American Journal of Applied Mathematics
2024; 12(6): 258-265
http://www.sciencepublishinggroup.com/j/ajam
doi: 10.11648/j.ajam.20241206.15
ISSN: 2330-0043 (Print); ISSN: 2330-006X (Online)

Combinatorial Properties and Invariants Associated with the Direct Product of Alternating and Dihedral Groups Acting on the Cartesian Product of Two Sets

John Mokaya Victor[1, *], Nyaga Lewis Namu[1], Gikunju David Muriuki[2]

[1]Department of Pure and Applied Mathematics, Jomo Kenyatta University of Agriculture and Technology, Nairobi, Kenya
[2]Department of Mathematical Sciences, The Co-operative University of Kenya, Nairobi, Kenya

Email address:

vm25953@gmail.com (John Mokaya Victor), lnyaga@jkuat.ac.ke (Nyaga Lewis Namu), dmuriuki@cuk.ac.ke (Gikunju David Muriuki)
[*]Corresponding author

To cite this article:

John Mokaya Victor, Nyaga Lewis Namu, Gikunju David Muriuki. (2024). Combinatorial Properties and Invariants Associated with the Direct Product of Alternating and Dihedral Groups Acting on the Cartesian Product of Two Sets. *American Journal of Applied Mathematics, 12*(6), 258-265. https://doi.org/10.11648/j.ajam.20241206.15

Received: 6 November 2024; **Accepted:** 19 November 2024; **Published:** 18 December, 2024

Abstract: Group theory, an area in mathematics, has undergone extensive research, but not exhaustively. In group theory, especially symmetric groups have been studied in terms of properties and actions. In this research, two groups of the Symmetric Group (Alternating and Dihedral Groups) are studied in terms of their properties and also their direct product between them. Also, ordered sets are studied in this research where their Cartesian product is looked at. Finally, this research combines the direct product of two symmetric groups (alternating group and dihedral group) and the Cartesian product of two ordered sets $(X \times Y)$ by group action. This research therefore focuses on determining the combinatorial properties (transitivity and primitivity), invariants (ranks and subdegrees), and structures (suborbital graphs) of this group action. To accomplish this, orbit-stabilizer theorem is used to compute transitivity, block action concept is applied to determine primitivity, and Cauchy-Frobenius lemma is used to compute ranks and subdegrees. For $n \geqslant 3$, it has been confirmed that the group action is transitive and also imprimitive. The rank of this group action is 6 and the subdegrees are obtained using the formula of Theorem 3.3. This research adds new concepts in group theory which will be useful in other areas like graph theory and also application in real life situations.

Keywords: Group Action, Ranks, Subdegrees, Transitivity, Primitivity, Alternating Group, Dihedral Group

1. Introduction

Suppose that (A_n, X) and (D_n, Y) are permutation groups, then the direct product $A_n \times D_n$ acting on the Cartesian product $X \times Y$ is given by the rule $(g_1, g_2)(x, y) = (g_1x, g_2y)$ $\forall$ $g_1 \in A_n,$ $g_2 \in D_n,$ $x \in X$ and $y \in Y.$

The direct product is a resourceful construction which allows groups to be combined in that they mantain their individual structures, while also making it possible to analyze and synthesize complex system in algebra, geometry and more.

This paper seeks to determine the transitivity, primitivity, ranks and subdegrees of the group action $A_n \times D_n$ on $X \times Y$ where $n \geq 3$.

2. Notation and Preliminary Results

Definition 2.1. A set is a collection of well defined objects. [1]

Definition 2.2. A group G is a set with a binary operation and satisfies the rules; identity, inverse, closure and associativity. [2]

Definition 2.3. A permutation of n elements is a one-to-one mapping from a set X of n elements onto itself. The set of all permutations of X form a group G under the operation of multiplication (composition) of permutations called a Symmetric group (S_n). It is denoted as (G, X).[3]

Definition 2.4. The group formed by the collection of the set of all even permutations in a Symmetric group (S_n) is called the Alternating group (A_n) and its order is $|A_n| = \frac{n!}{2}$.[4]

Definition 2.5. A Dihedral group, D_n is a group of symmetries and rigid motions of a regular polygon P_n of n sides. It has n number of sides (degree) and $2n$ number of elements (order). Dihedral group is composed of rotations and reflections that preserves the shape of the polygon. [5]

Definition 2.6. Let G be a permutation group and X be a non-empty set . Then, a group action of G on X is the function $G \times X \rightarrow X$ satisfying the algebraic law of identity and associativity:

1. Identity law $e \cdot x = x \,\forall\, x \in X$ and $e \in G$.
2. Associative law $(g \cdot h) \cdot x = g \cdot (h \cdot x) \,\forall\, g, h \in G$ and $x \in X$. [6]

Definition 2.7. Let G act on a set X. The set of elements of X fixed by $g \in G$ is called the fixed point set of g, denoted by $Fix\,(g)$. Thus $Fix\,(g) = \{x \in X\,|gx = x\}$. [7]

Definition 2.8. The stabilizer of $x \in X$ denoted by $Stab_G\,(x)$, is the set of all elements in G that fix x i.e. $Stab_G\,(x) = \{g \in G\,|gx = x\}$. [8]

Definition 2.9. If a group G acts on a set X then, X is partitioned into disjoint equivalence classes called orbits or transitivity classes of the action. For each $x \in X$ the orbit containing x is denoted by $Orb_G\,(x) = \{gx\,|g \in G\}$ which contains all the images of x under every g in G. If the action of a group G on a set X has a single orbit ($|orb_G\,(x)| = |X|$), then G is said to act transitively on X.i.e if x, $y \in X$ and $g \in G$ then *gx*=*y*. [9]

Definition 2.10. Let G be a transitive group acting on a set X. A subset Y of X is called a block if for any $g \in G$, either;

1. Y is invariant under g, ($gY = Y$) or
2. g translates everything out of Y, ($gY \cap Y = \emptyset$).[10]

Definition 2.11. All one - element subsets of X, empty set $\emptyset$ and X itself are obvious blocks and are called trivial blocks. If they are the only blocks, then G acts primitively on X, otherwise G acts imprimitively if the action has also non-trivial blocks. [11]

Definition 2.12. The Cartesian product of two sets A and B written as $A \times B$ is the set $A \times B = \{(a, b)\,|a \in Aandb \in B\}$. [12]

Definition 2.13. An element of a set X is said to be invariant under group action if the action maps the element into itself.[13]

Definition 2.14. Suppose (G_1, X_1) and (G_2, X_2) are permutation groups, then the direct product, $G_1 \times G_2$ acts on the Cartesian product $X_1 \times X_2$ by the rule $(g_1, g_2)\,(x_1, x_2) = (g_1x_1, g_2x_2)\,\forall g_1 \in G_1, g_2 \in G_2, x_1 \in X_1, x_2 \in X_2$.[14]

Theorem 2.1. (Orbit-Stabilizer Theorem) Suppose that a finite group G acts on X and for any $x \in X$. Then the number of elements in the orbit of x is the index of G over the stabilizer of x given by:

$$|orb_G(x)| = \frac{|G|}{|stab_G(x)|} \tag{1}$$

[15]

Lemma 2.1. (Cauchy-Frobeneus Lemma) Suppose that G is a finite permutation group that acts on set X. Then the number of orbits into which the set X is divided by the equivalence relation induced by G is;

$$\frac{1}{|G|}\sum_{g \in G}|Fix(g)| \tag{2}$$

Where $|Fix\,(g)| = |\{x \in X\,|gx = x\}|$. i.e. $|Fix\,(g)|$ is the number of elements that are invariant under $g \in G$. [16]

Theorem 2.2. Let A_n act on X, then $Stab_{A_n} = A_{n-1}$ for all $x \in X$. Therefore;

$$|Stab_G A_{n-1}| = \frac{(n-1)!}{2} \tag{3}$$

[17]

Definition 2.15. Let G act on X transitively and let $Stab_G\,(x)$ be the stabilizer of a point $x \in X$ then, the orbits $[\Delta_0 = \{x\}\,, \Delta_1, \Delta_2, ..., \Delta_{r-1}]$ of $Stab_G\,(x)$ on X are called suborbits of G. The total number of these suborbits r is called the rank of G. The subdegrees of G are the lengths of the suborbits $[n_i = |\Delta_i|\,(i = 0, 1, 2, ..., r-1)]$. [18]

3. Main Results

3.1. Transitivity of the Group $A_3 \times D_3$ Action on $X \times Y$

Lemma 3.1. The group $A_3 \times D_3$ action on $X \times Y$ is transitive.

Proof Let $G = A_3 \times D_3$, $A_3 = \{e_a, (a_1\; a_2\; a_3)\,, (a_1\; a_3\; a_2)\}$ and $D_3 = \{e_d, (d_2\; d_3)\,, (d_1\; d_2)\,, (d_1\; d_2\; d_3)\,, (d_1\; d_3\; d_2)\,, (d_1\; d_3)\}$.
Then;

$$G = \{(e_a, e_d), (e_a, (d_2\; d_3)), (e_a, (d_1\; d_2)), (e_a, (d_1\; d_2\; d_3)), (e_a, (d_1\; d_3\; d_2)), (e_a, (d_1\; d_3)), ((a_1\; a_2\; a_3), e_d),$$
$$((a_1\; a_2\; a_3), (d_2\; d_3)), ((a_1\; a_2\; a_3), (d_1\; d_2)), ((a_1\; a_2\; a_3), (d_1\; d_2\; d_3)), ((a_1\; a_2\; a_3), (d_1\; d_3\; d_2)),$$
$$((a_1\; a_2\; a_3), (d_1\; d_3)), ((a_1\; a_3\; a_2), e_d), ((a_1\; a_3\; a_2), (d_2\; d_3)), ((a_1\; a_3\; a_2), (d_1\; d_2)), ((a_1\; a_3\; a_2),$$
$$(d_1\; d_2\; d_3)), ((a_1\; a_3\; a_2), (d_1\; d_3\; d_2)), ((a_1\; a_3\; a_2), (d_1\; d_3))\}$$

$|G| = 18$

Suppose that $K = X \times Y$, $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ then;

$K = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_3, y_1), (x_3, y_2), (x_3, y_3)\}$

such that the order of K is 9.

From definition 2.8, $Stab_G(x_1, y_1) = \{(e_a, e_d), (e_a, (d_2\; d_3))\}$ implying that that the order of the stabilizer is 2 and from definition 2.9,

$Orb_G(x_1, y_1) = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_3, y_1), (x_3, y_2), (x_3, y_3)\}$ $|Orb_G(x_1, y_1)| = 9$
Using theorem 2.1;
$|Orb_G(x_1, y_1)| = \dfrac{|G|}{|Stab_G(x_1, y_1)|} = \dfrac{18}{2} = 9 = |X \times Y|$
$\therefore$ The group $A_3 \times D_3$ action on $X \times Y$ is transitive.

3.2. Transitivity of the Group $A_4 \times D_4$ Action on $X \times Y$

Lemma 3.2. The group action $A_4 \times D_4$ on $X \times Y$ is transitive.
Proof Let $G = A_4 \times D_4$,
$A_4 = \{(e_a), (a_2\ a_3\ a_4), (a_2\ a_4\ a_3), (a_1\ a_2)\ (a_3\ a_4), (a_1\ a_2\ a_3), (a_1\ a_2\ a_4), (a_1\ a_3\ a_2), (a_1\ a_3\ a_4), (a_1\ a_3)\ (a_2\ a_4), (a_1\ a_4\ a_2), (a_1\ a_4\ a_3), (a_1\ a_4)(a_2\ a_3)\}$
$D_4 = \{(e_d), (d_2\ d_4), (d_1\ d_2)(d_3\ d_4), (d_1\ d_2\ d_3\ d_4), (d_1\ d_3), (d_1\ d_3)(d_2\ d_4), (d_1\ d_4\ d_3\ d_2), (d_1\ d_4)(d_2\ d_3)\}$
and the order of $G = A_4 \times D_4$ is 96.
Also let;
$K = X \times Y, X = (x_1, x_2, x_3, x_4), Y = (y_1, y_2, y_3, y_4)$
therefore,
$K = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4), (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4), (x_3, y_1), (x_3, y_2), (x_3, y_3), (x_3, y_4), (x_4, y_1), (x_4, y_2), (x_4, y_3), (x_4, y_4)\}$
$|K| = 16$
From the definition 2.8;
$Stab_G(x_1 y_1) = \{(e_a, e_d), (e_a, (d_2\ d_4)), ((a_2\ a_3\ a_4), e_d), ((a_2\ a_3\ a_4), (d_2\ d_4)), ((a_2\ a_4\ a_3), e_d), ((a_2\ a_4\ a_3), (d_2\ d_4))\}$
Its order is 6. And from the definition 2.9;
$Orb_G(x_1, y_1) = \{(x_4, y_1), (x_4, y_2), (x_4, y_3), (x_4, y_4), (x_3, y_1), (x_3, y_2), (x_3, y_3), (x_3, y_4), (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4), (x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4)\}$
The order of orbit is 16. Applying theorem 2.1;
$|Orb_G(x_1, y_1)| = \dfrac{|G|}{|Stab_G(x_1, y_1)|} = \dfrac{96}{6} = 16 = |X \times Y|$
$\therefore$ The group action $A_4 \times D_4$ action on $X \times Y$ is transitive.

3.3. Transitivity of the Group $A_5 \times D_5$ Action on $X \times Y$

Lemma 3.3. The group $A_5 \times D_5$ action on $X \times Y$ is transitive.
Proof Let $G = A_5 \times D_5$,
$A_5 = \{(e_a), (a_3\ a_4\ a_5), (a_3\ a_5\ a_4), (a_2\ a_3)(a_4\ a_5), (a_2\ a_3\ a_4)(a_2\ a_3\ a_5), (a_2\ a_4\ a_3), (a_2\ a_4\ a_5), (a_2\ a_4)(a_3\ a_5), (a_2\ a_5\ a_3), (a_2\ a_5\ a_4), (a_2\ a_5)(a_3\ a_4), (a_1\ a_2)(a_4\ a_5), (a_1\ a_2)(a_3\ a_4), (a_1\ a_2)(a_3\ a_5), (a_1\ a_2\ a_3), (a_1\ a_2\ a_3\ a_4\ a_5), (a_1\ a_2\ a_3\ a_5\ a_4), (a_1\ a_2\ a_4\ a_5\ a_3), (a_1\ a_2\ a_4\ a_3\ a_5), (a_1\ a_2\ a_5\ a_4\ a_3), (a_1\ a_2\ a_4), (a_1\ a_2\ a_5), (a_1\ a_3\ a_2), (a_1\ a_2\ a_5\ a_3\ a_4), (a_1\ a_3\ a_4\ a_5\ a_2), (a_1\ a_3\ a_5\ a_4\ a_2), (a_1\ a_3)(a_4\ a_5), (a_1\ a_3)(a_2\ a_4), (a_1\ a_3)(a_2\ a_5), (a_1\ a_3\ a_4), (a_1\ a_3\ a_5), (a_1\ a_3\ a_2\ a_4\ a_5), (a_1\ a_3\ a_5\ a_2\ a_4), (a_1\ a_3\ a_2\ a_5\ a_4), (a_1\ a_3\ a_4\ a_2\ a_5), (a_1\ a_4\ a_2), (a_1\ a_4\ a_3), (a_1\ a_4)(a_3\ a_5), (a_1\ a_4\ a_5\ a_3\ a_2), (a_1\ a_4\ a_3\ a_5\ a_2), (a_1\ a_4\ a_5), (a_1\ a_4\ a_5\ a_2\ a_3), (a_1\ a_4)(a_2\ a_3), (a_1\ a_4\ a_2\ a_3\ a_5), (a_1\ a_4\ a_2\ a_5\ a_3), (a_1\ a_4\ a_3\ a_2\ a_5), (a_1\ a_5\ a_4\ a_3\ a_2), (a_1\ a_5\ a_3\ a_4\ a_2), (a_1\ a_5\ a_4\ a_2\ a_3), (a_1\ a_5\ a_2\ a_3\ a_4), (a_1\ a_5\ a_2\ a_4\ a_3), (a_1\ a_5\ a_3\ a_2\ a_4), (a_1\ a_4)(a_2\ a_5), (a_1\ a_5)(a_3\ a_4), (a_1\ a_5)(a_2\ a_3), (a_1\ a_5)(a_2\ a_4), (a_1\ a_5\ a_2), (a_1\ a_5\ a_3), (a_1\ a_5\ a_4)\}$,
$D_5 = \{e_d, (d_2\ d_5)(d_3\ d_4), (d_1\ d_2)(d_3\ d_5), (d_1\ d_3)(d_4\ d_5), (d_1\ d_4)(d_2\ d_3), (d_1\ d_5)(d_2\ d_4), (d_1\ d_2\ d_3\ d_4\ d_5), (d_1\ d_3\ d_5\ d_2\ d_4), (d_1\ d_4\ d_2\ d_5\ d_3), (d_1\ d_5\ d_4\ d_3\ d_2)\}$ $|G| = 600$
Let $K = X \times Y$ where $X = (x_1, x_2, x_3, x_4, x_5)$ and $Y = (y_1, y_2, y_3, y_4, y_5)$ then;
$K = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4), (x_1, y_5), (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4), (x_2, y_5), (x_3, y_1), (x_3, y_2), (x_3, y_3), (x_3, y_4), (x_3, y_5), (x_4, y_1), (x_4, y_2), (x_4, y_3), (x_4, y_4), (x_4, y_5), (x_5, y_1), (x_5, y_2), (x_5, y_3), (x_5, y_4), (x_5, y_5)\}$ $|K| = 25$
From the definition 2.8;

$$Stab_G(x_1,y_1) = \{(e_a,e_d),(e_a,(d_2\ d_5)(d_3\ d_4)),((a_2\ a_3)(a_4\ a_5),e_d),((a_3\ a_4\ a_5),e_d), ((a_3\ a_4\ a_5),(d_2\ d_5)(d_3\ d_4)),((a_3\ a_5\ a_4),e_d),((a_3\ a_5\ a_4),(d_2\ d_5)(d_3\ d_4)), ((a_2\ a_3)(a_4\ a_5),(d_2\ d_5)(d_3\ d_4)),((a_2\ a_3\ a_4),e_d),((a_2\ a_3\ a_4),(d_2\ d_5)(d_3\ d_4)), ((a_2\ a_3\ a_5),e_d)),((a_2\ a_3\ a_5),(d_2\ d_5)(d_3\ d_4)),((a_2\ a_4\ a_3),e_d),((a_2\ a_4\ a_3), (d_2\ d_5)(d_3\ d_4)),((a_2\ a_4\ a_5),e_d),((a_2\ a_4\ a_5),(d_2\ d_5)(d_3\ d_4)),((a_2\ a_4)(a_3\ a_5),e_d), ((a_2\ a_4)(a_3\ a_5),(d_2\ d_5)(d_3\ d_4)),(a_2\ a_5\ a_3),e_d),((a_2\ a_5\ a_3),(d_2\ d_5)(d_3\ d_4)), (a_2\ a_5\ a_4),e_d),((a_2\ a_5\ a_4),(d_2\ d_5)(d_3\ d_4)),((a_2\ a_5)(a_3\ a_4),e_d), ((a_2\ a_5)(a_3\ a_4),(d_2\ d_5)(d_3\ d_4))\} \quad |Stab_G(x_1,y_1)| = 24$$

From the definition 2.9;

$$Orb_G(x_1,y_1) = \{(x_1,y_1),(x_1,y_2),(x_1,y_3),(x_1,y_4),(x_1,y_5),(x_2,y_1),(x_2,y_2),(x_2,y_3),(x_2,y_4), (x_2,y_5),(x_3,y_1),(x_3,y_2),(x_3,y_3),(x_3,y_4),(x_3,y_5),(x_4,y_1),(x_4,y_2),(x_4,y_3), (x_4,y_4),(x_4,y_5),(x_5,y_1),(x_5,y_2),(x_5,y_3),(x_5,y_4),(x_5,y_5)\} \quad |Orb_G(x_1,y_1)| = 25$$

Applying theorem 2.1;

$$|Orb_G(x_1,y_1)| = \frac{|G|}{|Stab_G(x_1,y_1)|} = \frac{600}{24} = 25 = |X \times Y|$$

$\therefore$ The group $A_5 \times D_5$ action on $X \times Y$ is transitive.

3.4. Transitivity of the Group $A_n \times D_n$ Action on $X \times Y$

Theorem 3.1. For $n \geqslant 3$, the group $A_n \times D_n$ action on $X \times Y$ is transitive, where $X = \{x_1,x_2,x_3,...,x_n\}$ and $Y = \{y_1,y_2,y_3,...,y_n\}$

Proof For $n \geqslant 3$, it can be proofed that $|Orb_G(x_n,y_n)| = |X \times Y|$. If $G = A_n \times D_n$, $|A_n| = \frac{n!}{2}$,$|D_n| = 2n$ then, $|G| = |A_n \times D_n| = \frac{n! \times 2n}{2} = n! \times n$. Also, let $K = X \times Y$ and $|X| = n$, $|Y| = n$, then, $|K| = n \times n = n^2$. By theorem 2.2 the order of the stabilizer of $A_n = \frac{(n-1)!}{2}$ and that the order of stabilizer of D_n is always 2, using theorem 2.1;

$$\begin{aligned} |Orb_G(x_n,y_n)| &= \frac{|G|}{|Stab_G A_n||Stab_G D_n|} \\ &= \frac{n! \times n}{\frac{(n-1)!}{2} \times 2}\ since\ n! = n \cdot (n-1)! \\ &= \frac{n! \times n}{(n-1)!} \\ &= \frac{n \cdot (n-1)! \times n}{(n-1)!} \\ &= n \times n \\ &= n^2 \\ &= |X \times Y| \end{aligned}$$

$\therefore$ The group $A_n \times D_n$ action on $X \times Y$ is transitive.

3.5. Primitivity of the Group $A_3 \times D_3$ Action on $X \times Y$

Lemma 3.4. The group $A_3 \times D_3$ action on $X \times Y$ is imprimitive.

Proof. Let $G = A_3 \times D_3$, $K = X \times Y$, $X = (x_1,x_2,x_3)$, $Y = (y_1,y_2,y_3)$ and G acts transitively on K. The elements of;

$$K = \{(x_1,y_1),(x_1,y_2),(x_1,y_3),(x_2,y_1),(x_2,y_2),(x_2,y_3), (x_3,y_1),(x_3,y_2),(x_3,y_3)\}$$

$|K| = 9 = 3 \times 3 = 3^2$

From definition 2.10 and 2.11

Let $Q \subseteq K$ such that $|Q|$ divides $|K|$ by a divisor of $|K|$. Considering $|Q| = \{(x_1,y_1),(x_2,y_2),(x_3,y_3)\} = 3$ such that $\frac{|K|}{|Q|} = \frac{3^2}{3}$. There exist $g \in G$ such that it translates all elements of Q out of Q i.e. $gQ \cap Q = \emptyset$.

Since G acts transitively on K,

$g = ((a_1\ a_2\ a_3),(d_1\ d_3))$ then,

$gQ = ((a_1\ a_2\ a_3),(d_1\ d_3))(x_1,y_1) = (x_2,y_3)$

clearly $gQ \cap Q = \emptyset$, this is a proof that Q is a non-trivial block and therefore the group action is imprimitive

3.6. Primitivity of the Group $A_n \times D_n$ Action on $X \times Y$

Theorem 3.2. The group $A_n \times D_n$ action on $X \times Y$ is imprimitive.

Proof Consider $G = A_n \times D_n$ acting on $X \times Y$, $K = X \times Y$, $X = (x_1,x_2,x_3,...,x_n)$, $K = X \times Y$, $Y = (y_1,y_2,y_3,...,y_n)$ and G acts transitively on K therefore, $|K| = n \times n = n^2$. Suppose that $Q \subseteq K$ such that $|Q|$ divides $|K|$ by a divisor of $|K|$ and $|Q| = |\{(x_1,y_1),(x_2,y_2),(x_3,y_3),...,(x_n,y_n)\}| = n$ then, $\frac{|K|}{|Q|} = \frac{n^2}{n}$. There exist $g \in G$ such that g translates all elements of Q out of Q i.e. $gQ \cap Q = \emptyset$. It is clear that Q is a non-trivial block of the action concluding that the action is imprimitive.

3.7. Ranks and Subdegrees of the Group $A_3 \times D_3$ Action on $X \times Y$

Lemma 3.5. Suppose that the group $A_3 \times D_3$ action on $X \times Y$ is transitive, then the rank is 6 and the corresponding subdegrees are $1,2,2,2,1,1$.

Proof Since $G = A_3 \times D_3$ and $K = X \times Y$.

From lemma 3.1,

$$K = \{(x_1,y_1),(x_1,y_2),(x_1,y_3),(x_2,y_1),(x_2,y_2),(x_2,y_3),(x_3,y_1),(x_3,y_2),(x_3,y_3)\}$$

and
$Stab_G(x_1, y_1) = \{(e_a, e_d), (e_a, (d_2\ d_3))\}$.
Then, the number of elements in K fixed by each $g \in G$ is 9.

Table 1. *Permutations in $Stab_G(x_1, y_1)$ and the No. of Fixed Points.*

Permutation type in $Stab_G(x_1, y_1)$	Number of permutations ($\|G\|$)	$Fix_G(g)$
(e_a, e_d)	1	9
$(e_a, (ab))$	1	3
TOTAL	2	

By lemma 2.1, the number of suborbits of G acting on K is given by:
$$\frac{1}{|G|}\sum_{g\in G}|Fix(g)| = [\frac{1}{2}((9\times 1) + (1\times 3))] = \frac{1}{6}(12) = 6$$
This shows that there are 9 orbits classified into six suborbits i.e.
$\Delta_0 = Orb_G(x_1, y_1) = \{(x_1, y_1)\} = 1$ (Trivial Orbit)
$\Delta_1 = Orb_G(x_1, y_2) = \{(x_1, y_2), (x_1, y_3)\} = 2$
$\Delta_2 = Orb_G(x_2, y_2) = \{(x_1, y_2), (x_2, y_3)\} = 2$
$\Delta_3 = Orb_G(x_3, y_2) = \{(x_3, y_2), (x_3, y_3)\} = 2$
$\Delta_4 = Orb_G(x_2, y_1) = \{(x_2, y_1)\} = 1$
$\Delta_5 = Orb_G(x_3, y_1) = \{(x_3, y_1)\} = 1$
Therefore, the group action has rank 6 and the corresponding subdegrees are $1, 2, 2, 2, 1, 1$.

3.8. Ranks and Subdegrees of the Group $A_4 \times D_4$ Action on $X \times Y$

Lemma 3.6. Let the group $A_4 \times D_4$ action on $X \times Y$ be transitive, then the rank is 6 and the corresponding subdegrees are $1, 3, 3, 3, 3, 3$.
Proof Since $G = A_4 \times D_4$ and $K = X \times Y$.
From lemma 3.2,
$$K = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4), (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4), (x_3, y_1), (x_3, y_2), (x_3, y_3), (x_3, y_4), (x_4, y_1), (x_4, y_2), (x_4, y_3), (x_4, y_4)\}$$
and
$$Stab_G(x_1, y_1) = \{(e_a, e_d), (e_a, (d_2\ d_4)), ((a_2\ a_3\ a_4), e_d), ((a_2\ a_3\ a_4), (d_2\ d_4)), ((a_2\ a_4\ a_3), e_d), ((a_2\ a_4\ a_3), (d_2\ d_4))\}$$
Therefore, the number of elements in K fixed by each $g \in G$ is 16.

Table 2. *Permutations in $Stab_G(x_1, y_1)$ and the No. of Fixed Points.*

Permutation types in $Stab_G(x_1, y_1)$	Number of permutations ($\|G\|$)	$Fix_G(g)$
(e_a, e_d)	1	16
$(e_a, (ab))$	1	8
$((abc), e_d)$	1	4
$((abc), de)$	2	2
TOTAL	6	

By lemma 2.1, the number of suborbits of G acting on K is given as:
$$\frac{1}{|G|}\sum_{g\in G}|Fix(g)| = [\frac{1}{6}((1\times 16) + (1\times 8) + (2\times 4) + (2\times 2))] = \frac{1}{6}(36) = 6.$$
This shows that there are 16 orbits that can be classified into six suborbits i.e.
$\Delta_0 = Orb_G(x_1, y_1) = \{(x_1, y_1)\} = 1$ (Trivial Orbit)
$\Delta_1 = Orb_G(x_1, y_2) = \{(x_1, y_2), (x_1, y_3), (x_1, y_4)\} = 3$
$\Delta_2 = Orb_G(x_2, y_1) = \{(x_2, y_1), (x_3, y_1), (x_4, y_1)\} = 3$
$\Delta_3 = Orb_G(x_2, y_2) = \{(x_2, y_2), (x_3, y_2), (x_4, y_2)\} = 3$
$\Delta_4 = Orb_G(x_2, y_3) = \{(x_2, y_3), (x_3, y_3), (x_4, y_3)\} = 3$
$\Delta_5 = Orb_G(x_2, y_4) = \{(x_2, y_4), (x_3, y_4), (x_4, y_4)\} = 3$
Therefore, the group action has a rank 6 and the corresponding subdegrees are $1, 3, 3, 3, 3, 3$.

3.9. Ranks and Subdegrees of the Group $A_5 \times D_5$ Action on $X \times Y$

Lemma 3.7. Suppose that the group $A_5 \times D_5$ action on $X \times Y$ is transitive, then the rank is 6 and the corresponding subdegrees are $1, 5, 5, 5, 5, 4$.

Proof Since $G = A_5 \times D_5$ and $K = X \times Y$.

From lemma 3.3,

$$\begin{aligned} K = \{&(x_1,y_1),(x_1,y_2),(x_1,y_3),(x_1,y_4),(x_1,y_5),(x_2,y_1),(x_2,y_2),(x_2,y_3),(x_2,y_4),(x_2,y_5),(x_3,y_1),(x_3,y_2),(x_3,y_3),\\ &(x_3,y_4),(x_3,y_5),(x_4,y_1),(x_4,y_2),(x_4,y_3),(x_4,y_4),(x_4,y_5),(x_5,y_1),(x_5,y_2),(x_5,y_3),(x_5,y_4),(x_5,y_5)\}\end{aligned}$$

and

$$\begin{aligned} Stab_G(x_1,y_1) = \{&(e_a,e_d),(e_a,(d_2\ d_5)(d_3\ d_4)),((a_2\ a_3)(a_4\ a_5),e_d),((a_3\ a_4\ a_5),e_d),((a_3\ a_4\ a_5),(d_2\ d_5)(d_3\ d_4)),\\ &((a_3\ a_5\ a_4),e_d),((a_3\ a_5\ a_4),(d_2\ d_5)(d_3\ d_4)),((a_2\ a_3)(a_4\ a_5),(d_2\ d_5)(d_3\ d_4)),((a_2\ a_3\ a_4),e_d),\\ &((a_2\ a_3\ a_4),(d_2\ d_5)(d_3\ d_4)),((a_2\ a_3\ a_5),e_d)),((a_2\ a_3\ a_5),(d_2\ d_5)(d_3\ d_4)),((a_2\ a_4\ a_3),e_d)\\ &((a_2\ a_4\ a_3),(d_2\ d_5)(d_3\ d_4)),((a_2\ a_4\ a_5),e_d),((a_2\ a_4\ a_5),(d_2\ d_5)(d_3\ d_4)),((a_2\ a_4)(a_3\ a_5),e_d),\\ &((a_2\ a_4)(a_3\ a_5),(d_2\ d_5)(d_3\ d_4)),(a_2\ a_5\ a_3),e_d),((a_2\ a_5\ a_3),(d_2\ d_5)(d_3\ d_4)),(a_2\ a_5\ a_4),e_d),\\ &((a_2\ a_5\ a_4),(d_2\ d_5)(d_3\ d_4)),((a_2\ a_5)(a_3\ a_4),e_d),((a_2\ a_5)(a_3\ a_4),(d_2\ d_5)(d_3\ d_4))\}\end{aligned}$$

Therefore, the number of elements in K fixed by each $g \in G$ is 25.

Table 3. *Permutations in $Stab_G(x_1,y_1)$ and the No. of Fixed Points.*

Permutation types in $Stab_G(x_1,y_1)$	**Number of permutations ($\|G\|$)**	$Fix_G(g)$
(e_a,e_d)	1	25
$(e_a,(gj)(hi))$	1	10
$((cde),e_d)$	1	10
$((cde),(gj)(hi))$	1	10
$((ced),e_d)$	1	10
$((ced),(gj))$	1	5
$((bc)(de),e_d)$	1	10
$((bc)(de),(gj)(hi))$	1	5
$((bcd),e_d)$	1	10
$((bcd),(gj))$	1	10
$((bce),e_d)$	1	5
$((bce),(gj)(hi))$	1	10
$((bdc),(gj)(hi))$	1	2
$((bdc),e_d)$	1	5
$((bde),e_d)$	1	2
$((bde),(gj)(hi))$	1	2
$((bd)(ce),e_d)$	1	2
$((bd)(ce),(gj)(hi))$	1	1
$((bec),e_d)$	1	2
$((bec),(gj)(hi))$	1	1
$((bed),e_d)$	1	2
$((bed),(gj)(hi))$	1	2
$((be)(cd),e_d)$	1	2
$((be)(cd),(gj)(hi))$	1	2
TOTAL	24	

By lemma 2.1, the number of suborbits of G acting on K is given as:

$= \frac{1}{|G|} \sum_{g \in G} |Fix\,(g)|$
$= \frac{1}{24}(25+10+10+10+10+5+10+5+10+10+5+10+5+2+2+2+2+1+2+1+2+2+1+2)$
$= \frac{1}{24}(144)$
$= 6.$

This shows that there are 25 suborbits that can be classified into six categories, that is
$\Delta_0 = Orb_G(x_1, y_1) = \{(x_1, y_1)\} = 1$ (Trivial Orbit)
$\Delta_1 = Orb_G(x_2, y_1)\} = \{(x_2, y_1), (x_2, y_2), (x_2, y_3), (x_2, y_4), (x_2, y_5)\} = 5$
$\Delta_2 = Orb_G(x_3, y_1) = \{(x_3, y_1), (x_3, y_2), (x_3, y_3), (x_3, y_4), (x_3, y_5)\} = 5$
$\Delta_3 = Orb_G(x_4, y_1) = \{(x_4, y_1), (x_4, y_2), (x_4, y_3), (x_4, y_4), (x_4, y_5)\} = 5$
$\Delta_4 = Orb_G(x_5, y_1) = \{(x_5, y_1), (x_5, y_2), (x_5, y_3), (x_5, y_4), (x_5, y_5)\} = 5$
$\Delta_5 = Orb_G(x_1, y_2) = \{(x_1, y_2), (x_1, y_3), (x_1, y_4), (x_1, y_5)\} = 4$
Therefore, the group action has a rank 6 and the corresponding subdegrees are $1, 5, 5, 5, 5, 4$.

3.10. Ranks and Subdegrees of Group $A_n \times D_n$ Action on $X \times Y$

Theorem 3.3. Suppose that the group $A_n \times D_n$ acts on $X \times Y$, then the rank is 6 for $n \geqslant 3$ and the corresponding subdegrees are;

$$|\Delta_i| = \begin{cases} 1, & \\ p, & \text{if } i \leqslant r \\ p-1, & \text{if } i > r \end{cases}$$

$\forall i = 1, 2, ..., 5$, $p = \left[\frac{|K|-1}{5}\right]$ and $r = (|K|-1) \mod 5$

Proof Using lemma 3.5, lemma 3.6 and lemma 3.7, the number of orbits in each suborbit can be classified into two categories;

The first suborbit which contains only the trivial orbit;
$\Delta_0 = Orb_G(x_1, y_1) = (x_1, y_1) = 1$

The remaining 5 suborbits that is, $(\Delta_1, \Delta_2, ..., \Delta_5)$ will have orbits depending on the division of remaining orbits $(|K|-1)$ by 5 (number of the remaining suborbits);

If $(|K|-1)$ is perfectly divisible by 5, then each of the remaining five suborbits $(\Delta_1, \Delta_2, ..., \Delta_5)$ will have equal number of $p - orbits$.

If $(|K|-1)$ is not perfectly divisible by 5 and the remainder is r, then the first r suborbits will have $p - orbits$ and the remaining $(5-r) - suborbits$ will have $(p-1) - orbits$.

Corollary 5.5.1 Determine the rank and subdegrees of the group $A_6 \times D_6$ action on $X \times Y$

Proof. From theorem 3.3 the rank is 6 and the corresponding subdegrees are obtained by; $|K| = 36$ and $p = \frac{36-1}{5} = \frac{35}{5} = 7$ therefore each of the five suborbits will have 7 orbits. Implying that the subdegrees are $1, 7, 7, 7, 7, 7$.

Corollary 5.5.2 Determine the rank and subdegrees of the group $A_7 \times D_7$ action on $X \times Y$

Proof. From theorem 3.3 the rank is 6 and the corresponding subdegrees are obtained by; $|K| = 49$,$p = \frac{49-1}{5} = \frac{48}{5} = 9.6 \approx 10$ and $r = 48 \bmod 5 = 3$. This implies that the first three suborbits will each have 10 orbits and the remaining two suborbits each will have 9 orbits. Therefore the subdegrees will be $1, 10, 10, 10, 9, 9$.

4. Conclusion

Combinatorial properties (transitivity and primitivity) have been studied and it has been proofed that the group action $(A_n \times D_n)$ on set $(X \times Y)$ is transitive and imprimitive. Also, invariant properties (ranks and subdegrees) have been investigated and and it was found that for $n \geqslant 3$, the rank is 6 and the subdegrees are obtained according to theorem 3.3

Symbols and Abbreviations

A_n	Alternating group (degree n and order is $\frac{n!}{2}$)
D_n	Dihedral group of order $2n$
e_a	An identity element in Alternating group
e_d	An identity element in Dihedral group
$K = X \times Y$	Cartesian product between set X and set Y
$\text{Stab}_G(x)$	Stabilizer of a point x in G
$\text{Fix}_G(x)$	A point x fixed an element in G
$\text{Orb}_G(x)$	The orbit of a point x in G
Δ	Suborbit of G on set K
$\lvert K \rvert$	Order of K
$\forall$	for all
$\emptyset$	An empty set
$G = A_n \times D_n$	Direct product between Alternating group and Dihedral group

ORCID

0009-0000-5457-6011 (John Mokaya Victor)
0009-0000-0004-9187 (Nyaga Lewis Namu)
0009-0001-0285-6692 (Gikunju David Muriuki)

Author Contributions

John Mokaya Victor: Conceptualization, Formal Analysis, Funding acquisition, Investigation, Methodology, Writing ¨C original draft, Writing - review & editing

Nyaga Lewis Namu: Conceptualization, Formal Analysis, Project administration, Supervision, Validation

Gikunju David Muriuki: Conceptualization, Formal Analysis, Project administration, Supervision, Validation

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Lowen, R. (1996). Elementary Set Theory. *Fuzzy Set Theory*, $1-19$. https://doi.org/10.1007/978-94-015-8741-9_1

[2] Kurzweil, H. and Stellmacher, B. (2004). Permutation Groups. *The Theory of FiniteGroups* (pp. 77-97). New York: Springer New York. https://doi.org/10.1007/0-387-21768-1_4

[3] Burnside William. (1911). Theory of Groups of finite Order. *Cambridge University Press*.2.1. https://doi.org/10.1017/CBO9781139237253.023

[4] Thomas, W. J. (2020). Abstract Algebra (Theory and Applications). *Stephen F. Austin University*. https://scholarworks.sfasu.edu/ebooks/23

[5] abstractGroups2014 Abstract Groups. (2014). *A First Course in Abstract Algebra*, 224–235. https://doi.org/10.1201/b17673-28

[6] Gachogu, R., Kamuti, I. N., Gachuki, M. N. (2017). Properties of Suborbitals of Dihedral group acting on ordered subset. *Advances in Pure Mathematic*s, 7, 375-382. https://doi.org/10.4236/apm.2017.78024

[7] Orina, M. D., Namu, N. L., Muriuki, G. D. (2024). Combinatorial Properties, Invariants and Structures Associated with the Direct Product of Alternating and Cyclic Groups Acting on the Cartesian Product of Two Sets. American Journal of Applied Mathematics, 12(5), 167-174. https://doi.org/10.11648/j.ajam.20241205.16

[8] Gachimu, R., kamuti, I., Nyaga, L., Rimberia, J., Kamaku, P. (2016).Properties invariants Associated with the Action of the Alternating Group on Unordered subsets. *International Journal of Pure and Applied Mathematics*, 106(1), 333-346. https://doi.org/10.12732/ijpam.v106i1.27

[9] Njagi, L. (2016). Ranks and Subdegrees of Suborbital Graphs of Symmetric Group Acting on Ordered Pairs. *Journal of Advanced Research in Applied Science*, 3(2). https://doi.org/10.53555/nnas.v3i2.664

[10] Siro, M., Kamuti, I, Maingi, M. (2014). Actions of Symmetric Group $S_n(n \leqslant 7)$ on Unordered Quadruples. *International Journal of Algebra*, 8(3),115-120. https://doi.org/10.12988/ija.2014.311112

[11] Gikunju, M. D., Nyaga, N. L., Rimberia, K. J. (2017). Ranks, Subdegree and Suborbital Graph of Direct Product of Symmetric Group Acting on the Cartesian Product of Three Sets. *Pure and Applied Mathematics Journal*, 6(1), 1-4. https://doi.org/10.11648/j.pamj.20170601.11

[12] Scott, W. R. (1987). Group Theory.*Dover Publications Inc*. New York. Second Edition. Page 2.

[13] Joseph, A. G. (2012). Contemporary Abstract ALgebra.Brooks *Cole Cengage Learning*. 8th Edition.

[14] Gustavo, M. L. (2013). Direct product of the group. Mathematics Stack Exchange. https://math.stackexchange.com/q/390352

[15] Rose, J. S. (1978). A course on group theory. *Cambridge University Press*, Cambridge.

[16] Harary, F. (1969). Graph theory. *Addison - Wesley Publishing Company*, New York. https://doi.org/10.2307/3615048

[17] Wambui, M. T., Rimberia, J. (2019). Ranks, Subdegrees, Suborbital Graphs and Cycle Indices Associated with the Product Action of $A_n \times A_n \times A_n\,(4 \leqslant n \leqslant 8)$ on Cartesian Product $X \times Y \times Z$. *Kenyatta University Institutional Repository*. http://ir-library.ku.ac.ke/handle/123456789/20050

[18] Ivanov, A. A (1983). On The Problem of Computing Subdegrees of Transitive Permutation Groups. *Soviet Mathemetical Survey*. 38. 123-124. https://doi.org/10.1070/rm1983v038n06abeh003460