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Research Article

An Accurate Two-Step Optimized Hybrid Block Method for Integrating Stiff Differential Equations

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Abstract

An accurate two-step optimized hybrid block method is proposed for integrating stiff initial value problems of ordinary differential equations. The techniques of interpolation and collocation were applied to a power series polynomial for the derivation of the method using a three-parameter approximation of the hybrid points. The hybrid points were obtained by minimizing the local truncation error of the main method. The discrete schemes were produced as by-products of the continuous scheme and used to simultaneously solve initial value problems (IVPs) in block mode. The analysis of the basic properties of the method revealed that the schemes are self-starting, consistent, zero-stable, and A-stable. Furthermore, the analysis of the order of accuracy of the method showed that there is a gain of one order of accuracy in the main scheme where the optimization was carried out thereby enhancing the accuracy of the whole method. The accuracy of the method was ascertained using several numerical experiments. Comparison of the numerical results of the new method with those of the existing methods revealed that the newly developed method performed better than some of the existing hybrid block methods. Hence, the new method should be employed for the numerical solution of stiff ordinary differential equations to obtain more accurate results.

Keywords

Linear Stability, Local Truncation Error (LTE), Parameter Approximations, Initial Value Problems (IVPs), Ordinary Differential Equations (ODEs)

1. Introduction

The first order initial value problems of ordinary differential equations are used in mathematical formulation of many physical processes such as growth and decay, electrical circuit, falling body problem, prey-predictor model, radioactive decay, etc. Most of these physical phenomenon produced differential equations that are stiff in nature. and thus finding exact solutions to the differential equations is often challenging. The utilization of numerical techniques was necessary in order to obtain an approximate solution. Various approaches, such as collocation, interpolation, integration, and interpolation polynomials, have been thoroughly investigated in academic literature to construct continuous linear multistep

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methods (LMMs) for the direct solution of initial value problems in ordinary differential equations (see [1-5] and the literature therein). Most of the traditional methods such as Runge-Kutta [6], multi-step Adams family [7-9], and higher-order multi-derivative types [10, 11] did not yield desirable results in solving stiff differential equations because a large amount of computational effort was required or conditional stability was obtained. This necessitated the adoption of implicit block methods which possess the attribute of being self-starting, highly accurate, and absolutely stabile. One of such notable methods in this category are the hybrid block methods. Hybrid linear multi-step methods were introduced a few decades ago to overcome the first Dahlquist barrier on the step number and order of stable LMMs [12]. Dahlquist stated that for a k-step LMM to be stable, the order cannot exceed k + 1 (if k is odd) or k + 2 (if k is even) [12]. Hybrid methods allow access to information at intra-step points thereby providing solutions at those points. Therefore, restriction on the order and hence accuracy of the method is removed.

An initial value problem of the form:

$$x' = f(t,x), x(t_0) = x_0 \tag{1}$$

is considered, where, $t \in [t_0, T]$, $f\colon [t_0, T] \times \Re \to \Re$. It is assumed that equation (1) satisfies the conditions of the existence and uniqueness theorem for initial value problems [9].

The desire for improved accuracy of numerical methods has led to the development of new methods which are derived by minimization of the Local Truncation Errors (LTEs). In their study, Singh et al. [12] introduced an optimized hybrid block approach with distinct characteristics for numerically integrating initial value problems of ordinary differential systems. The method successfully overcomes the first Dahlquist barrier on Linear Multi-Step Methods (LMMs) by incorporating both block and hybrid characteristics. The method of interpolation and collocation was employed by utilizing an approximate polynomial representation of the theoretical solution of the problem. Three intermediate points were added within a single block, with one point being fixed and the other two optimized to minimize the errors in the primary formula and an additional formula. The resulting scheme had a fifth order accuracy and possessed the attribute of A-stability. The study conducted by Ramos [13] proposed a two-step method that involved the selection of two intermediate points through the optimization of the LTEs. However, the most optimal formulation was attained through the process of reformulating the method in a manner that decreases the frequency of instances of the source term f. Kashkari and Syam [14] presented a novel optimized one-step hybrid block technique that is specifically tailored for the optimization of first-order initial value problems (IVPs). The methodology entailed the careful selection of three hybrid points to optimize the LTEs of the basic equations governing the behavior of the block. Furthermore, Tassaddiq et al. [15] introduced a novel one-step implicit block approach that incorporates three intra-step grid points. The principal term of the LTE was minimized to identify one of the three intra-step points as the optimal solution. A modification of the technique resulted in a substantial reduction in computational expenses while preserving the identical levels of consistency, zero-stability, A-stability, and convergence. The method was employed to address practical issues and its outcomes were compared with those of other methods documented in literature to demonstrate the superiority of the new approach. Ramos et al. [16] employed an enhanced hybrid block technique in conjunction with a modified cubic B-spline method to solve non-linear partial differential equations. No linearization was necessary in the approach, and the time step-size was optimized without compromising accuracy. Singla et al. [17] devised a set of one-step hybrid block methods that incorporate two intra-step points. These methods are designed to solve first-order initial value stiff differential systems. Within each family, there is an intrastep point that determines the sequence of the main technique, and a second point that governs the stability characteristics of the method. The approaches were also devised as Runge-Kutta methods. Yakubu and Sibanda [18] proposed a novel approach for solving first-order stiff initial value problems through the development of a one-step family of three optimized second-derivative hybrid block methods. The optimization process was integrated into the derivation of the methods to achieve maximal accuracy. The analysis revealed that the methods exhibit convergence and A-stability. Some other recent and notable contributions on optimized hybrid block method may be found in [19-23] and the literature therein.

The new Two Step Optimized Hybrid Block Method (TSOHBM) proposed in this research incorporates five hybrid points with a three-parameter approximation. The interval of integration is allowed to determine the optimal hybrid points through the optimization of the principal term of the LTE of the main method. Previous works have not considered up to five intra-step points with three unknown parameters in an optimization technique of this nature.

This article is organized as follows: Derivation of the two step optimized hybrid block method is done in section 2, and analysis of the basic properties of the method is carried out in section 3. In section 4, numerical examples are solved to ascertain the performance of the new method, and discussion of the results is presented in section 5.

2. Materials and Methods

The theoretical solution $x(t)$ of equation (1) is approximated by the polynomial Q(t) of the form

$$Q(t) = \sum_{j=0}^{m} b_j t^j. \tag{2}$$

where $b_j \in R$ are real unknown coefficients to be determined. $m = (C + I) - 1$, I and C denote the number of interpolation and collocation points respectively. The first derivative of (2) is obtained

$$Q'(t) = \sum_{j=0}^{m} j b_j t^{j-1}, \tag{3}$$

Interpolating equation (2) at $t_{n+j}, j = 0$ collocating equation (3) at $t_{n+j}, j = 0, p, \frac{2}{3}\ q, \frac{4}{3}, r, 2$, where $p, \frac{2}{3}\ q, \frac{4}{3}, r$ are the hybrid points such that $0 < p < \frac{2}{3} < q < \frac{4}{3} < r < 2$. This yields a system of linear equations given in (4).

$$\begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & t_n^6 & t_n^6 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^4 & 7t_n^4 \\ 0 & 1 & 2t_{n+p} & 3t_{n+p}^2 & 4t_{n+p}^3 & 5t_{n+p}^4 & 6t_{n+p}^4 & 7t_{n+p}^4 \\ 0 & 1 & 2t_{n+\frac{2}{3}} & 3t^2_{n+\frac{2}{3}} & 4t^3_{n+\frac{2}{3}} & 5t^4_{n+\frac{2}{3}} & 6t^4_{n+\frac{2}{3}} & 7t^4_{n+\frac{2}{3}} \\ 0 & 1 & 2t_{n+q} & 3t_{n+q}^2 & 4t_{n+q}^3 & 5t_{n+q}^4 & 6t_{n+q}^4 & 7t_{n+q}^4 \\ 0 & 1 & 2t_{n+\frac{4}{3}} & 3t^2_{n+\frac{4}{3}} & 4t^3_{n+\frac{4}{3}} & 5t^4_{n+\frac{4}{3}} & 6t^4_{n+\frac{4}{3}} & 7t^4_{n+\frac{4}{3}} \\ 0 & 1 & 2t_{n+r} & 3t_{n+r}^2 & 4t_{n+r}^3 & 5t_{n+r}^4 & 6t_{n+r}^4 & 7t_{n+r}^4 \\ 0 & 1 & 2t_{n+2} & 3t_{n+2}^2 & 4t_{n+2}^3 & 5t_{n+2}^4 & 6t_{n+2}^4 & 7t_{n+2}^4 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{pmatrix} = \begin{pmatrix} x_n \\ f_n \\ f_{n+p} \\ f_{n+\frac{2}{3}} \\ f_{n+q} \\ f_{n+\frac{4}{3}} \\ f_{n+r} \\ f_{n+2} \end{pmatrix} \tag{4}$$

Solving the system in (4) by Gaussian Elimination method to obtain the coefficients b_j 's, $j = 0, 1, \ldots, 7$ and putting back into equation (2) to obtain the implicit scheme of the form

$$x(t) = \alpha_0(t) x_n + h(\beta_0(t) f_n + \beta_p\ (t) f_{n+p} + \beta_{\frac{2}{3}}\ (t) f_{n+\frac{2}{3}} + \beta_q\ (t) f_{n+q} + \beta_{\frac{4}{3}}\ (t) f_{n+\frac{4}{3}} + \beta_r\ (t) f_{n+r} + \beta_2\ (t) f_{n+2}. \tag{5}$$

where, $\alpha_0(t)$, and $\beta_j(t), j = 0, p, \frac{2}{3}, q, \frac{4}{3}, r, 2$ are continuous coefficients.

Evaluating equation (5) at the points $t = t_{n+p}, t_{n+\frac{2}{3}}, t_{n+q}, t_{n+\frac{4}{3}}, t_{n+r}, t_{n+2}$, yield the respective formulas for $x_{n+p}, x_{n+\frac{2}{3}}, x_{n+q}, x_{n+\frac{4}{3}}, x_{n+r}, x_{n+2}$. Expanding the main formula $x(t_{n+2})$ in the Taylor series around t_n yield after some simplification the following local truncation error.

$$\mathcal{L}(x(t_{n+2}); h) = \frac{1}{297675}\left(r\big((7p)q\big) - (7p)q - (7p)r + 12p - (7q)r + 12q + 12r - 22\right) h^8 x^{(8)}(t_n) + O(h^8). \tag{6}$$

Setting the principal term of the LTE in (6) to zero yields the following equation in three unknowns:

$$r\big((7p)q\big) - (7p)q - (7p)r + 12p - (7q)r + 12q + 12r - 22 = 0. \tag{7}$$

Solving the equation gives the optimized values of p, q, r as

$$p = \tfrac{1}{7}(7 - \sqrt{35}); \quad q = 1; \quad r = \tfrac{1}{7}(7 + \sqrt{35}) \tag{8}$$

Substituting the values of p, q, r into the local truncation error formulae (6) gives

$$\mathcal{L}(x(t_{n+2}); h) = \frac{1}{4167450} h^9 x^{(9)}(t_n) + O(h^{10}). \tag{9}$$

Lastly, putting the values of the parameters p, q, r into the equations for $x_{n+p}, x_{n+\frac{2}{3}}, x_{n+q}, x_{n+\frac{4}{3}}, x_{n+r}, x_{n+2}$, we get the following two-step optimal hybrid block method:

$$x_{n+p} = x_n + \frac{h}{10948560}\Big(817(539 + 50\sqrt{35}) f_n + (1882384 - 130928\sqrt{35}) f_{n+p} + (3441123 - 619650\sqrt{35}) f_{n+\frac{2}{3}} + 36480\sqrt{35} f_{n+q} + (3560193 - 619650\sqrt{35}) f_{n+\frac{4}{3}} + (1882384 - 130928\sqrt{35}) f_{n+r} + 19(-13573 + 2150\sqrt{35}) f_{n+2}\Big),$$

$$x_{n+\frac{2}{3}} = x_n + \frac{h}{92340}\Big(323 f_n + (16072 - 3136\sqrt{35}) f_{n+p} + 39393 f_{n+\frac{2}{3}} + 19456 f_{n+q} + 8073 f_{n+\frac{4}{3}} + (16072 - 3136\sqrt{35}) f_{n+r} + 1083 f_{n+2}\Big),$$

$$x_{n+q} = x_n + \frac{h}{4560}\Big(38 f_n + (784 + 147\sqrt{35}) f_{n+p} + 2673 f_{n+\frac{2}{3}} + 243 f_{n+\frac{4}{3}} + (784 - 147\sqrt{35}) f_{n+r} + 38 f_{n+2}\Big), x_{n+\frac{4}{3}} = x_n + \frac{h}{23085}\Big(114 f_n + (3920 - 784\sqrt{35}) f_{n+p} + 12744 f_{n+\frac{2}{3}} + 4864 f_{n+q} + 4914 f_{n+\frac{4}{3}} + (3920 - 784\sqrt{35}) f_{n+r} + 304 f_{n+2}\Big),$$

$$x_{n+r} = x_n + \frac{h}{10948560}\Big(-817(50\sqrt{35}-539)f_n + (1882384+130928\sqrt{35})f_{n+p} + (3441123+619650\sqrt{35})f_{n+\frac{2}{3}} - 36480\sqrt{35}f_{n+q} + (3560193+619650\sqrt{35})f_{n+\frac{4}{3}} + (1882384+130928\sqrt{35})f_{n+r} - 19(13573+2150\sqrt{35})f_{n+2}\Big),$$

$$x_{n+2} = x_n + \frac{h}{1140}\Big(19f_n + 392f_{n+p} + 729f_{n+\frac{2}{3}} + 729f_{n+\frac{4}{3}} + 392f_{n+r} + 19f_{n+2}\Big), \tag{10}$$

3. Analysis of the Basic Properties of the TSOHBM

Here, the basic properties of the TSOHBM (10) namely; accuracy, consistency, zero-stability, convergence, linear stability, and A-stability are investigated.

3.1. Order of Accuracy and Consistency

Rewriting the TSOHBM (10) in the matrix difference form yields

$$A_1X_n = A_0X_{n-1} + h(B_0F_{n-1} + B_1F_n), \tag{11}$$

Where $A_0, A_1, B_0,$ and B_1 are 6×6 matrices given by

$$A_0 = \begin{bmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1 \end{bmatrix};\ A_1 = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix};\ B_0 = \begin{bmatrix} 0&0&0&0&0&\frac{817(539+50\sqrt{35})}{10948560}\\ 0&0&0&0&0&\frac{323}{92340}\\ 0&0&0&0&0&\frac{38}{4560}\\ 0&0&0&0&0&\frac{114}{23085}\\ 0&0&0&0&0&\frac{-817(50\sqrt{35}-539)}{10948560}\\ 0&0&0&0&0&\frac{19}{1140} \end{bmatrix} \tag{12}$$

$$B_1 = \begin{bmatrix} \frac{1882384-130928\sqrt{35}}{10948560} & \frac{3441123-619650\sqrt{35}}{10948560} & \frac{36480\sqrt{35}}{10948560} & \frac{3560193-619650\sqrt{35}}{10948560} & \frac{1882384-130928\sqrt{35}}{10948560} & \frac{19(-13573+2150\sqrt{35})}{10948560}\\ \frac{16072-3136\sqrt{35}}{92340} & \frac{39393}{92340} & \frac{19456}{92340} & \frac{8073}{92340} & \frac{16072-3136\sqrt{35}}{92340} & \frac{1083}{92340}\\ \frac{784+147\sqrt{35}}{4560} & \frac{2673}{4560} & 0 & \frac{243}{4560} & \frac{784-147\sqrt{35}}{4560} & \frac{38}{4560}\\ \frac{3920-784\sqrt{35}}{23085} & \frac{12744}{23085} & \frac{4864}{23085} & \frac{4914}{23085} & \frac{3920-784\sqrt{35}}{23085} & \frac{304}{23085}\\ \frac{1882384+130928\sqrt{35}}{10948560} & \frac{3441123+619650\sqrt{35}}{10948560} & \frac{-36480\sqrt{35}}{10948560} & \frac{3560193+619650\sqrt{35}}{10948560} & \frac{1882384+130928\sqrt{35}}{10948560} & \frac{19(-13573-2150\sqrt{35})}{10948560}\\ \frac{392}{1140} & \frac{729}{1140} & 0 & \frac{729}{1140} & \frac{392}{1140} & \frac{19}{1140} \end{bmatrix} \tag{13}$$

$$X_n = (x_{n+p}, x_{n+\frac{2}{3}}, x_{n+q}, x_{n+\frac{4}{3}}, x_{n+r}, x_{n+1})^T, X_{n-1} = (x_{n-2+p}, x_{n-2+\frac{2}{3}}, x_{n-2+q}, x_{n-2+\frac{4}{3}}, x_{n-2+r}, x_n)^T, F_n = (f_{n+p}, f_{n+\frac{2}{3}}, f_{n+q}, f_{n+\frac{4}{3}}, f_{n+r}, f_{n+1})^T, F_{n-1} = (f_{n-2+p}, f_{n-2+\frac{2}{3}}, f_{n-2+q}, f_{n-2+\frac{4}{3}}, f_{n-2+r}, f_n)^T. \tag{14}$$

For a sufficiently differentiable test function $\varphi(t_n)$ in the interval $[0, T\ ]$, Let the difference operator $\bar{D}$ for the TSOHBM in (10) be given as

$$\bar{D}(\varphi(t_n); h) = \sum_{j\ =\omega}\left[\bar{\xi}_j(t_n + jh) - h\bar{\mu}_j\varphi'(t_n + jh)\right], \omega = 0, p, \tfrac{2}{3}\ \ q, \tfrac{4}{3}, r, 2. \tag{15}$$

Where, $\bar{\xi}_j$ and $\bar{\mu}_j$ are column vectors of the matrices A_0 and A_1, respectively. The Taylor series expansion about t_n for $x(t_n + jh)$ and $x'(t_n + jh)$ yield

$$\bar{\mathcal{L}}(\varphi(t_n); h) = c_0x(t_n) + c_1hx'^{(t_n)} + c_2h^2x^{(2)}(t_n) + \cdots + c_ph^px^{(p)}(t_n) + \cdots \tag{16}$$

where $c_i, i = 0,1,2, \ldots$ are vectors. From equation (16), the order of the TSOHBM is $p = (7,7,7,7,7,8)^T$ with the error constant

$$c_{p+1} = \frac{47}{163364040}, \frac{-16}{8680203}, \frac{-11}{7620480}, \frac{-16}{8680203}, \frac{47}{163364040}, \frac{1}{4167450}. \tag{17}$$

showing that the TSOHBM has at least seventh order accuracy.

Since $p \geq 1$, then the block method TSOHBM (10) is consistent (see [5]).

3.2. Zero-stability and Convergence

The zero-stability pertains to the stability of the difference system in (11) in the limit as $h \to 0$. As $h \to 0$, (11) becomes

$$A_1 X_n - A_0 X_{n-1} = 0 \tag{18}$$

The first characteristic polynomial $\rho(\sigma) = det\ (\sigma A_1 - A_0) = \sigma^5(\sigma - 1) = 0$. Thus, $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = 0, \sigma_6 = 1$. Hence the block method (10) is zero-stable.

Since the TSOHBM satisfy the properties of consistency and zero-stability, then the method is convergent according to [24].

3.3. Linear Stability

Consider the linearized test problem

$$x'(t) = \sigma x(t), Re(\sigma) < 0 \tag{19}$$

Applying the proposed block method to the trial problem (19), we obtain the recurrence relation

$$X_n = H(\hbar) X_{n-1}, \hbar = \sigma h \tag{20}$$

where the matrix $H(\hbar)$ is given by $(A_1 - rB_0)^{-1}(A_0 - rB_0)$. The stability property of this matrix's eigenvalues, which governs how the numerical solution behaves, is the spectral radius, $H(\hbar)$, which is used in the method to define the region of absolute stability S. The method is A-stable if

$$S = \{\hbar \in C: |\rho[H(\hbar)]| < 1\} \tag{21}$$

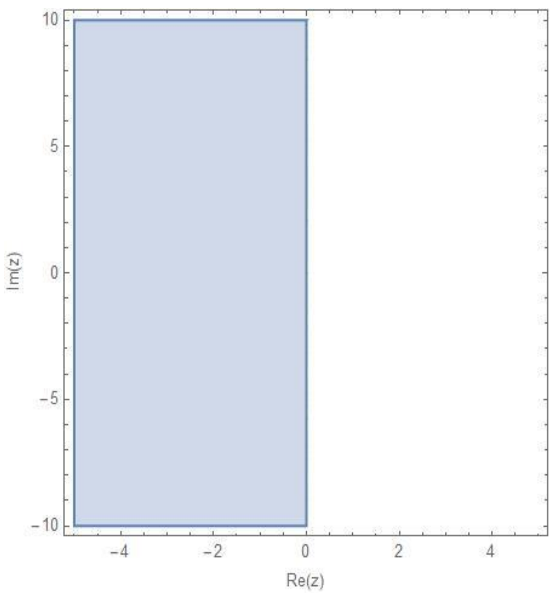

Figure 1. *Region of absolute stability of the TSOHBM.*

Upon performing various calculations, it becomes evident that the predominant eigenvalue can be expressed as a quotient function.

$$\rho[H(\hbar)] = \frac{2\hbar^6+43\hbar^5+417\hbar^4+2445\hbar^3+9060\hbar^2+19845\hbar+19845}{2\hbar^6-43\hbar^5+417\hbar^4-2445\hbar^3+9060\hbar^2-19845\hbar+19845} \tag{22}$$

which has a modulus of less than one in C− (see Figure 1). Hence, the TSOHBM (10) is A-stable.

4. Results

The accuracy of the TSOHBM is shown by applying the method to solve some popular applied problems of the form (1) in literature. The methods being compared are the TSOHBM (10), the TDHM) in [25], the TOBBDF in [26], and the RBHMO in [13].

Example 1

Consider the Prothero-Robinson problem which has appeared in [13]:

$$x'(t) = 10^{-6}(x - \sin x) + \cos x, x(0) = 0. \tag{23}$$

The exact solution is $x(t) = \sin x$. The problem is solved in the interval $[0,10]$ for number of steps $n = 50, 100, 200, ,400$.

Example 2

Given the highly stiff problem investigated by [26]:

$$x'(t) = -\lambda x, x(0) = 1, \lambda = 10. \tag{24}$$

with exact solution $x(t) = e^{\lambda t}$, The problem is solved in the interval $[0,0.1]$ for step size $h = 0.1$.

Example 3

Consider the first order stiff initial value problem has appeared in [26]:

$$x'(t) = t - x, x(0) = 0. \tag{25}$$

The exact solution is $x(t) = t + e^{-t} - 1$. The problem is solved in the interval $[0,1]$ for step size $h = 0.1$.

Example 4

We consider the stiff problem investigated by [25]:

$$x_1' = -x_1 + 95x_2, x_1(0) = 1, x_2' = -x_1 - 97x_2, x_2(0) = 1. \tag{26}$$

with exact solution $x_1(t) = \frac{1}{47}(95e^{-2t} - 48e^{-96t})$, , $x_2(t) = \frac{1}{47}(48e^{-96t} - e^{-2t})$, The problem is solved in the interval $[0,1]$ for step sizes h= $1/8, 1/16, 1/32, 1/64$.

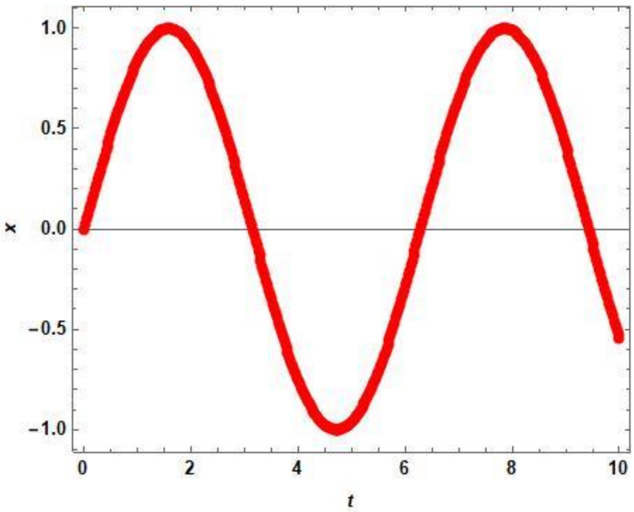

Figure 2. Solution plot for example 1.

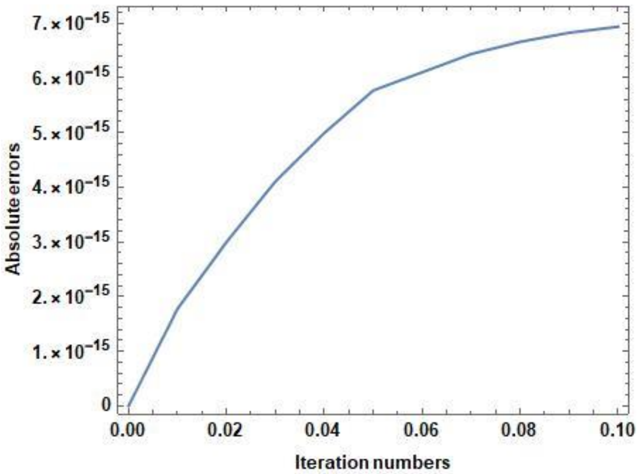

Figure 5. Error plot for example 2.

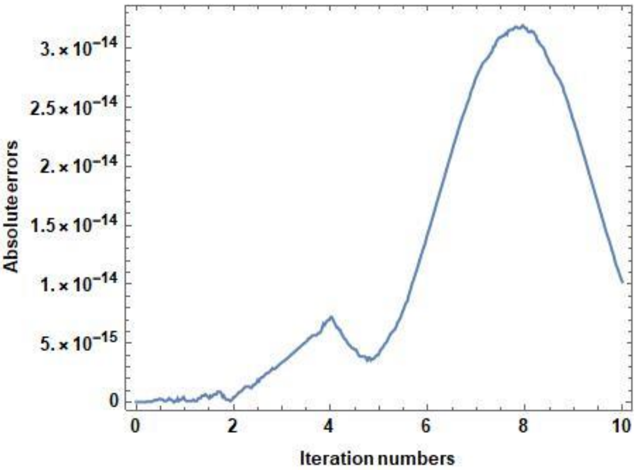

Figure 3. Error plot for example 1.

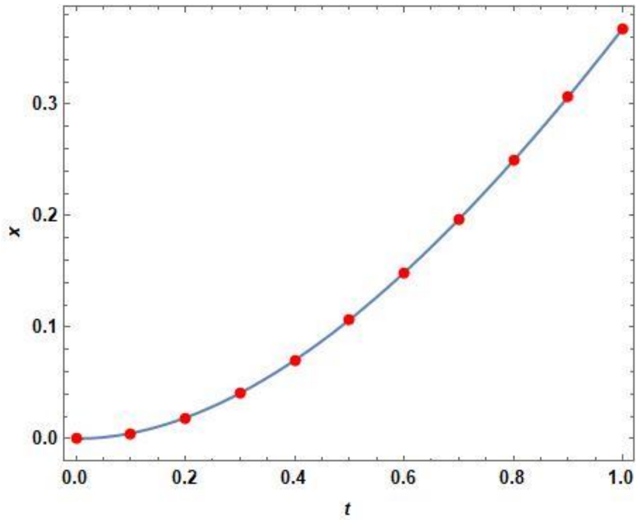

Figure 6. Solution plot for example 3.

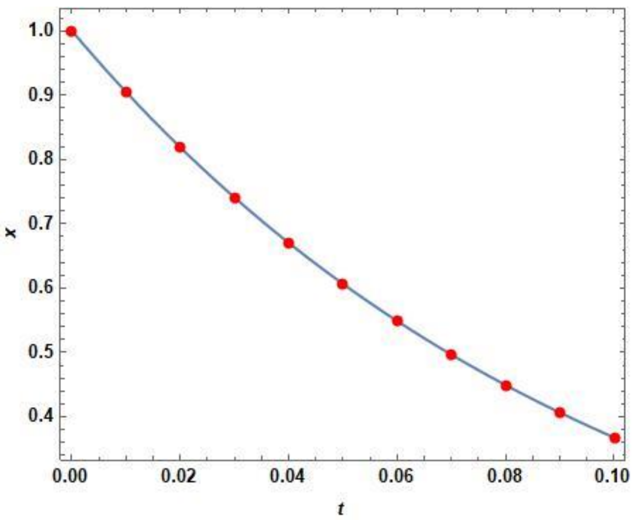

Figure 4. Solution plot for example 2.

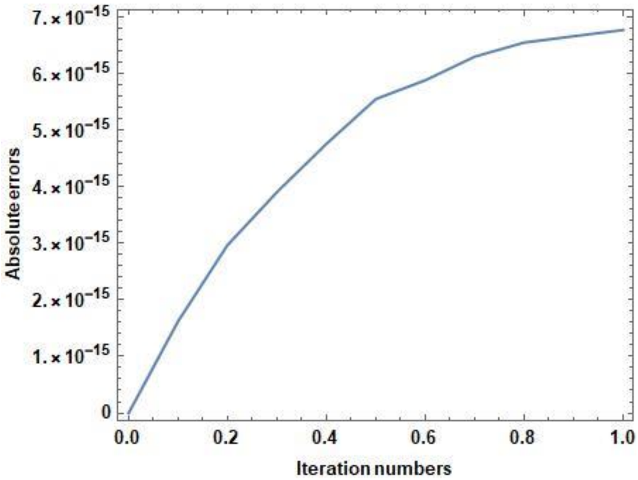

Figure 7. Error plot for example 3.

5. Discussion

Table 1 shows the comparison of maximum absolute errors in TSOHBM and RBHMO. The first, second, third, fourth and fifth columns indicate number of steps, exact solution, computed solution, error in TSOHBM, and error in RBHMO respectively. The solution from TSOHBM agrees with the exact solution up to at least 12 decimal places while that of RBHMO agrees up to 10 decimal places indicating that the current method is more accurate.

Figures 2 and 3 shows the solution plot and error plots respectively for example 1. Table 2 shows the comparison of maximum absolute errors in TSOHBM and TOBBDF. The first, second, third, fourth and fifth columns indicate step size, exact solution, computed solution, error in TSOHBM, and error in TOBBDF respectively. The solution from TSOHBM coincides with the exact solution up to at least 15 decimal places while that of TOBBDF coincides up to 10 decimal places indicating that the current method is more accurate.

Figures 4 and 5 shows the solution plot and error plots respectively for example 2. Table 3 shows the comparison of maximum absolute errors in TSOHBM and TOBBDF. The first, second, third, fourth and fifth columns indicate step size, exact solution, computed solution, error in TSOHBM, and error in TOBBDF respectively. The solution from TSOHBM coincides with the exact solution up to at least 15 decimal places while that of TOBBDF coincides up to 10 decimal places indicating that the current method is more accurate.

Figures 6 and 7 shows the solution plot and error plots respectively for example 2. Table 4 shows the comparison of maximum absolute errors in TSOHBM and TDHM. The first, second, third, fourth and fifth columns indicate step size, method, error in x_1, and error in x_2 respectively. The solution from TSOHBM coincides with the exact solution up to at least 15 decimal places while that of TOBBDF coincides up to 10 decimal places indicating that the current method is more accurate.

Table 1. *Comparative analysis of results of example 1 with the method in Ramos [13].*

n	Exact solution	Computed solution	Error in TSOHBM	Error in RBHMO
50	-0.544021	-0.544021	4.69580E-12	2.3770E-10
100	-0.544021	-0.544021	3.59712E-14	1.4550E-11
200	-0.544021	-0.544021	2.27596E-14	9.0252E-13
400	-0.544021	-0.544021	1.02141E-14	5.6413E-14

Table 2. *Comparative analysis of results of example 2 with the method in Ukpebor [26].*

h	Exact solution	Computed solution	Error in TSOHBM	Error in TOBBDF
0.01	0.904837	0.904837	1.77636E-15	1.000000E-11
0.02	0.818731	0.818731	2.9976E-15	0.000000E+00
0.03	0.740818	0.740818	4.10783E-15	0.000000E+00
0.08	0.449329	0.449329	6.66134E-15	2.000000E-10
0.09	0.40657	0.40657	6.82787E-15	2.000000E-10
0.10	0.367879	0.367879	6.93889E-15	3.000000E-10

Table 3. *Comparative analysis of results of example 3 with the method in Ukpebor [26].*

h	Exact solution	Computed solution	Error in TSOHBM	Error in TOBBDF
0.1	0.00483742	0.00483742	1.62283E-15	3.000000E-11
0.2	0.0187308	0.0187308	2.96985E-15	2.000000E-11
0.3	0.0408182	0.0408182	3.90660E-15	1.900000E-10
0.8	0.249329	0.249329	6.55032E-15	2.000000E-10
0.9	0.30657	0.30657	6.66134E-15	2.000000E-10
1.0	0.367879	0.367879	6.77236E-15	3.000000E-10

Table 4. *Comparative analysis of results of example 4 with the method in Adogbe [25].*

Step	Method	Error_ x_1	Error_ x_2
1/8	TSOHBM	4.08317E-11	4.11881E-13
	TDHM	1.61313E-10	1.6880E-9
1/16	TSOHBM	3.46612E-13	3.64856E-15
	TDHM	3.12445E-10	2.8709E-12
1/32	TSOHBM	2.9976E-15	3.20924E-17
	TDHM	2.28712E-10	1.90027E-12
1/64	TSOHBM	5.55112E-17	4.33681E-19
	TDHM	1.38512E-10	2.00324E-12

6. Conclusions

This work has presented an accurate two-step optimized hybrid block method for integrating stiff differential equations. The method incorporated five hybrid points with a three-parameter approximation. The technique was designed such that the interval of integration determines the best hybrid points through the optimization of the principal term of the LTE of the main method. Consequently, the accuracy of the resulting numerical scheme was greatly enhanced as demonstrated in the numerical results obtained when the method was implemented to solve some well-known stiff differential equations. It was also established through rigorous analysis that the method is consistent, convergent, zero stable and efficient for solving first-order ordinary differential equations. Hence, the new method is strongly suggested for general use.

Abbreviations

TSOHBM	Two Step Optimized Hybrid Block Method
TDHM	Third Derivative Hybrid Method
TOBBDF	Three-step Optimized Block Backward Differentiation Formula
RBHMO	Reformulated Block Hybrid Method

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Adelegan Lukuman Momoh: Formal Analysis, Writing - review & editing

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Data Availability Statement

The data supporting the research is included in the article.

Conflicts of Interest

The authors declare no conflicts of interest.

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Research Field

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