int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{h(s,u(s))}{s}ds-\frac{1}{\Gamma(\alpha)}\int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\frac{h(s,u(s))}{s}ds\right|\\
\leqslant\ &\left|\frac{1}{\Gamma(\alpha)}\int_1^{\tau_1}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{h(s,u(s))}{s}ds+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{h(s,u(s))}{s}ds-\frac{1}{\Gamma(\alpha)}\int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\frac{h(s,u(s))}{s}ds\right|\\
\leqslant\ &\left|\frac{1}{\Gamma(\alpha)}\int_1^{\tau_1}\left[\left(\log\frac{\tau_2}{s}\right)-\left(\log\frac{\tau_1}{s}\right)\right]^{\alpha-1}\frac{h(s,u(s))}{s}ds+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{h(s,u(s))}{s}ds\right|\\
\leqslant\ &\frac{1}{\Gamma(\alpha)}\int_1^{\tau_1}\left[\left(\log\frac{\tau_2}{s}\right)-\left(\log\frac{\tau_1}{s}\right)\right]^{\alpha-1}\frac{p(s)\varphi(r)}{s}ds+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{p(s)\varphi(r)}{s}ds.
\end{aligned}
$$

Wich is independent of $u\in S$. as $\tau_1\mapsto\tau_2$ the right side of the above inequality tends to zero. Therefore, it follows from Arzela-Ascoli theorem that B is a completely continuous operator on S.

Step 3. Hypothesis (3) of Lemma 2.3 is satisfied

Let $u\in E$ and $v\in S$ be arbitrary elements such that $u=AuBv+Cu$. Then we have

$$\begin{aligned}
u(t) &\leqslant |Au(t)|\,|Bv(t)| + |Cu(t)| \\
&\leqslant |g(t,u(t))|\,|\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{h(s,v(s))}{s}ds + \frac{\mu(v) - \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)}\int_1^e \left(\log\frac{e}{s}\right)^{\beta_i-1}\frac{f_i(e,\mu(v))}{s}ds}{g(e,\mu(v))} \\
&\quad -\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{h(e,\mu(v))}{s}ds| + |\sum_{i=1}^m \frac{1}{\Gamma(\beta_i)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta_i-1}\frac{f_i(s,u(s))}{s}ds| \\
&\leqslant |g(t,u(t))| \left\{\frac{2\,\|p\|\,\varphi(r)}{\Gamma(\alpha+1)} + M_1 + \frac{M_0\sum_{i=1}^m \frac{1}{\Gamma(\beta_i+1)}}{G}\right\} + |\sum_{i=1}^m \frac{1}{\Gamma(\beta_i)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta_i-1}\frac{f_i(s,u(s))}{s}ds| \\
&\leqslant (|g(t,u(t)) - g(t,0)| + |g(t,0)|)\left\{\frac{2\,\|p\|\,\varphi(r)}{\Gamma(\alpha+1)} + M_1 + \frac{M_0\sum_{i=1}^m \frac{1}{\Gamma(\beta_i+1)}}{G}\right\} \\
&\quad + \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta_i-1}\frac{(|f_i(s,u(s)) - f_i(s,0)| + |f_i(s,0)|)}{s}ds \\
&\leqslant (r\,\|\psi\| + G_0)\left\{\frac{2\,\|p\|\,\varphi(r)}{\Gamma(\alpha+1)} + M_1 + \frac{M_0\sum_{i=1}^m \frac{1}{\Gamma(\beta_i+1)}}{G}\right\} + \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta_i-1}\frac{(r\,\|\varphi_i\| + F_0)}{s}ds \\
&\leqslant (r\,\|\psi\| + G_0)\left\{\frac{2\,\|p\|\,\varphi(r)}{\Gamma(\alpha+1)} + M_1 + \frac{M_0\sum_{i=1}^m \frac{1}{\Gamma(\beta_i+1)}}{G}\right\} + \sum_{i=1}^m \frac{r\,\|\varphi_i\| + F_0}{\Gamma(\beta_i+1)}.
\end{aligned}$$

which leads to

$$\begin{aligned}
\|u\| &\leqslant (r\,\|\psi\| + G_0)\left(\frac{2\,\|p\|\,\varphi(r)}{\Gamma(\alpha+1)} + M_1 + \frac{M_0\sum_{i=1}^m \frac{1}{\Gamma(\beta_i+1)}}{G}\right) + \sum_{i=1}^m \frac{r\,\|\varphi_i\| + F_0}{\Gamma(\beta_i+1)} \\
&\leqslant r.
\end{aligned}$$

Therefore $u \in S$.
Step 4. Finally we show that $\lambda M + \rho \leqslant 1$, that is, (H_4) of Lemma 2.3 hold.
Since

$$M = \|B(S)\| = \sup_{u\in S}\left\{\sup_{t\in J}|Bu(t)|\right\} \leqslant \frac{2\,\|p\|\,\varphi(r)}{\Gamma(\alpha+1)} + M_1 + \frac{M_0\sum_{i=1}^m \frac{1}{\Gamma(\beta_i+1)}}{G}.$$

Therefore, by (H_5) we have

$$\|\psi\|\,M + \sum_{i=1}^m \frac{\|\phi_i\|}{\Gamma(\beta_i+1)} < 1,$$

where $\lambda = \|\psi\|$ and $\rho = \sum_{i=1}^m \frac{\|\phi_i\|}{\Gamma(\beta_i+1)}$.
Thus all the conditions of Lemma 2.3 are satisfied and hence the operator equation $u = AuBu + Cu$ has a solution in S. In consequence, the problem $(1)(2)$ has a solution on J. Thus completes the proof.

4. Existence and Uniqueness Results

This section is for the study of the existence and uniqueness of solution of the non local boundary value problem (1). This result is obtained by using the Banach fixed point theorem and Arzila-Ascolii theorem.
Theorem 4.1. Assune that conditions $(H_1 - H_3)$ holds. If

$$\left[\,\|\psi\|\left[\frac{2\,\|p\|\,\varphi(r)}{\Gamma(\alpha+1)} + M_1 + \frac{M_0\sum_{i=1}^m \frac{1}{\Gamma(\beta_i+1)}}{G}\right] + \sum_{i=1}^m \frac{\|\phi_i\|}{\Gamma(\beta_i+1)}\right] < 1.$$

Then the non-local boundary value problem of Caputo-type Hadamard have a unique solution.
Proof. Define the operator $T : X \mapsto X$ associated with the problem (1) by

$$Tu(t) = g(t,u(t))I^{\alpha}h(t,u(t)) + g(t,u(t))\left[\frac{\mu(u) - \sum_{i=1}^{m} I^{\beta_i} f_i(e,\mu(u))}{g(e,\mu(u))} - I^{\alpha}h(e,\mu(u))\right] + \sum_{i=1}^{m} I^{\beta_i} f_i(t,u(t)). \tag{8}$$

Now, let show that the operator T have a fixed point in B_ρ which represents a unique solution of the non local boundary value problem (1). So the proof will be given in two steps

Step 1. Define the set B_ρ as follow

$$B_\rho = \{u \in X; \|u\|_X \leq \rho\},$$

where ρ is a positive real constant chosen so that

$$\frac{2G\|p\|\varphi(\rho)}{\Gamma(\alpha+1)} + M_1 G + +2M_0 \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i+1)} < \rho.$$

First, show that $TB_\rho \subset B_\rho$. For $t \in J$ and $u \in B_\rho$,

$$\begin{aligned}
|Tu(t)| &= \left| g(t,u(t))I^{\alpha}h(t,u(t)) + g(t,u(t))\left[\frac{\mu(u) - \sum_{i=1}^{m} I^{\beta_i} f_i(e,\mu(u))}{g(e,\mu(u))} - I^{\alpha}h(e,\mu(u))\right] + \sum_{i=1}^{m} I^{\beta_i} f_i(t,u(t))\right| \\
&= \Bigg| g(t,u(t))\Bigg[\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{h(s,u(s))}{s}ds + \frac{\mu(u) - \sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i)}\int_1^e \left(\log\frac{e}{s}\right)^{\beta_i-1}\frac{f_i(e,\mu(u))}{s}ds}{g(e,\mu(u))} \\
&\quad -\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{h(e,\mu(u))}{s}ds\Bigg] + \sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta_i-1}\frac{f_i(t,u(t))}{s}ds\Bigg| \\
&\leq |g(t,u(t))|\left[\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|h(s,u(s))|}{s}ds + \frac{|\mu(u)| + \sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i)}\int_1^e \left(\log\frac{e}{s}\right)^{\beta_i-1}\frac{|f_i(e,\mu(u))|}{s}ds}{|g(e,\mu(u))|}\right. \\
&\quad \left.+\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|h(e,\mu(u))|}{s}ds\right] + \sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta_i-1}\frac{|f_i(t,u(t))|}{s}ds \\
&\leq |g(t,u(t))|\left[\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|p(s)\varphi(u)|}{s}ds + \left|\frac{\mu(u)}{g(e,\mu(u))}\right| + \frac{\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i)}\int_1^e \left(\log\frac{e}{s}\right)^{\beta_i-1}\frac{M_0}{s}ds}{|g(e,\mu(u))|}\right. \\
&\quad \left.+\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|p(s)\varphi(u)|}{s}ds\right] + \sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta_i-1}\frac{M_0}{s}ds \\
&\leq |g(t,u(t))|\left[\frac{2}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|p(s)\varphi(u)|}{s}ds + M_1 + \frac{\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i)}\int_1^e \left(\log\frac{e}{s}\right)^{\beta_i-1}\frac{M_0}{s}ds}{|g(e,\mu(u))|}\right] \\
&\quad +\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta_i-1}\frac{M_0}{s}ds \\
&\leq G\left[\frac{2\|p\|\varphi(r)}{\Gamma(\alpha+1)} + M_1 + \frac{M_0\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+1)}}{G}\right] + M_0\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+1)} \\
&\leq \frac{2G\|p\|\varphi(r)}{\Gamma(\alpha+1)} + M_1 G + +2M_0\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+1)} < \rho.
\end{aligned}$$

Which show that T maps B_ρ into itself.

Step 2. Let show that the operator $T : B_\rho \mapsto B_\rho$ is a contraction.

Let $u, v \in X$ and $t \in J$. The use of assumptions gives

$$
\begin{aligned}
& |Tu(t) - Tv(t)\| \\
= \; & \left| g(t,u(t)) I^{\alpha} h(t,u(t)) + g(t,u(t)) \left[\frac{\mu(u) - \sum_{i=1}^{m} I^{\beta_i} f_i(e,\mu(u))}{g(e,\mu(u))} - I^{\alpha} h(e,\mu(u)) \right] + \sum_{i=1}^{m} I^{\beta_i} f_i(t,u(t)) \right. \\
& \left. - g(t,v(t)) I^{\alpha} h(t,v(t)) + g(t,v(t)) \left[\frac{\mu(v) - \sum_{i=1}^{m} I^{\beta_i} f_i(e,\mu(v))}{g(e,\mu(v))} - I^{\alpha} h(e,\mu(v)) \right] - \sum_{i=1}^{m} I^{\beta_i} f_i(t,v(t)) \right| \\
= \; & \left| g(t,u(t)) \left[\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{h(s,u(s))}{s} ds + \frac{\mu(u) - \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^e \left(\log \frac{e}{s} \right)^{\beta_i - 1} \frac{f_i(e,\mu(u))}{s} ds}{g(e,\mu(u))} \right. \right. \\
& \left. - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s} \right)^{\alpha-1} \frac{h(e,\mu(u))}{s} ds \right] + \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t \left(\log \frac{t}{s} \right)^{\beta_i - 1} \frac{f_i(t,u(t))}{s} ds \\
& - g(t,v(t)) \left[\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{h(s,v(s))}{s} ds + \frac{\mu(v) - \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^e \left(\log \frac{e}{s} \right)^{\beta_i - 1} \frac{f_i(e,\mu(v))}{s} ds}{g(e,\mu(v))} \right. \\
& \left. \left. - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s} \right)^{\alpha-1} \frac{h(e,\mu(v))}{s} ds \right] + \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t \left(\log \frac{t}{s} \right)^{\beta_i - 1} \frac{f_i(t,v(t))}{s} ds \right| \\
\leq \; & \left| g(t,u(t)) \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{h(s,u(s))}{s} ds - g(t,v(t)) \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{h(s,v(s))}{s} ds \right| \\
& + \left| g(t,u(t)) \frac{\mu(u)}{g(e,\mu(u))} - g(t,v(t)) \frac{\mu(v)}{g(e,\mu(v))} \right| + \left| g(t,u(t)) \frac{\sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^e \left(\log \frac{e}{s} \right)^{\beta_i - 1} \frac{f_i(e,\mu(u))}{s} ds}{g(e,\mu(u))} \right. \\
& \left. - g(t,v(t)) \frac{\sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^e \left(\log \frac{e}{s} \right)^{\beta_i - 1} \frac{f_i(e,\mu(u))}{s} ds}{g(e,\mu(v))} \right| + \left| g(t,u(t)) \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s} \right)^{\alpha-1} \frac{h(e,\mu(u))}{s} ds \right. \\
& \left. - g(t,v(t)) \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s} \right)^{\alpha-1} \frac{h(e,\mu(v))}{s} ds \right| + \left| \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t \left(\log \frac{t}{s} \right)^{\beta_i - 1} \frac{f_i(t,u(t))}{s} ds \right. \\
& \left. - \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t \left(\log \frac{t}{s} \right)^{\beta_i - 1} \frac{f_i(t,v(t))}{s} ds \right| \\
\leq \; & \left| g(t,u(t)) \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{|h(s,u(s))|}{s} ds - g(t,v(t)) \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{|h(s,v(s))|}{s} ds \right| \\
& + |g(t,u(t)) - g(t,v(t))| \, M_1 + \left| g(t,u(t)) \frac{\sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^e \left(\log \frac{e}{s} \right)^{\beta_i - 1} \frac{|f_i(e,\mu(u))|}{s} ds}{g(e,\mu(u))} \right. \\
& \left. - g(t,v(t)) \frac{\sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^e \left(\log \frac{e}{s} \right)^{\beta_i - 1} \frac{|f_i(e,\mu(u))|}{s} ds}{g(e,\mu(v))} \right| + \left| g(t,u(t)) \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s} \right)^{\alpha-1} \frac{|h(e,\mu(u))|}{s} ds \right. \\
& \left. - g(t,v(t)) \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s} \right)^{\alpha-1} \frac{|h(e,\mu(v))|}{s} ds \right| + \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t \left(\log \frac{t}{s} \right)^{\beta_i - 1} \frac{|f_i(t,u(t)) - f_i(t,v(t))|}{s} ds \\
\leq \; & \psi(t) \, |u(t) - v(t)| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{|p(s)\varphi(u)|}{s} ds + \psi(t) \, |u(t) - v(t)| \, M_1 \\
& + \psi(t) \, |u(t) - v(t)| \frac{M_0 \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i+1)}}{G} + \psi(t) \, |u(t) - v(t)| \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s} \right)^{\alpha-1} \frac{|p(e)\varphi(u)|}{s} ds \\
& + \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t \left(\log \frac{t}{s} \right)^{\beta_i - 1} \frac{|\phi_i(s)| \, |u(t) - v(t)|}{s} ds \\
\leq \; & \frac{2 \, \|\psi\| \, \|p\| \, \varphi(r)}{\Gamma(\alpha+1)} \|u - v\| + M_1 \|\psi\| \, \|u - v\| + \frac{M_0 \, \|\psi\| \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i+1)}}{G} \|u - v\| + \sum_{i=1}^{m} \frac{\|\phi_i\|}{\Gamma(\beta_i+1)} \|u - v\| \\
\leq \; & \left[\|\psi\| \left[\frac{2 \, \|p\| \, \varphi(r)}{\Gamma(\alpha+1)} + M_1 + \frac{M_0 \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i+1)}}{G} \right] + \sum_{i=1}^{m} \frac{\|\phi_i\|}{\Gamma(\beta_i+1)} \right] \|u - v\| .
\end{aligned}
$$

This implies that the operator T is contraction.

Then by the Banach fixed point theorem, there exists a unique point $u \in X$, such that $Tu = u$, it is a unique solution of the nonlocal boundary value problem (1).

The proof completed.

5. Example

Consider the following nonlocal boundary value problem of Caputo-type Hadamard hybrid fractional integro-differential equation

$$
\begin{aligned}
{}^{H}_{c}D^{\frac{1}{3}}\left[\frac{u(t)-\sum_{i=1}^{3} I^{\frac{1}{2i}} f_i(t,u(t))}{g(t,u(t))}\right] &= \frac{1}{8(1+t)}(\cos(u(t))+\sin(u(t))) \qquad (9)\\
u(1)=0, u(e) = \sum_{i=1}^{n} \frac{1}{8n}\,|u|
\end{aligned}
$$

The problem (9) is a particular case of (1) with $\alpha = \frac{1}{3}$, and

$$
\begin{aligned}
f_i(t,u(t)) &= \frac{t\,|u(t)|}{8i},\\
g(t,u(t)) &= \frac{t}{8}\cos(u(t)),\\
h(t,u(t)) &= \frac{1}{8(1+t)}(\cos(u(t))+\sin(u(t))).
\end{aligned}
$$

Clearly, f_i $i = 1,..,3$ and g, h are continuous functions and satisfy conditions (H_1),(H_2) with $\phi_i(t) = \frac{t}{8i}$, $\psi(t) = \frac{t}{8}$ and $p(t) = \frac{1}{8(1+t)}$ and $M = \frac{1}{16}$.
Also

$$
\begin{aligned}
|f_i(t,u) - f_i(t,v)| &\leq \frac{t}{8i}\,|u-v|\\
&\leq \frac{e}{8i}\,\|u-v\|\\
|g(t,u) - g(t,v)| &\leq \frac{t}{8}\,|\cos(u)-\cos(v)|\\
&\leq \frac{e}{8}\,\|u-v\|
\end{aligned}
$$

and

$$
\begin{aligned}
|h(t,u) - h(t,v)| &\leq \frac{1}{8(1+t)}\,|u-v|\\
&\leq \frac{1}{16}\,\|u-v\|
\end{aligned}
$$

Now, we have

$$
\|\psi\|\,M + \sum_{i=1}^{m}\frac{\|\phi_i\|}{\Gamma(\beta_i+1)} = 0.36754333 < 1.
$$

Since all assumptions are hold. According to Theorem 3.1 the problem (9) has at least one solution. To see if this solution is unique, note that the assumptions are hold, also the condition of Theorem 4.1

$$
\left[\|\psi\|\left[\frac{2\,\|p\|\,\varphi(r)}{\Gamma(\alpha+1)} + M_1 + \frac{M_0\sum_{i=1}^{m}\frac{1}{\Gamma(\beta_i+1)}}{G}\right] + \sum_{i=1}^{m}\frac{\|\phi_i\|}{\Gamma(\beta_i+1)}\right] = 0.16856373 < 1,
$$

are satisfied. Therefore, from Theorem 4.1 the problem (9) has a unique solution.

6. Results

This paper, discuss two important results, before the prove of results the problem was trandformed to Hadamard type problem. The first one based on Dhage fixed point theorem, after transforming our nonlocal boundary value problem into integral equation we defined operator equation, then we applied the fixed point theorem to get the existence resutl. The second result was the existence and uniqueness of solution for our nonlocal boundary value problem, we get this result using the Banach fixed point theorem.

This work gives as results The existence and uniqueness results for a nonlocal bounbary value problem of Caputo-type Hadamard hybrid fractional integro-differential equations in Banach space, using the fixed point theory.

7. Discussion

In view of the research cited in the introduction we can see that the problem we studied in this paper is new subject for fractional differential equations involving the Caputo-type Hadamard fractional derivative, this studies based on fixed point theory, more than this we perturbated our problem with two different type of perturbations (Hybrid type and integro-differential term) , previous research studied this kind of problem without perturbation.

8. Conclusions

This work consider the existence and uniqueness results for a nonlocal bounbary value problem of Caputo-type Hadamard hybrid fractional integro-differential equations in Banach space, the problem was perturbated twice and with new fractional differential derivative. By transforming the problem into a Volttera integral equation and using the Dhage fixed point theorem, the existence results of solutions for the boundary value problem (1) was proved under some conditions. Then, using the Banach fixed point theorem, the existence and uniqueness of solution for the boundary value problem, after transforming the problem into a fixed point problem.

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Conflicts of Interest

The authors declare no conflicts of interest.

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