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Research Article

New Six Formulas of Radical Roots Developed by Using an Engineering Methodology to Solve Sixth Degree Polynomial Equation in General Forms by Calculating All Solutions Nearly in Parallel

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Abstract

This paper proposes new six formulas allowing to calculate all roots of sixth degree polynomial equation nearly in parallel while including the use of radical expressions, which is extending a new engineering methodology to solve polynomial equations of nth degree where the value of n can exceed five. This methodology is based on developing the roots of nth degree polynomial equation according to a distributed structure of radical terms, where each term is built by multiplying two radicals presenting the roots of polynomial equations with inferior degrees. This distributed structure of terms is allowing them to neutralize each other during multiplications, which forward calculations toward eliminating radicalities, suppressing complex terms and reducing degrees. As a result, this paper is proposing new two theorems solving sixth degree polynomial equation in complete forms while relying on two different approaches built on the same engineering methodology of roots architecting, which allow calculating solutions nearly in parallel. This engineering methodology is scalable to solve higher degrees of polynomial equations while extending the same distributed architecture of terms whereas re-engineering the expressions of included sub-terms in order to manifest the same outcomes of reciprocal neutralization, radicality suppression and degrees reduction during calculations. Therefore, this paper is also presenting the engineered requirements and techniques along with details in order to scale the used methodology by projecting it on nth degree polynomial equations where the possibility of calculating the values of all roots nearly in parallel whereas the polynomial degrees can exceed the quantic form. The new proposed engineering methodology in this paper is listing all necessary logic, techniques and formulas to solve nth degree polynomial equations in general forms stage-by-stage while relying on the use of radical expressions, which will scale the results of this paper toward solving highly complex equations.

Keywords

New Formulas, New Six Roots, New Engineered Theorems, Sixth Degree Polynomial Equation, Solving Sixth Degree Equation, Solving Nth Degree Polynomial Equations

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1. Introduction

Solvability of nth degree polynomial equations has been challenging to mathematicians over centuries, especially when looking for algebraic terms to express the roots. This challenge is due to the complexity of calculations that can exceed the previsions of human mind especially when degrees of polynomials are making them surpassing the quartic form.

The complexity of calculations during the attempts of solving polynomial equations of high degrees is mostly due to using radicalities while adopting specific approaches. In addition, reaching a point of having high complexity of outcomes where the form of equation is far from foreseeing a reduction; will lead to conduct a radical change on used approach or even replace it by adopting a different one.

Adopting a specific approach to solve a specific degree of polynomial equations lead to have limitation in resulted forms of equations where reductions are harder to be conducted by comparison to the starting point. In addition, conducted calculations may augment in high rate when searching for the solution by trying different approaches that may lead to restart calculations from scratch. Furthermore, the complexity of resulted forms of equations may lead toward adopting a limited solution to be used on a specific form of polynomial equations that have specific conditions stated by the values of included coefficients.

As a results, finding unified solution formulas for polynomial equations in general forms and in complete forms is more challenging when relying on a research methodology where calculations and logic may to be restarted from scratch when the approach is radically modified or even replaced.

Therefore, we rely on an engineering methodology where we build the appropriate approach step-by-step basing on patterns and characteristics that should be met. Then, we use this approach to architect the adequate roots and to structure their involved terms and sub-terms. Then, we forward logic and calculations toward engineering the formulas of all roots, in order to allow the calculation of all expected solutions nearly in parallel.

Relying on this engineering methodology to solve polynomial equations avoid us restarting the logic and calculations from scratch, and allow us to keep relying on the same approach and results of calculations while conducting only slight modifications, when necessary, toward reaching the final forms of unified formulas. In addition, this engineering methodology allows us to project the same approach and extend the same logic toward solving higher degrees of polynomial equations.

In this engineering methodology, the axe of focus is building and architecting according to a scalable logic starting from requirements engineering toward designing the starting point, the path, the destination and the structure of expected final results. As a result, conducted calculations and adopted reasoning follow a pr-designed path toward structuring the unified formulas of niched roots.

The advantage of this paper is presenting two new theorems solving sixth degree polynomial equations in general forms by using radical expressions where the possibility of calculating the values of all roots nearly in parallel. One among these two new theorems is solving sixth degree polynomial equations without the quantic terms, whereas the other theorem is solving sixth degree polynomial equations in complete forms that include the quantic terms.

This paper is also presenting the engineered requirements and techniques that lead to architect the results of proposed theorems. Furthermore, this paper is presenting another theorem solving quartic equations in general forms which is essentially used to structure all the six roots of sixth degree polynomial equations in complete forms.

Each proposed theorem in this paper is developed according to a scaled logic along with a detailed proof step-by-step, where we rely on the results of engineered requirements and techniques to forward the design and development of expressions, formulas and proofs.

This paper is a principal step in our work of solving nth degree polynomial equations basing on projecting the presented methods and results in this paper on other polynomial equations with degrees higher than five, which will be presented in other articles.

The mathematician Ferrari Lodovico is historically attributed the credit of discovering a principal solution for fourth degree polynomial equation by the year 1540. However, since his discovered root expression required having a solution for equations in cubic forms, which was not published yet by Cardano Gerolamo at that time, Ferrari Lodovico could not publish his discovery officially and immediately. Nevertheless, Ferrari's mentor, Gerolamo Cardano, did publish the discovered quartic solution by Ferrari along with the cubic solution in the historical book of mathematics; Ars Magna [1].

The discovered cubic root by the mathematician Cardano Gerolamo [2] for third degree equations under the polynomial form $x^3 + px + q = 0$ does help Ferrari's solution to solve quartic equations by reducing their expressions from the fourth degree to the second degree, but it does not directly help to properly define the four roots of any quartic equation. Therefore, when having a fourth-degree polynomial equation in general form, it is necessary to conduct further calculations to determine its four roots while relying on the solution of Ferrari Lodovico.

The proposed method by Cardano Gerolamo was also relied-on as a foundation base to solve particular forms of n^{th} degree equations; such as by using specific radical expressions under the form $\sqrt{na + \sqrt{b}} + \sqrt{na - \sqrt{b}}$ where the values of n can vary in the range $2,3,4,\ldots$, etc. [3].

Over the history of mathematics, there have been many elaborated methods and published solutions to solve polynomial forms of quartic equations such as Euler's solution [4], Galois's method [5], Descartes's method [6], Lagrange's method [7] and algebraic geometry [8].

The cubic solution was almost-always considered as an essential milestone for further research to solve polynomial equations of fourth degree and above [9, 10], such as the case of trying to elaborate a proof demonstrating that quantic equations do not accept quadratic expressions as solutions, which is further discussed in the published research results in [11-14].

A particular solution for third degree polynomial equation was first found in the year 1515 by Scipione del Ferro (1465–1526), which was presented as a solution for some specific cases determined by the values of involved coefficients in the considered polynomials. However, the official root-form considered as a practical solution for cubic equations; is the discovered root by the mathematician Cardano Gerolamo, which is generally recognized as the base of further historical research on solving quartic equations and other specific forms of polynomial equations.

There have been other recent publications presenting research results dedicated to solve quartic equations where the used methods are built on reducing polynomial expression [15, 17], whereas other published results are relying on the use of computational algorithms and numerical analysis in order to find the roots of polynomial equations with degrees exceeding the quartic form [17, 18].

In the published research article by Tschirnhaus [19], he proposed a particular innovative method to solve polynomial equation $P_n(x)$ of n^{th} degree by relying on its transformation into a reduced polynomial form $Q_n(y)$ with fewer terms by scaling the proposed idea of Descartes; in which a polynomial of n^{th} degree is practically reducible by removing its term in the degree $(n-1)$. The projection of this published method on quantic forms of polynomial equations is presented with more details in [20].

There are some other published articles treating resolvable forms of quantic polynomials by relying on the use of radical expressions [21, 22]; in which we can find the description of specific criteria predetermining whether a quantic polynomial form may accept plausible roots with radical expressions or not. In addition, there are other published papers treating the resolvability of some equations with polynomial degrees higher than five while using factorization or by conditioning the forms of coefficients to be depending on each other [23, 24], which do not solve neither quantic equations nor sixth degree equations in general forms.

Some polynomial equations of sixth degree in simple forms, such as $ax^6 + dx^3 + g = 0$, can be solved by factorization or by relying on variable change, but other sixth degree equations in complete forms could not be solved over history [25, 26], until nowadays by using the presented solutions in this paper, which make the content of this article valuable.

The proposed concept in this article about architecting solutions according to a distributed structure of terms can extend itself to different problems in geometry, numbers theory and algebra in general; because this concept is introducing an engineering methodology based on identifying patterns and characteristics that allow forwarding calculations and expressions toward specific converging points where the results are built step-by-step and not only searched.

The used engineering methodology in this paper can also be projected on prime numbers by architecting the expressions of odd numbers according to a distributed architecture of terms, which may reveal (or prove) further characteristics about prime numbers. Furthermore, this engineering methodology can be scaled on Collatz conjecture in order to architect odd numbers according to converged trees where odd numbers are presented in form of distributed structures of terms, which may provide insights about prime numbers and their distribution.

Because the contained results in this article are original, consisting of many new codependent formulas, mathematical expressions and new theorems interconnected in a scaling manner basing on extendable logic; every proposed formula will be proved mathematically and used to build the rest of presented content, and we will scale through them by relying on logical analysis and deduction which are following a structured development architecture guided by the proposed engineering methodology in this paper.

This engineering methodology was first used to develop the solution of quartic polynomial equations in general forms [27], by building the unified formulas of the four roots of any quartic equation, which allow calculating the four solutions nearly simultaneously. Then, the same engineering methodology was scaled to solve quantic equations [28] in complete forms by proposing the necessary unified formulas of roots to calculate the five solutions nearly in parallel. Therefore, this paper is allowing to scale this engineering methodology from quartic and quantic polynomials toward solving sixth degree polynomial equations in general forms whereas enabling values calculation of all roots nearly in parallel.

The contents of this paper are structured as follow: Section 2 presenting the used methodology and its results of engineered requirements and techniques to solve polynomial equations of nth degree. Section 3 presenting four formulary solutions for fourth degree polynomial equations in general forms, which enable calculating the roots of these equations nearly in parallel. Section 4, presenting new six formulary solutions using radical expressions to solve sixth degree polynomial equation in general forms where the coefficient of fifth degree part is different from zero. Section 5, presenting new six roots solving sixth degree polynomial equation in general forms where the coefficient of fifth degree part is equal zero. Finally, Section 6 for conclusion.

2. Engineered Methodology and Techniques to Solve Nth Degree Polynomial Equations

The used methodology in this paper to solve nth degree polynomial equations in general forms is based on architecting roots according to a distributed structure of terms while relying on radicality.

In addition, this used methodology is relying on developing specific patterns into the structure of roots in order to help converging calculations whereas eliminating degrees.

Furthermore, this methodology is built on an engineering logic where roots are predesigned before being expressed according to unified formulas, which support the expressions structuring for all roots of polynomial equation.

The used methodology in this paper lead to define a list of engineered requirements and techniques, which are helping to develop the necessary unified formulas to calculate the roots of nth degree polynomial equations in general forms whereas enabling to calculate the values of possible roots nearly in parallel.

The results of our engineered requirements and techniques according to the used methodology are described as follow:

1. Roots should be expressed according to a distributed structure of terms $\{\sum_{i=0}^{i=u} \mathrm{T_i}\}$, which will be multiplied by each other during calculations.
2. Each included term in the distributed structure of roots should be expressed according to the simplest possible radicality.
3. All included terms in the distributed structure of roots should either be constants or be radical expressions.
4. The included constant terms in the distributed structure of roots should allow eliminating specific parts with specific degrees from a polynomial equation.
5. We adapt a polynomial equation of nth degree $\left\{\left(\sum_{i=0}^{i=n} a_i X^i\right) = 0\right\}$ where $\{a_n \neq 0\}$ by presenting it as $\left\{\left(\sum_{i=0}^{i=n} \frac{a_i}{a_n} X^i\right) = 0\right\}$.
6. We use the expression $\left\{X = \frac{-a_{n-1}}{n a_n} + \frac{Y}{n}\right\}$ to eliminate the term of degree $(n-1)$ from a polynomial equation of nth degree $\left\{\left(\sum_{i=0}^{i=n} \frac{a_i}{a_n} X^i\right) = 0\right\}$ when $(n-1)$ is an odd value or when this elimination is simplifying calculations.
7. All included radical terms in each root should have the same radicality, in order to converge resulted expressions during calculations. Therefore, we choose them to have a radicality of square root.
8. Each included radical term in the distributed structure of a root $\{\sum_{i=0}^{i=u} \mathrm{T_i}\}$ should be expressed according to a sum of simple radical terms $\left\{\left(\sum_{i=0}^{i=u} T_i\right) = \left(\sum_{i=0}^{i=u} x_i\right) = \left(\sum_{i=0}^{i=u} \sqrt{y_i}\right)\right\}$ when the degree of polynomial equation is equal four.
9. Each included radical term in the distributed structure of a root $\{\sum_{i=0}^{i=u} T_i\}$ should be expressed according to a multiplication of at least two different sub-terms $\left\{\left(\sum_{i=0}^{i=u} T_i\right) = \left(\sum_{i\neq j} x_i \ x_j\right)\right\}$ when the degree of polynomial equation is surpassing four.
10. When the degree of polynomial equation is surpassing four, each included sub-term $\{x_l\}$ in the distributed structure of a root $\left\{\left(\sum_{i=0}^{i=u} T_i\right) = \left(\sum_{i\neq j} x_i \ x_j\right)\right\}$ should appear in multiple distributed terms in order to allow further factorizations.
11. When the degree of polynomial equation is surpassing four, each included sub-term $\{x_l\}$ in the distributed structure of terms $\left\{\left(\sum_{i=0}^{i=u} T_i\right) = \left(\sum_{i\neq j} x_i \ x_j\right)\right\}$ should be presented according to a radical expression of cubic root, quadratic root or a constant.
12. Combinations among included sub-terms in a root should allow expressing the values of involved coefficients in a polynomial equation.
13. The included sub-terms in the distributed structure of terms should allow neutralizing their contents when they are multiplied by each other in order to have simplified results.
14. The included sub-terms in the distributed structure of terms should allow eliminating radicality when they are raised to the power of higher polynomial degrees.
15. The included sub-terms in the distributed structure of terms should allow eliminating radicality when they are multiplied by each other.
16. The included sub-terms in the distributed structure of terms should allow forwarded calculations to supress terms that have odd values of polynomial degrees.
17. The included sub-terms in the distributed structure of terms should allow forwarded calculations to either supressing terms of the highest degrees or supressing terms of the lowest degrees.
18. The distributed structure of terms $\left\{\left(\sum_{i=0}^{i=u} T_i\right) = \left(\sum_{i\neq j} x_i \ x_j\right)\right\}$ should include a sub-term $\{x_1\}$ presented according to a

radical expression of cubic root where $\left(x_1 = \sqrt{\frac{-b}{3} + \frac{1}{3}\sqrt[3]{-\frac{D}{2} + \sqrt{\left(\frac{D}{2}\right)^2 + \left(\frac{C}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D}{2} - \sqrt{\left(\frac{D}{2}\right)^2 + \left(\frac{C}{3}\right)^3}}}\right)$.

19. The distributed structure of terms should include two sub-terms $\{x_2, x_3\}$ presented according to radical expressions of quadratic roots where $\left(x_2 = \sqrt{-\frac{\frac{P}{2}+x_1}{2} + \sqrt{\left(\frac{\frac{P}{2}+x_1}{2}\right)^2 - \frac{Q^2}{64x_1}}}\right)$ and $\left(x_3 = \sqrt{-\frac{\frac{P}{2}+x_1}{2} - \sqrt{\left(\frac{\frac{P}{2}+x_1}{2}\right)^2 - \frac{Q^2}{64x_1}}}\right)$.
20. In order to eliminate high degree expressions in a polynomial equation whereas allowing calculations to converge, we use a constant value $\{\alpha_1\}$ expressed by using included sub-terms in the distributed structure of root where $\{\alpha_1 = \sum x_i^2\}$.
21. In order to eliminate average degree expressions in a polynomial equation whereas allowing calculations to converge, we use a constant value $\{\alpha_2\}$ expressed by using included sub-terms in the distributed structure of root where $\{\alpha_2 = \sum_{i\neq\ j} x_i^2\ \ x_j^2\}$
22. In order to eliminate low degree expressions in a polynomial equation whereas allowing calculations to converge, we use a constant value $\{\alpha_3\}$ expressed by using included sub-terms in the distributed structure of root where $\{\alpha_3 = \sum_{i\neq\ j\neq k} x_i x_j x_k\}$
23. In order to eliminate the lowest degree expressions in a polynomial equation whereas allowing calculations to converge, we use a constant value $\{\alpha_4\}$ expressed by using included sub-terms in the distributed structure of root where $\{\alpha_4 = \sum_{i\neq\ j\neq k\neq l} x_i x_j x_k\ x_l\}$.
24. In order to eliminate odd degrees of expressions in a polynomial equation whereas allowing calculations to converge, we re-formulate the solution $\{X = (\sum_{i\neq\ j} x_i\ \ x_j)\}$ to be presented as $\{X = (\sum x_i)^2 - \sum x_i^2 = (\sum x_i)^2 - \alpha_1\}$.
25. In order to reduce degrees of expressions in a polynomial equation whereas allowing calculations to converge, we re-formulate the second-degree form $\left\{X^2 = \left(\sum_{i\neq\ j} x_i\ \ x_j\right)^2\right\}$ to be presented as $\{X^2 = \alpha_2 + 2\alpha_3(\sum x_i) + 6\alpha_4\}$.
26. In order to reduce degrees of complex expressions in a polynomial equation whereas allowing calculations to converge, we re-formulate the quartic form $\left\{X^4 = \left(\sum_{i\neq\ j} x_i\ \ x_j\right)^4\right\}$ to be presented as $\{X^4 = 4(\sum x_i)^2\ \ \alpha_3^2\ + 4\alpha_3(\sum x_i)[\alpha_2 + 6\alpha_4] +\ [\alpha_2 + 6\alpha_4]^2\}$.
27. We use the proposed constants $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and the expression $\{X = (\sum_{i\neq\ j} x_i\ \ x_j)\}$ in order re-express the polynomial equation $\left\{\left(\sum_{i=0}^{i=n} \frac{a_i}{a_n} X^i\right) = 0\right\}$ to be represented as $\left\{\left(\sum_{i=0}^{i=n} \gamma_i Z^i\right) = 0\right\}$ where $\{Z = (\sum x_i)\}$.
28. When the solution of nth degree polynomial equation is expressed as $\{X = (\sum_{i\neq\ j} x_i\ \ x_j)\}$, we adopt a constant value $\left\{\frac{\Gamma}{\alpha_3} = \frac{\Gamma}{\sum_{i\neq\ j\neq k} x_i x_j x_k} = V\right\}$ where Γ is expressed in function of $\{(\sum x_k)\}$; in order to converge calculations during the process of equations solving.
29. The resulted polynomial expression at the final stages of forwarded calculations should have only one unknown variable expressed by including the use of one sub-term incorporated in the distributed structure of roots which to be considered as an unknown variable.
30. The resulted polynomial form at the final stages of forwarded calculations should have less degree than the staring point, or should not have the term of constant value (the term with zero degree). Otherwise, this resulted polynomial form should not have any terms with odd degrees.
31. The included sub-terms $\{x_i\}$ in the distributed structure of a root $\{(\sum \mathrm{T_i} = \sum x_i)\ or\ (\sum \mathrm{T_i} = \sum_{i\neq\ j} x_i\ \ x_j\ )\}$ should also be used in the calculation of all other roots by changing signs of these sub-terms whereas exploiting the involved coefficients in the polynomial equation.
32. Reusing the included sub-terms in the structure of a root while only changing their signs should allow calculating the values of different roots $\{Solution_k = \sum \pm \mathrm{T_i}\}$ nearly in parallel.

3. Solutions and Theorems for Fourth Degree Polynomial Equation in General Form

This section presents new unified formulary solutions for fourth degree polynomial equation in general form.

3.1. First Proposed Theorem

In this subsection, we propose a new theorem to solve fourth degree polynomial equations that may be presented as shown in (eq.1). The expressions of proposed solutions are dependent on the value of the expression $\left(8\left(\frac{b}{a}\right)^3-\frac{32cb}{a^2}+\frac{64d}{a}\right)$. We are proposing four solutions for $\left(8\left(\frac{b}{a}\right)^3-\frac{32cb}{a^2}+\frac{64d}{a}\right)<0$, four solutions for $\left(8\left(\frac{b}{a}\right)^3-\frac{32cb}{a^2}+\frac{64d}{a}\right)>0$ and four solutions for $\left(8\left(\frac{b}{a}\right)^3-\frac{32cb}{a^2}+\frac{64d}{a}\right)=0$.

The proposed solutions are expressed by using $y_{0,1}$, which is presented in (eq.17), and by using P, Q and R shown in (eq.6).

The proof of this theorem is presented in an independent subsection, because it is long and it contains the full expressions of proposed solutions along with details, in order to highlight the used logic to develop those solutions and to build the ground that we use to prove other proposed theorems in this paper.

Theorem 1

A fourth-degree polynomial equation under the expression (eq.1), where coefficients belong to the group of numbers $\mathbb{R}$, has four solutions:

$$ax^4+bx^3+cx^2+dx+e=0 \;\; where \;\; a\neq 0 \tag{1}$$

If $\left(8\left(\frac{b}{a}\right)^3-\frac{32cb}{a^2}+\frac{64d}{a}\right)<0$, and by using the expressions of $y_{0,1}$ in (eq.17), P in (Eq.6) and Q in (Eq.6):

a) Solution 1: $S_{1,1}$ presented in the expression (eq.35);
b) Solution 2: $S_{1,2}$ presented in the expression (eq.36);
c) Solution 3: $S_{1,3}$ presented in the expression (eq.37);
d) Solution 4: $S_{1,4}$ presented in the expression (eq.38).

If $\left(8\left(\frac{b}{a}\right)^3-\frac{32cb}{a^2}+\frac{64d}{a}\right)>0$, and by using the expressions of $y_{0,1}$ in (eq.17), P in (Eq.6) and Q in (Eq.6):

a) Solution 1: $S_{2,1}$ presented in the expression (eq.39);
b) Solution 2: $S_{2,2}$ presented in the expression (eq.40);
c) Solution 3: $S_{2,3}$ presented in the expression (eq.41);
d) Solution 4: $S_{2,4}$ presented in the expression (eq.42).

If $\left(8\left(\frac{b}{a}\right)^3-\frac{32cb}{a^2}+\frac{64d}{a}\right)=0$, and by using the expressions of $y_{0,1}$ in (eq.28), P in (Eq.6) and Q in (Eq.6):

a) Solution 1: $S_{3,1}$ presented in the expression (eq.43);
b) Solution 2: $S_{3,2}$ presented in the expression (eq.44);
c) Solution 3: $S_{3,3}$ presented in the expression (eq.45);
d) Solution 4: $S_{3,4}$ presented in the expression (eq.46).

3.2. Proof of Theorem 1

By dividing the polynomial (eq.1) on the coefficient a, we have the next form:

$$x^4+\frac{b}{a}x^3+\frac{c}{a}x^2+\frac{d}{a}x+\frac{e}{a}=0 \;\; with \;\; a\neq 0 \tag{2}$$

We suppose that x is expressed as shown in (eq.3):

$$x=\frac{\left(\frac{-b}{a}+y\right)}{4} \tag{3}$$

We replace x with supposed expression in (eq.3) to reduce the form of presented polynomial in (eq.2). Thereby, we have the presented expression in (eq.4).

$$y^4+y^2\left[-6\left(\frac{b}{a}\right)^2+\frac{16c}{a}\right]+y\left[8\left(\frac{b}{a}\right)^3-\frac{32cb}{a^2}+\frac{64d}{a}\right]-3\left(\frac{b}{a}\right)^4+\frac{16cb^2}{a^3}-\frac{64db}{a^2}+\frac{256e}{a}=0 \tag{4}$$

To simplify the expression of shown polynomial equation in (eq.4), we replace the expressions of used coefficients as shown in (eq.5) where the values of those coefficients are defined in (eq.6).

$$y^4+Py^2+Qy+R=0 \tag{5}$$

$$P=-6\left(\frac{b}{a}\right)^2+\frac{16c}{a};Q=8\left(\frac{b}{a}\right)^3-\frac{32cb}{a^2}+\frac{64d}{a};R=-3\left(\frac{b}{a}\right)^4+\frac{16cb^2}{a^3}-\frac{64db}{a^2}+\frac{256e}{a} \tag{6}$$

To solve the shown polynomial equation in (eq.5), we propose new expressions for the variable y; expressions (eq.7) and (eq.8):

$$\text{For } Q\leq 0:\; y=\sqrt{y_0}+\sqrt{y_1}+\sqrt{y_2} \tag{7}$$

$$\text{For } Q\geq 0:\; y=-\sqrt{y_0}-\sqrt{y_1}-\sqrt{y_2} \tag{8}$$

We propose the expressions (eq.9) and (eq.10) for y_1 and y_2 successively, in condition of $y_0\neq 0$. Those expressions of y_1 and y_2 are based on quadratic solutions.

$$y_1=-\frac{\frac{P}{2}+y_0}{2}+\sqrt{\left(\frac{\frac{P}{2}+y_0}{2}\right)^2-\frac{Q^2}{64y_0}} \tag{9}$$

$$y_2=-\frac{\frac{P}{2}+y_0}{2}-\sqrt{\left(\frac{\frac{P}{2}+y_0}{2}\right)^2-\frac{Q^2}{64y_0}} \tag{10}$$

To reduce the expression of equation (eq.5) and find a way to solve it, we propose the following expressions for the coefficients P and Q:

$$-2\left[{y_0}^2+{y_1}^2+{y_2}^2\right]=P \tag{11}$$

For $Q \le 0$: $-8\sqrt{y_0}\sqrt{y_1}\sqrt{y_2} = Q$ (12)

For $Q \ge 0$: $8\sqrt{y_0}\sqrt{y_1}\sqrt{y_2} = Q$ (13)

In the following calculation, we replace the variable y with the expression (eq.7) where we suppose $Q < 0$, and we replace P and Q with their shown expressions in (eq.11) and (eq.12):

$$y^4 + Py^2 + Qy + R = -\left[\sqrt{y_0}^4 + \sqrt{y_1}^4 + \sqrt{y_2}^4\right] + 2\left[\sqrt{y_0}^2\sqrt{y_1}^2 + \sqrt{y_0}^2\sqrt{y_2}^2 + \sqrt{y_1}^2\sqrt{y_2}^2\right] + R$$

$$= -\left[\sqrt{y_0}^2 + \sqrt{y_1}^2 + \sqrt{y_2}^2\right]^2 + 4\left[\sqrt{y_0}^2\sqrt{y_1}^2 + \sqrt{y_0}^2\sqrt{y_2}^2 + \sqrt{y_1}^2\sqrt{y_2}^2\right] + R$$

$$= 4\left[- y_0\left(\frac{P}{2} + y_0\right) + \frac{Q^2}{64y_0}\right] + R - \frac{P^2}{4} = 0$$

$$-\left[\sqrt{y_0}^2 + \sqrt{y_1}^2 + \sqrt{y_2}^2\right]^2 + 4\left[\sqrt{y_0}^2\sqrt{y_1}^2 + \sqrt{y_0}^2\sqrt{y_2}^2 + \sqrt{y_1}^2\sqrt{y_2}^2\right] + R = 0 \Rightarrow y_0^3 + \frac{P}{2}y_0^2 + \frac{P^2-4R}{16}y_0 - \frac{Q^2}{64} = 0 \quad (14)$$

To solve the resulted expression in (eq.14), we use Cardano's solution for third degree polynomial equations.
For $w^3 + cw + d = 0$, Cardano's solution is as follow:

$$w = \sqrt[3]{\frac{-d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3}} + \sqrt[3]{\frac{-d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3}} \quad (15)$$

For $y^3 + by^2 + cy + d = 0$, we use the form $y = \frac{-b+w}{3}$, and we suppose $\mathrm{D} = 27d + 2b^3 - 9cb$ and $\mathrm{C} = 9c - 3b^2$ to express the cubic solution as follow:

$$y = \frac{-b}{3} + \frac{1}{3}\sqrt[3]{-\frac{\mathrm{D}}{2} + \sqrt{\left(\frac{\mathrm{D}}{2}\right)^2 + \left(\frac{\mathrm{C}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{\mathrm{D}}{2} - \sqrt{\left(\frac{\mathrm{D}}{2}\right)^2 + \left(\frac{\mathrm{C}}{3}\right)^3}} \quad (16)$$

By using the expression (eq.16), the solutions of third degree polynomial equation shown in (eq.14) are $y_{0,1}$ in (eq.17), $y_{0,2}$ in (eq.18) and $y_{0,3}$ in (eq.19), where $P^{;} = \frac{P}{2}$, $R^{;} = \frac{-27Q^2-2P^3+72PR}{64}$ and $Q^{;} = -\frac{3P^2+36R}{16}$

$$y_{0,1} = -\frac{P^{;}}{3} + \frac{1}{3}\sqrt[3]{-\frac{R^{;}}{2} + \sqrt{\left(\frac{R^{;}}{2}\right)^2 + \left(\frac{Q^{;}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{R^{;}}{2} - \sqrt{\left(\frac{R^{;}}{2}\right)^2 + \left(\frac{Q^{;}}{3}\right)^3}} \quad (17)$$

In condition of $y_{0,1} \neq 0$, $y_{0,2}$ and $y_{0,3}$ are as follow:

$$y_{0,2} = -\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} \quad (18)$$

$$y_{0,3} = -\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} \quad (19)$$

We deduce that when y_0 takes the value $y_{0,1}$, the value of $y_{0,2}$ is equal to the shown value of y_1 in (eq.9), and the value of $y_{0,3}$ is equal to the shown value of y_2 in (eq.10).

There are three other possible expressions for y which respect the proposition $-8\sqrt{y_0}\sqrt{y_1}\sqrt{y_2} = Q$ when $Q \le 0$, and they give the same results of calculations toward having the shown third degree polynomial in (eq.14). Thereby, they give the same values for roots $y_{0,1}, y_{0,2}$ and $y_{0,3}$. These three expressions are $y = -\sqrt{y_0} - \sqrt{y_1} + \sqrt{y_2}$, $y = -\sqrt{y_0} + \sqrt{y_1} - \sqrt{y_2}$ and $y = \sqrt{y_0} - \sqrt{y_1} - \sqrt{y_2}$.

By using the expressions in (eq.6) and (eq.17), the solutions of presented fourth degree polynomial equation in (eq.5) when

$\left(8\left(\frac{b}{a}\right)^3-\frac{32cb}{a^2}+\frac{64d}{a}\right)<0$ are as shown in (eq.20), (eq.21), (eq.22) and (eq.23).

$$\text{Solution 1: } s_{1,1}=\sqrt{y_{0,1}}+\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}+\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}}+\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}-\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}} \tag{20}$$

$$\text{Solution 2: } s_{1,2}=-\sqrt{y_{0,1}}-\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}+\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}}+\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}-\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}} \tag{21}$$

$$\text{Solution 3: } s_{1,3}=-\sqrt{y_{0,1}}+\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}+\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}}-\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}-\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}} \tag{22}$$

$$\text{Solution 4: } s_{1,4}=\sqrt{y_{0,1}}-\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}+\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}}-\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}-\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}} \tag{23}$$

There are three other possible expressions for y which respect the proposition $8\sqrt{y_0}\sqrt{y_1}\sqrt{y_2}=Q$ when $Q\geq 0$, and they give the same third degree polynomial shown in (eq.14) after calculations. These three expressions are $y=-\sqrt{y_0}+\sqrt{y_1}+\sqrt{y_2}$, $y=\sqrt{y_0}-\sqrt{y_1}+\sqrt{y_2}$ and $y=\sqrt{y_0}+\sqrt{y_1}-\sqrt{y_2}$.

By using the shown expressions in (eq.6) and (eq.17), the solutions of presented fourth degree polynomial equation in (eq.5) when $\left(8\left(\frac{b}{a}\right)^3-\frac{32cb}{a^2}+\frac{64d}{a}\right)>0$ are as shown in (eq.24), (eq.25), (eq.26) and (eq.27).

$$\text{Solution 1: } s_{2,1}=-\sqrt{y_{0,1}}-\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}+\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}}-\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}-\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}} \tag{24}$$

$$\text{Solution 2: } s_{2,2}=-\sqrt{y_{0,1}}+\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}+\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}}+\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}-\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}} \tag{25}$$

$$\text{Solution 3: } s_{2,3}=\sqrt{y_{0,1}}-\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}+\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}}+\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}-\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}} \tag{26}$$

$$\text{Solution 4: } s_{2,4}=\sqrt{y_{0,1}}+\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}+\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}}-\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}-\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}} \tag{27}$$

Concerning $\left(8\left(\frac{b}{a}\right)^3-\frac{32cb}{a^2}+\frac{64d}{a}\right)=0$:

The expression of $y_{0,1}$ is as shown in (eq.28) where $P'=\frac{P}{2}, R'=\frac{-2P^3+72PR}{64}$ and $Q'=-\frac{3P^2+36R}{16}$, whereas $y_{0,2}$ and $y_{0,3}$ are as shown in (eq.29) and (eq.30).

$$y_{0,1} = -\frac{P^{;}}{3} + \frac{1}{3}\sqrt[3]{-\frac{R^{;}}{2} + \sqrt{\left(\frac{R^{;}}{2}\right)^2 + \left(\frac{Q^{;}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{R^{;}}{2} - \sqrt{\left(\frac{R^{;}}{2}\right)^2 + \left(\frac{Q^{;}}{3}\right)^3}} \tag{28}$$

$$y_{0,2} = -\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} = 0 \;\; or \;\; y_{0,2} = -\left(\frac{P}{2} + y_{0,2}\right) \tag{29}$$

$$y_{0,3} = -\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} = -\left(\frac{P}{2} + y_{0,2}\right) \;\; or \;\; y_{0,3} = 0 \tag{30}$$

Because of having the expressions $y_{0,2} = 0$ or $y_{0,3} = 0$, and having the intersection between the forms (eq.7) and (eq.8) for Q=0 $\left(y = \sqrt{y_0} + \sqrt{y_1} + \sqrt{y_2} \;\; for \;\; Q \le 0 \;\; and \;\; y = -\sqrt{y_0} - \sqrt{y_1} - \sqrt{y_2} \;\; for \;\; Q \ge 0\right)$, there are four solutions for the polynomial equation shown in (eq.5) when Q=0 and they are as shown in (eq.31), (eq.32), (eq.33) and (eq.34).

$$\text{Solution 1: } s_{3,1} = \sqrt{y_{0,1}} + \sqrt{-\left(\frac{P}{2} + y_{0,1}\right)} \tag{31}$$

$$\text{Solution 2: } s_{3,2} = -\sqrt{y_{0,1}} - \sqrt{-\left(\frac{P}{2} + y_{0,1}\right)} \tag{32}$$

$$\text{Solution 3: } s_{3,3} = -\sqrt{y_{0,1}} + \sqrt{-\left(\frac{P}{2} + y_{0,1}\right)} \tag{33}$$

$$\text{Solution 4: } s_{3,4} = \sqrt{y_{0,1}} - \sqrt{-\left(\frac{P}{2} + y_{0,1}\right)} \tag{34}$$

When we give the value $y_{0,1}$ in (eq.17) to y_0, the values of $y_{0,2}$ in (eq.18) and $y_{0,3}$ in (eq.19) are equal to the shown values of y_1 in (eq.9) and (y_3) in (eq.10) respectively. Thereby, even when we replace the value of $y_{0,1}$ in the expressions of proposed solutions by the values of $y_{0,2}$ or $y_{0,3}$, the results are only redundancies of proposed solutions, because the value of P in the precedent expressions and in the proposed solutions is as follow:

$$\frac{P}{2} = -\left(\sqrt{y_0}^2 + \sqrt{y_1}^2 + \sqrt{y_2}^2\right) = -\left(\sqrt{y_{0,1}}^2 + \sqrt{y_{0,2}}^2 + \sqrt{y_{0,3}}^2\right)$$

In order to solve the polynomial equation shown in (eq.2), we use the expression $x = \frac{-\frac{b}{a}+y}{4}$ where y is the unknown variable in polynomial equation (eq.5). By using expressions (eq.6) and (eq.17) for $\left(8\left(\frac{b}{a}\right)^3 - \frac{32cb}{a^2} + \frac{64d}{a}\right) < 0$, the solutions for equation (eq.1) are as shown in (eq.35), (eq.36), (eq.37) and (eq.38).

$$\text{Solution 1: } S_{1,1} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \tag{35}$$

$$\text{Solution 2: } S_{1,2} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \tag{36}$$

$$\text{Solution 3: } S_{1,3} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \tag{37}$$

$$\text{Solution 4: } S_{1,4}=-\frac{b}{4a}+\frac{1}{4}\sqrt{y_{0,1}}-\frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}+\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}}-\frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}-\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}} \quad (38)$$

By using the expression $x=\frac{-\frac{b}{a}+y}{4}$ while relying on expressions (eq.6) and (eq.17) for $\left(8\left(\frac{b}{a}\right)^3-\frac{32cb}{a^2}+\frac{64d}{a}\right)>0$, the proposed solutions for equation (eq.1) are as shown in (eq.39), (eq.40), (eq.41) and (eq.42).

$$\text{Solution 1: } S_{2,1}=-\frac{b}{4a}-\frac{1}{4}\sqrt{y_{0,1}}-\frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}+\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}}-\frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}-\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}} \quad (39)$$

$$\text{Solution 2: } S_{2,2}=-\frac{b}{4a}-\frac{1}{4}\sqrt{y_{0,1}}+\frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}+\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}}+\frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}-\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}} \quad (40)$$

$$\text{Solution 3: } S_{2,3}=-\frac{b}{4a}+\frac{1}{4}\sqrt{y_{0,1}}-\frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}+\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}}+\frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}-\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}} \quad (41)$$

$$\text{Solution 4: } S_{2,4}=-\frac{b}{4a}+\frac{1}{4}\sqrt{y_{0,1}}+\frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}+\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}}-\frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2}-\sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2-\frac{Q^2}{64y_{0,1}}}} \quad (42)$$

By using the expression $x=\frac{-\frac{b}{a}+y}{4}$ while relying on expressions (eq.6) and (eq.28) for $\left(8\left(\frac{b}{a}\right)^3-\frac{32cb}{a^2}+\frac{64d}{a}\right)=0$, the proposed solutions for equation (eq.1) are as shown in (eq.43), (eq.44), (eq.45) and (eq.46).

$$\text{Solution 1: } S_{3,1}=-\frac{b}{4a}+\frac{1}{4}\sqrt{y_{0,1}}+\frac{1}{4}\sqrt{-\left(\frac{P}{2}+y_{0,1}\right)} \quad (43)$$

$$\text{Solution 2: } S_{3,2}=-\frac{b}{4a}-\frac{1}{4}\sqrt{y_{0,1}}-\frac{1}{4}\sqrt{-\left(\frac{P}{2}+y_{0,1}\right)} \quad (44)$$

$$\text{Solution 3: } S_{3,3}=-\frac{b}{4a}-\frac{1}{4}\sqrt{y_{0,1}}+\frac{1}{4}\sqrt{-\left(\frac{P}{2}+y_{0,1}\right)} \quad (45)$$

$$\text{Solution 4: } S_{3,4}=-\frac{b}{4a}+\frac{1}{4}\sqrt{y_{0,1}}-\frac{1}{4}\sqrt{-\left(\frac{P}{2}+y_{0,1}\right)} \quad (46)$$

4. New Six Solutions for Sixth Degree Polynomial Equation in General Form

In this section, we propose six new solutions for sixth degree polynomial equation in general form shown in (eq.47), where we rely on our proposed solutions for quartic polynomial equations in previous section to structure the expressions of proposed solutions for sixth degree polynomial equations. We extend the used logic in precedent theorem (Theorem 1) by projection on sixth degree equations to prove the expressions of developed roots.

4.1. Second Proposed Theorem

In this subsection, we present our second proposed theorem to introduce new six formulary solutions for sixth degree polynomial equation in general form shown in (eq.47), where coefficients belong to the group ℝ whereas the coefficient of fifth degree part is different from zero. First, we divide the polynomial (eq.47) on coefficient A to reduce its expression to the simplified form shown in (eq.48) where coefficients are expressed as shown in (eq.49).

$$\mathrm{A}x^6 + \mathrm{B}x^5 + \mathrm{C}x^4 + \mathrm{D}x^3 + \mathrm{E}x^2 + \mathrm{F}x + \mathrm{G} = 0 \;\; with \;\; \mathrm{A} \neq 0 \;\; and \;\; \mathrm{B} \neq 0 \tag{47}$$

$$x^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g = 0 \;\; with \;\; b \neq 0 \tag{48}$$

$$b = \frac{\mathrm{B}}{\mathrm{A}};\;\; c = \frac{\mathrm{C}}{\mathrm{A}};\;\; D = \frac{\mathrm{D}}{\mathrm{A}};\, e = \frac{\mathrm{E}}{\mathrm{A}};\;\; f = \frac{\mathrm{F}}{\mathrm{A}};\;\; g = \frac{\mathrm{G}}{\mathrm{A}}; \tag{49}$$

$$z^4 + \Gamma_3 z^3 + \Gamma_2 z^2 + \Gamma_1 z + \Gamma_0 = 0 \tag{50}$$

Theorem 2

After reducing the form of sixth degree polynomial shown in (eq.47) to the presented form in (eq.48) where coefficients are as expressed in (eq.49); the sixth-degree polynomial equation shown in (eq.48), where coefficients belong to the group of numbers ℝ, can be reduced to a fourth-degree polynomial equation, which may be expressed as shown in (eq.50). The reduction from sixth degree polynomial to quartic polynomial is conducted by supposing $x = x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3$, whereas supposing $z = (x_0 + x_1 + x_2 + x_3)$ is the solution for fourth degree polynomial equation in (eq.50) by using Theorem 1 and relying on the expression $x_3 = -\frac{\Gamma_3}{4}$. The variable Γ_3 is defined as shown in (eq.51) where α_3 is presented in (eq.52) and Γ_4 is the solution for the polynomial equation (eq.53), which relies on the coefficients (eq.54), (eq.55), (eq.56) and (eq.57). The shown coefficients in (eq.54), (eq.55), (eq.56) and (eq.57) are expressed by using the constant V which is presented in (eq.58). The coefficients Γ_3, Γ_2, Γ_1 and Γ_0 of quartic equation (eq.50), which is used to calculate z, are determined by using the shown expressions in (51), (eq.59), (eq.60) and (eq.61) while using calculated values of Γ_4 and V.

As a result, we have twelve calculated values as potential solutions for sixth degree polynomial equation shown in (eq.48), where many of them are only redundancies of others, because there are only six official solutions to determine.

The twelve solutions to calculate for sixth degree equation (eq.48) are as shown in the groups (eq.99), (eq.100) and (eq.101). The proposed six values as official solutions for sixth degree polynomial equation shown in (eq.48) are as presented in (eq.102), (eq.103), (eq.104), (eq.105), (eq.106) and (eq.107).

$$\Gamma_3 = \frac{4\alpha_3}{b} + \Gamma_4 \tag{51}$$

$$\alpha_3 = -\frac{\frac{4\Gamma_4\left(f - \frac{d^2}{4b}\right)}{b}}{\frac{32f}{b^2} + \frac{40d^2}{b^3} - \frac{64cd}{b^2} + \frac{64e}{b}} \tag{52}$$

$$\lambda_3\Gamma_4^6 + \lambda_2\Gamma_4^4 + \lambda_1\Gamma_4^2 + \lambda_0 = 0 \tag{53}$$

$$\lambda_3 = -\frac{40960}{V^4b^4} + \frac{16384}{V^3b^3} - \frac{1536}{V^2b^2} \tag{54}$$

$$\lambda_2 = -\frac{24576d}{V^2b^4} + \frac{16384c}{V^2b^3} + \frac{3072d}{Vb^3} - \frac{2048c}{Vb^2} + \frac{1024}{V} \tag{55}$$

$$\lambda_1 = -\frac{512d}{b} + \frac{1536f}{b^3} + \frac{28V^2f}{b} - \frac{7V^2d^2}{b^2} + \frac{96Vf}{b^2} - \frac{168d^2V}{b^3} + \frac{192cdV}{b^2} - \frac{192Ve}{b} - \frac{3456d^2}{b^4} + \frac{4096cd}{b^3} - \frac{1024e}{b^2} - \frac{1024c^2}{b^2} \tag{56}$$

$$\lambda_0 = -\frac{64V^2d^3}{b^4} + \frac{64cd^2V^2}{b^3} - \frac{64eV^2d}{b^2} + \frac{128V^2g}{b} + \frac{192V^2df}{b^3} - \frac{128V^2cf}{b^2} \tag{57}$$

$$V = -\frac{\frac{32f}{b^2} + \frac{40d^2}{b^3} - \frac{64cd}{b^2} + \frac{64e}{b}}{4\frac{\left(f - \frac{d^2}{4b}\right)}{b}} \tag{58}$$

$$\Gamma_2 = \frac{8\Gamma_4^2}{Vb} - \frac{6d}{b^2} + \frac{4c}{b} + \frac{\left(f - \frac{d^2}{4b}\right)V^2}{2b\Gamma_4^2} - \frac{8\Gamma_4^2}{V^2b^2} \tag{59}$$

$$\Gamma_1 = \frac{5\Gamma_4^3}{Vb} + \frac{3Vd^2}{4b^3\Gamma_4} - \frac{6d\Gamma_4}{b^2} + \frac{4c\Gamma_4}{b} - \frac{dcV}{\mathrm{b}^2\Gamma_4} + \frac{eV}{b\Gamma_4} - \frac{\Gamma_4^3}{4} - \frac{8\Gamma_4^3}{V^2b^2} + \frac{f - \frac{d^2}{4b}}{4\Gamma_4 b}V^2 \tag{60}$$

$$\begin{aligned}\Gamma_0 = {} & \frac{\Gamma_4^4}{2Vb} - \frac{V^2d^3}{16b^4\Gamma_4^2} + \frac{3Vd^2}{8b^3} - \frac{3d\Gamma_4^2}{4b^2} + \frac{c\Gamma_4^2}{2b} + \frac{cd^2V^2}{8\mathrm{b}^3\Gamma_4^2} - \frac{cdV}{2b^2} + \frac{eV}{2b} - \frac{eV^2d}{4b^2\Gamma_4^2} + \frac{gV^2}{2b\Gamma_4^2} \\ & - \left(\frac{\Gamma_4^2}{4} + V^2\frac{f - \frac{d^2}{4b}}{4b\Gamma_4^2}\right)\left(\frac{\Gamma_4^2}{4} + \frac{8\Gamma_4^2}{V^2b^2} - \frac{2\Gamma_4^2}{VB} + \frac{3d}{b^2} - \frac{2c}{b} - \frac{\left(f - \frac{d^2}{4b}\right)}{4b\Gamma_4^2}V^2\right)\end{aligned} \tag{61}$$

4.2. Proof of Theorem 2

Considering the sixth-degree polynomial equation shown in (eq.48), we propose the expression (eq.62) in order to reduce the form from sixth degree to a fourth-degree polynomial. We also propose the expression (eq.63), which presents the solution form for quartic equation by extending the used logic and presented solutions in Theorem 1.

$$x = x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3 \tag{62}$$

$$z = x_0 + x_1 + x_2 + x_3 \tag{63}$$

We replace x with its proposed value in (eq.62), in order to end by calculations to the reduced form shown in (eq.50).

In the shown expressions in (eq.64), we rely on the use of x_i, x_j and x_k where $\{x_i, x_j, x_k\} \in \{x_0, x_1, x_2, x_3\}$ and $i \neq j \neq k$.

$$\alpha_1 = \sum_{i=0}^{i=3} x_i^2\,;\;\; \alpha_2 = \sum_{i\neq j} x_i^2 x_j^2\,;\;\; \alpha_3 = \sum_{i\neq j\neq k} x_i x_j x_k\,;\;\; \alpha_4 = x_0x_1x_2x_3 \tag{64}$$

$$x = \frac{[z^2-\alpha_1]}{2} \tag{65}$$

$$x^2 = \alpha_2 + 2\alpha_3 z + 6\alpha_4 \tag{66}$$

$$x^3 = z^3\alpha_3 + \frac{1}{2}z^2[\alpha_2 + 6\alpha_4] - z\alpha_1\alpha_3 - \frac{1}{2}[\alpha_2 + 6\alpha_4]\alpha_1 \tag{67}$$

$$x^4 = 4z^2\alpha_3^2 + 4\alpha_3 z[\alpha_2 + 6\alpha_4] + [\alpha_2 + 6\alpha_4]^2 \tag{68}$$

$$x^5 = 2z^4\alpha_3^2 + 2z^3[\alpha_2 + 6\alpha_4]\alpha_3 + \frac{1}{2}z^2[(\alpha_2 + 6\alpha_4)^2 - 4\alpha_3^2\alpha_1] - 2z\alpha_3\alpha_1[\alpha_2 + 6\alpha_4] - \frac{1}{2}\alpha_1[\alpha_2 + 6\alpha_4]^2 \tag{69}$$

$$x^6 = 8z^3\alpha_3^3 + 12z^2[\alpha_2 + 6\alpha_4]\alpha_3^2 + 6z[(\alpha_2 + 6\alpha_4)]^2\alpha_3 + [\alpha_2 + 6\alpha_4]^3 \tag{70}$$

$$\gamma_4 z^4 + \gamma_3 z^3 + \gamma_2 z^2 + \gamma_1 z + \gamma_0 = 0 \tag{71}$$

We use the expressions of $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ in (eq.64), x in (eq.65), x^2 in (eq.66), x^3 in (eq.67), x^4 in (eq.68), x^5 in (eq.69) and x^6 in (eq.70), to have the fourth degree polynomial shown in (eq.71) where the values of coefficients are as follow:

$$\gamma_4 = 2b\alpha_3^2$$

$$\gamma_3 = 8\alpha_3^3 + d\alpha_3 + 2b[\alpha_2 + 6\alpha_4]\alpha_3$$

$$\gamma_2 = 12[\alpha_2 + 6\alpha_4]\alpha_3^2 + \frac{1}{2}b[(\alpha_2 + 6\alpha_4)^2 - 4\alpha_3^2\alpha_1] + 4c\alpha_3^2 + \frac{1}{2}d[\alpha_2 + 6\alpha_4] + \frac{1}{2}f$$

$$\gamma_1 = 6[(\alpha_2 + 6\alpha_4)]^2\alpha_3 - 2b\alpha_3\alpha_1[\alpha_2 + 6\alpha_4] + 4c[\alpha_2 + 6\alpha_4]\alpha_3 - d\alpha_1\alpha_3 + 2e\alpha_3$$

$$\gamma_0 = [\alpha_2 + 6\alpha_4]^3 - \frac{1}{2}b\alpha_1[\alpha_2 + 6\alpha_4]^2 + c[\alpha_2 + 6\alpha_4]^2 - \frac{1}{2}d[\alpha_2 + 6\alpha_4]\alpha_1 + e[\alpha_2 + 6\alpha_4] - \frac{1}{2}f\alpha_1 + g$$

We divide the polynomial equation shown in (eq.71) on γ_4 to simplify its expression. As a result, we have the shown equation in (eq.50) where the values of coefficients are as follow:

$$\Gamma_4 = \frac{[\alpha_2 + 6\alpha_4] + \frac{d}{2b}}{\alpha_3} \Rightarrow \alpha_2 = \alpha_3\Gamma_4 - \frac{d}{2b} - 6\alpha_4$$

$$\Gamma_3 = \frac{4\alpha_3}{b} + \Gamma_4$$

$$\Gamma_2 = \frac{6\left[\Gamma_4\alpha_3 - \frac{d}{2b}\right]}{b} + \frac{\Gamma_4^2}{4} - \alpha_1 + \frac{2c}{b} + \frac{f - \frac{d^2}{4b}}{4b\alpha_3^2}$$

$$\Gamma_1 = \frac{3\left[\Gamma_4\alpha_3 - \frac{d}{2b}\right]^2}{b\alpha_3} - \Gamma_4\alpha_1 + \frac{2c\left[\alpha_3\Gamma_4 - \frac{d}{2b}\right]}{b\alpha_3} + \frac{e}{b\alpha_3}$$

$$\Gamma_0 = \frac{\left[\Gamma_4\alpha_3 - \frac{d}{2b}\right]^3}{2b\alpha_3^2} - \frac{1}{4}\alpha_1\Gamma_4^2 + \frac{c\left[\alpha_3\Gamma_4 - \frac{d}{2b}\right]^2}{2b\alpha_3^2} + \frac{e\left[\alpha_3\Gamma_4 - \frac{d}{2b}\right]}{2b\alpha_3^2} + \frac{g - \frac{f\alpha_1}{2} + \frac{d^2\alpha_1}{8b}}{2b\alpha_3^2}$$

From precedent section (section 2), we ended with solutions expressed as $y = y_0 + y_1 + y_2$ for fourth degree polynomial equation in simple form shown in (eq.5), whereas the solution for fourth degree polynomial equation in complete form is expressed as $z = -\frac{b}{4a} + \frac{1}{4}y_0 + \frac{1}{4}y_1 + \frac{1}{4}y_2$.

We replace z with $\frac{-\Gamma_3 + y}{4}$ in order to reduce the form of quartic equation from expression (eq.50) to expression (eq.72), where the values of coefficients are as shown in (eq.73).

$$y^4 + Py^2 + Qy + R = 0 \tag{72}$$

$$P = -6\Gamma_3^2 + 16\Gamma_2; Q = 8\Gamma_3^3 - 32\Gamma_2\ \Gamma_3 + 64\Gamma_1; R = -3\Gamma_3^4 + 16\Gamma_2\ \Gamma_3^2 - 64\Gamma_1\ \Gamma_3 + 256\Gamma_0 \tag{73}$$

Concerning the fourth degree polynomial equation in (eq.50), where z is as shown in (eq.63), the principal proposed expressions for the solutions are $z = -\frac{\Gamma_3}{4} + \frac{1}{4}\sqrt{y_0} + \frac{1}{4}\sqrt{y_1} + \frac{1}{4}\sqrt{y_2}$ when $Q \leq 0$ and $z = -\frac{\Gamma_3}{4} - \frac{1}{4}\sqrt{y_0} - \frac{1}{4}\sqrt{y_1} - \frac{1}{4}\sqrt{y_2}$ *when* $Q \geq 0$; where $x_3 = -\frac{\Gamma_3}{4}, x_0 = \pm\frac{1}{4}\sqrt{y_0}, x_1 = \pm\frac{1}{4}\sqrt{y_1}$ and $x_2 = \pm\frac{1}{4}\sqrt{y_2}$. These two principal expressions are sufficient to conduct the calculations of proof, and then generalize the results by using the other expressed forms of solutions in Theorem 1.

We replace $\Gamma_3, \Gamma_2, \Gamma_1$ *and* Γ_0 with their values in function of $\{\Gamma_4\ , \alpha_1\ , \alpha_3\}$ in order to have the expressions of P in (eq.74), Q in (eq.75) and R in (eq.76).

$$P = -2\Gamma_4^2 - \frac{96\alpha_3^2}{b^2} + \frac{48\Gamma_4\alpha_3}{b} - \frac{48d}{b^2} - 16\alpha_1 + \frac{32c}{b} + 4\frac{\left(f - \frac{d^2}{4b}\right)}{b\alpha_3^2} \tag{74}$$

$$\begin{aligned} Q = {} & \frac{512\alpha_3^3}{b^3} - \frac{384\alpha_3^2\Gamma_4}{b^2} + \frac{64\alpha_3\Gamma_4^2}{b} + \frac{384d\alpha_3}{b^3} + \frac{128\alpha_3\alpha_1}{b} - \frac{256c\alpha_3}{b^2} - \frac{32f}{b^2\alpha_3} - \frac{96d\Gamma_4}{b^2} - 32\Gamma_4\alpha_1 \\ & + \frac{64c\Gamma_4}{b} - \frac{8\Gamma_4\left(f - \frac{d^2}{4b}\right)}{b\alpha_3^2} + \frac{56d^2}{b^3\alpha_3} - \frac{64cd}{b^2\alpha_3} + \frac{64e}{b\alpha_3} \end{aligned} \tag{75}$$

$$\begin{aligned} R = {} & -\frac{768\alpha_3^4}{b^4} + \Gamma_4^4 + \frac{768\Gamma_4\alpha_3^3}{b^3} + \frac{16\Gamma_4^3\alpha_3}{b} - \frac{224\Gamma_4^2\alpha_3^2}{b^2} - \frac{768d\alpha_3^2}{b^4} - \frac{256\alpha_3^2\alpha_1}{b^2} + \frac{512c\alpha_3^2}{b^3} + \frac{64f}{b^3} - \frac{48d\Gamma_4^2}{b^2} - 16\alpha_1\Gamma_4^2 + \frac{32c\Gamma_4^2}{b} \\ & + 4\frac{\left(f - \frac{d^2}{4b}\right)}{b\alpha_3^2}\Gamma_4^2 + \frac{384d\Gamma_4\alpha_3}{b^3} + \frac{128\Gamma_4\alpha_3\alpha_1}{b} - \frac{256c\Gamma_4\alpha_3}{b^2} + \frac{32f\Gamma_4}{b^2\alpha_3} + \frac{40d^2\Gamma_4}{b^3\alpha_3} - \frac{208d^2}{b^4} + \frac{256cd}{b^3} - \frac{256e}{b^2} - \frac{64cd\Gamma_4}{b^2\alpha_3} + \frac{64e\Gamma_4}{b\alpha_3} - \frac{16d^3}{b^4\alpha_3^2} \\ & + \frac{32cd^2}{b^3\alpha_3^2} - \frac{64ed}{b^2\alpha_3^2} + 128\frac{\left(g - \frac{f\alpha_1}{2} + \frac{d^2\alpha_1}{8b}\right)}{b\alpha_3^2} \end{aligned} \tag{76}$$

By using the expressions (eq.11), (eq.12), (eq.13) and (eq.14) from the proof of first proposed theorem (Theorem 1), the values of P, Q and R are as shown in (eq.77), (eq.78) and (eq.79) successively.

$$P = -2[(4x_0)^2 + (4x_1)^2 + (4x_2)^2] \;\Rightarrow\; P = -32\left[\alpha_1 - \frac{\Gamma_3^2}{16}\right] = 2\Gamma_4^2 + \frac{16\Gamma_4\alpha_3}{b} + \frac{32\alpha_3^2}{b^2} - 32\alpha_1 \tag{77}$$

$$Q = -8(4x_0)(4x_1)(4x_2) \;\Rightarrow\; Q = -\frac{8(4x_0)(4x_1)(4x_2)(4x_3)}{4x_3} = \frac{2048\alpha_4}{\Gamma_3} = \frac{2048\alpha_4}{\frac{4\alpha_3}{b} + \Gamma_4} \tag{78}$$

$$R = [(4x_0)^2 + (4x_1)^2 + (4x_2)^2]^2 - 4[(4x_0)^2(4x_1)^2 + (4x_0)^2(4x_2)^2 + (4x_1)^2(4x_2)^2]$$

$$\Rightarrow\; R = 256\left[\alpha_1 - \frac{\Gamma_3^2}{16}\right]^2 - 1024\left[\alpha_2 - \frac{\Gamma_3^2}{16}\left(\alpha_1 - \frac{\Gamma_3^2}{16}\right)\right]$$

$$\Rightarrow\ R = -\frac{768\alpha_3^4}{b^4} - 3\Gamma_4^4 - \frac{768\alpha_3^3\Gamma_4}{b^3} - \frac{48\Gamma_4^3\alpha_3}{b} - \frac{288\Gamma_4^2\alpha_3^2}{b^2} + \frac{512\alpha_3^2\alpha_1}{b^2} + 32\alpha_1\Gamma_4^2 + \frac{256\alpha_1\alpha_3\Gamma_4}{b} + 256\alpha_1^2 - 1024\alpha_2 \quad (79)$$

We have a group of four variables $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, whereas we have a group of only three equations to solve $\{P, Q, R\}$ where all of them are dependent on the value of Γ_4. Thereby, the next step is about using the appropriate logic of analysis and calculation to find the value of Γ_4 while taking advantage of the fact that having a group of four variables enables us to solve four equations.

In order to reduce the expression of R in (eq.76), and find a way to determine the value of Γ_4, we suppose that $\left(\frac{4\Gamma_4^2\left(f-\frac{d^2}{4b}\right)}{b\alpha_3^2} + \frac{32f\Gamma_4}{b^2\alpha_3} + \frac{40d^2\Gamma_4}{b^3\alpha_3} - \frac{64cd\Gamma_4}{b^2\alpha_3} + \frac{64e\Gamma_4}{b\alpha_3}\right) = 0$ where $\frac{\Gamma_4}{\alpha_3} \neq 0$. As a result, we have the shown expression in (eq.80).

$$\frac{\Gamma_4}{\alpha_3} = V = -\frac{\frac{32f}{b^2} + \frac{40d^2}{b^3} - \frac{64cd}{b^2} + \frac{64e}{b}}{\frac{4\left(f-\frac{d^2}{4b}\right)}{b}} \quad (80)$$

From expression (eq.80), we can see that $\frac{\Gamma_4}{\alpha_3}$ has a constant value. Therefore, in the rest of calculation, we will replace $\frac{\Gamma_4}{\alpha_3}$ by the constant V, which is shown in (eq.58).

From precedent calculations of dividing the polynomial equation shown in (eq.71) on γ_4 to simplify its expression, along using the expression of Γ_4, we have $\alpha_2 = (\alpha_3\Gamma_4 - \frac{d}{2b} - 6\alpha_4)$.

We have the resulted equation in (eq.81) by using the expressions of P in (eq.74) and (eq.77), which we use to define the shown value of $\alpha_1$in (eq.82).

$$-4\Gamma_4^2 - \frac{128\Gamma_4^2}{V^2b^2} + \frac{32\Gamma_4^2}{Vb} - \frac{48d}{b^2} + 16\alpha_1 + \frac{32c}{b} + \frac{4V^2\left(f-\frac{d^2}{4b}\right)}{\Gamma_4^2 b} = 0 \quad (81)$$

$$\alpha_1 = \frac{\Gamma_4^4 + \frac{32\Gamma_4^4}{V^2b^2} - \frac{8\Gamma_4^4}{Vb} + \frac{12d\Gamma_4^2}{b^2} - \frac{8c\Gamma_4^2}{b} - \frac{V^2\left(f-\frac{d^2}{4b}\right)}{b}}{4\Gamma_4^2} \quad (82)$$

We have the presented polynomial equation in (eq.83) by using the expressions of R in (eq.76) and (eq.79).

$$4\Gamma_4^4 + \frac{1536\Gamma_4\alpha_3^3}{b^3} + \frac{64\Gamma_4^3\alpha_3}{b} + \frac{64\Gamma_4^2\alpha_3^2}{b^2} - \frac{768d\alpha_3^2}{b^4} - \frac{768\alpha_3^2\alpha_1}{b^2} + \frac{512c\alpha_3^2}{b^3} + \frac{64f}{b^3} - \frac{48d\Gamma_4^2}{b^2} - 48\alpha_1\Gamma_4^2 + \frac{32c\Gamma_4^2}{b} + \frac{384d\Gamma_4\alpha_3}{b^3}$$
$$- \frac{128\Gamma_4\alpha_3\alpha_1}{b} - \frac{256c\Gamma_4\alpha_3}{b^2} - \frac{208d^2}{b^4} + \frac{256cd}{b^3} - \frac{256e}{b^2} - \frac{16d^3}{b^4\alpha_3^2} + \frac{32cd^2}{b^3\alpha_3^2} - \frac{64ed}{b^2\alpha_3^2} - 256\alpha_1^2 + 1024\alpha_2 + 128\frac{\left(g-\frac{f\alpha_1}{2}+\frac{d^2\alpha_1}{8b}\right)}{b\alpha_3^2} = 0 \quad (83)$$

We replace $\frac{\Gamma_4}{\alpha_3}$ with its shown expression in (eq.80) to pass from equation (eq.83) to equation (eq.84).

$$4\Gamma_4^4 + \frac{1536\Gamma_4^4}{V^3b^3} + \frac{64\Gamma_4^4}{Vb} + \frac{64\Gamma_4^4}{V^2b^2} - \frac{768d\Gamma_4^2}{V^2b^4} - \frac{768\Gamma_4^2\alpha_1}{V^2b^2} + \frac{512c\Gamma_4^2}{V^2b^3} + \frac{64f}{b^3} - \frac{48d\Gamma_4^2}{b^2} - 48\alpha_1\Gamma_4^2 + \frac{32c\Gamma_4^2}{b} + \frac{384d\Gamma_4^2}{Vb^3} - \frac{128\Gamma_4^2\alpha_1}{Vb}$$
$$- \frac{256c\Gamma_4^2}{Vb^2} - \frac{208d^2}{b^4} + \frac{256cd}{b^3} - \frac{256e}{b^2} - \frac{16V^2d^3}{b^4\Gamma_4^2} + \frac{32cd^2V^2}{b^3\Gamma_4^2} - \frac{64edV^2}{b^2\Gamma_4^2} - 256\alpha_1^2 + 1024\alpha_2 + \frac{128V^2\left(g-\frac{f\alpha_1}{2}+\frac{d^2\alpha_1}{8b}\right)}{b\Gamma_4^2} = 0 \quad (84)$$

We use equations (eq.75) and (eq.78) to calculate the value of α_4 and then express α_2 as shown in (eq.85) where we rely on replacing $\frac{\Gamma_4}{\alpha_3}$ with its constant value V, which is presented in (eq.80).

$$1024\alpha_2 = \frac{1024\Gamma^2}{V} - \frac{512d}{b} - \frac{6144\Gamma^4}{V^4b^4} + \frac{3072\Gamma^4}{V^3b^3} + \frac{384\Gamma^4}{V^2b^2} - \frac{4608d\Gamma^2}{V^2b^4} - \frac{1536\Gamma^2\alpha_1}{V^2b^2} + \frac{3072c\Gamma^2}{V^2b^3} + \frac{384f}{b^3} + \frac{96V\left(f-\frac{d^2}{4b}\right)}{b^2} - \frac{672d^2}{b^4}$$

$$+\frac{768cd}{b^3}-\frac{768e}{b^2}-\frac{192\Gamma^4}{Vb}+\frac{96fV}{b^2}+\frac{288d\Gamma^2}{b^2}+96\Gamma^2\alpha_1-\frac{192c\Gamma^2}{b}+\frac{24V^2(f-\frac{d^2}{4b})}{b}-\frac{168(d^2V)}{b^3}+\frac{192(cdV)}{b^2}-\frac{192eV}{b} \tag{85}$$

In order to pass from equation (eq.84) to equation (eq.86), we replace α_2 with its shown expression in (eq.85).

$$4\Gamma_4^4+\frac{1024\Gamma_4^2}{V}-\frac{512d}{b}-\frac{6144\Gamma_4^4}{V^4b^4}+\frac{4608\Gamma_4^4}{V^3b^3}-\frac{128\Gamma_4^4}{Vb}+\frac{448\Gamma_4^4}{V^2b^2}-\frac{5376d\Gamma_4^2}{V^2b^4}-\frac{2304\Gamma_4^2\alpha_1}{V^2b^2}+\frac{3584c\Gamma_4^2}{V^2b^3}+\frac{448f}{b^3}+\frac{24V^2\left(f-\frac{d^2}{4b}\right)}{b}+\frac{192fV}{b^2}-\frac{192d^2V}{b^3}$$
$$+\frac{192cdV}{b^2}-\frac{192eV}{b}+\frac{240d\Gamma_4^2}{b^2}+48\alpha_1\ \Gamma_4^2-\frac{160c\Gamma_4^2}{b}+\frac{384d\Gamma_4^2}{Vb^3}-\frac{128\Gamma_4^2\alpha_1}{Vb}-\frac{256c\Gamma_4^2}{Vb^2}-\frac{880d^2}{b^4}+\frac{1024cd}{b^3}-\frac{1024e}{b^2}-\frac{16V^2d^3}{b^4\Gamma_4^2}$$
$$+\frac{32cd^2V^2}{b^3\Gamma_4^2}-\frac{64edV^2}{b^2\Gamma_4^2}-256\alpha_1^2+\frac{128V^2g}{b\Gamma_4^2}-\frac{64V^2f\alpha_1}{b\Gamma_4^2}+\frac{16V^2d^2\alpha_1}{b^2\Gamma_4^2}=0 \tag{86}$$

We replace α_1 with its shown expression in (eq.82), and then we assemble terms of equation (eq.86) in function of degrees, in order to have the expression (eq.88) where coefficients are presented in (eq.54), (eq.55), (eq.56) and (eq.57).

$$\alpha_4=x_1x_2x_3x_4 \Rightarrow\ 2048\alpha_4=-32\Gamma_3^2\alpha_1-\frac{8\Gamma_3^2(e-\frac{c^2}{4})}{\alpha_3^2}+\frac{64\Gamma_3 d}{\alpha_3} \tag{87}$$

$$\lambda_3(\Gamma_4^2)^4+\lambda_2(\Gamma_4^2)^3+\lambda_1(\Gamma_4^2)^2+\lambda_0(\Gamma_4^2)=0 \tag{88}$$

Since we supposed that $\left(\frac{4\Gamma_4^2\left(f-\frac{d^2}{4b}\right)}{b\alpha_3^2}+\frac{32f\Gamma_4}{b^2\alpha_3}+\frac{40(d^2\Gamma_4)}{b^3\alpha_3}-\frac{64cd\Gamma_4}{b^2\alpha_3}+\frac{64e\Gamma_4}{b\alpha_3}\right)=0$ whereas adopting $\frac{\Gamma_4}{\alpha_3}\neq 0$, we eliminate the root zero as solution for polynomial equation (eq.88) and we use the cubic solution to solve the polynomial equation $(\lambda_3(\Gamma_4^2)^3+\lambda_2(\Gamma_4^2)^2+\lambda_1(\Gamma_4^2)+\lambda_0)=0$, because all coefficients are expressed only in function of c, d, e, f, g and the constant V. As a result, we have six possible values for Γ_4 as solutions for polynomial equation shown in (eq.53).

Supposing that $G_{\{\Gamma_4\}}$ is the group of solutions for shown equation in (eq.53), where these solutions are expressed as $\Gamma_{4,i}$ and $-\Gamma_{4,i}$ with $1\leq\ i\leq\ 3$. The group of solutions $G_{\{\Gamma_4\}}$ is determined by relying on cubic root shown in (eq.90) and quadratic roots (eq.91) and (eq.92).

$$G_{\{\Gamma_4\}}=\{\Gamma_{4,1},\Gamma_{4,2},\Gamma_{4,3},-\Gamma_{4,1},-\Gamma_{4,2},-\Gamma_{4,3}\} \tag{89}$$

We suppose that $b'=\frac{\lambda_2}{\lambda_3}$, $c'=\frac{\lambda_1}{\lambda_3}$ and $d'=\frac{\lambda_0}{\lambda_3}$, whereas using the expressions (eq.54), (eq.55), (eq.56) and (eq.57). We suppose also that $D'=27d'+2b'^3-9c'b'$ and $C'=9c'-3b'^2$. The solutions $\Gamma_{4,1}$, $\Gamma_{4,2}$ and $\Gamma_{4,3}$ for shown equation in (eq.53) are as follow:

$$\Gamma_{4,1}^2=\frac{-b'}{3}+\frac{1}{3}\sqrt[3]{-\frac{D'}{2}+\sqrt{\left(\frac{D'}{2}\right)^2+\left(\frac{C'}{3}\right)^3}}+\frac{1}{3}\sqrt[3]{-\frac{D'}{2}-\sqrt{\left(\frac{D'}{2}\right)^2+\left(\frac{C'}{3}\right)^3}} \tag{90}$$

$$\Gamma_{4,2}^2=-\frac{\frac{b'}{2}+\Gamma_{4,1}^2}{2}+\sqrt{\left(\frac{\frac{b'}{2}+\Gamma_{4,1}^2}{2}\right)^2-\frac{D'}{\Gamma_{4,1}^2}} \tag{91}$$

$$\Gamma_{4,3}^2=-\frac{\frac{b'}{2}+\Gamma_{4,1}^2}{2}-\sqrt{\left(\frac{\frac{b'}{2}+\Gamma_{4,1}^2}{2}\right)^2-\frac{D'}{\Gamma_{4,1}^2}} \tag{92}$$

After determining the values of Γ_4 by using cubic solution and quadratic solutions, the following step consists of solving the polynomial equation shown in (eq.72).

The coefficients P in (eq.74) and R in (eq.76) are dependent on Γ_4^2 and α_1. The coefficient α_1 in (eq.82) is dependent on Γ_4^2, whereas the coefficient Q shown in (eq.75) is dependent on Γ_4 and α_1. Therefore, we are going to use only $\Gamma_{4,1}$, $\Gamma_{4,2}$ and $\Gamma_{4,3}$ to calculate the potential values of z, because $\{-\Gamma_{4,1},-\Gamma_{4,2},-\Gamma_{4,3}\}$ are going only to inverse the sign of coefficient Q and thereby inversing the signs of potential values of z as solutions for polynomial equation shown in (eq.50), which will not influence the potential values of x as solutions for sixth degree polynomial equation shown in (eq.48) because $x=\frac{1}{2}(z^2-\alpha_1)$.

We use the first proposed theorem (Theorem 1) to calculate the four solutions for fourth degree polynomial equation shown in (eq.72) after calculating the coefficients P, Q and R for each value of Γ_4 from the group $\{\Gamma_{4,1},\Gamma_{4,2},\Gamma_{4,3}\}$. Thereby, we have twelve values to calculate as potential solutions for the polynomial equation shown in (eq.72).

After using first proposed theorem on (eq.72) for each value of Γ_4 from the group $\{\Gamma_{4,1}, \Gamma_{4,2}, \Gamma_{4,3}\}$, we have three groups of potential solutions for polynomial equation shown in (eq.72), where each group is dependent on different value of Γ_4. We express these groups of solutions as follow: $K_{\{\Gamma_{4,1}\}}, K_{\{\Gamma_{4,2}\}}, K_{\{\Gamma_{4,3}\}}$.

$$K_{\{\Gamma_{4,1}\}} = \left\{S_{(\Gamma_{4,1},1)}, S_{(\Gamma_{4,1},2)}, S_{(\Gamma_{4,1},3)}, S_{(\Gamma_{4,1},4)}\right\} \quad (93)$$

$$K_{\{\Gamma_{4,2}\}} = \left\{S_{(\Gamma_{4,2},1)}, S_{(\Gamma_{4,2},2)}, S_{(\Gamma_{4,2},3)}, S_{(\Gamma_{4,2},4)}\right\} \quad (94)$$

$$K_{\{\Gamma_{4,3}\}} = \left\{S_{(\Gamma_{4,3},1)}, S_{(\Gamma_{4,3},2)}, S_{(\Gamma_{4,3},3)}, S_{(\Gamma_{4,3},4)}\right\} \quad (95)$$

Concerning the fourth-degree polynomial equation shown in (eq.50), we have three groups of solutions where each group is dependent on different value of Γ_4; as shown in (eq.96), (eq.97) and (eq.98). The values of $S_{(\Gamma_{4,i},j)}$, where $1 \leq i \leq 3$ and $1 \leq j \leq 4$, are from the expressed solutions in the groups (eq.93), (eq.94) and (eq.95).

$$M_{\{\Gamma_{4,1}\}} = \left\{-\frac{1}{4}\left[\frac{4\Gamma_{4,1}}{V} + \Gamma_{4,1}\right] + \frac{1}{4}S_{(\Gamma_{4,1},1)}, -\frac{1}{4}\left[\frac{4\Gamma_{4,1}}{V} + \Gamma_{4,1}\right] + \frac{1}{4}S_{(\Gamma_{4,1},2)}, -\frac{1}{4}\left[\frac{4\Gamma_{4,1}}{V} + \Gamma_{4,1}\right] + \frac{1}{4}S_{(\Gamma_{4,1},3)}, -\frac{1}{4}\left[\frac{4\Gamma_{4,1}}{V} + \Gamma_{4,1}\right] + \frac{1}{4}S_{(\Gamma_{4,1},4)}\right\} \quad (96)$$

$$M_{\{\Gamma_{4,2}\}} = \left\{-\frac{1}{4}\left[\frac{4\Gamma_{4,2}}{V} + \Gamma_{4,2}\right] + \frac{1}{4}S_{(\Gamma_{4,2},1)}, -\frac{1}{4}\left[\frac{4\Gamma_{4,2}}{V} + \Gamma_{4,2}\right] + \frac{1}{4}S_{(\Gamma_{4,2},2)}, -\frac{1}{4}\left[\frac{4\Gamma_{4,2}}{V} + \Gamma_{4,2}\right] + \frac{1}{4}S_{(\Gamma_{4,2},3)}, -\frac{1}{4}\left[\frac{4\Gamma_{4,2}}{V} + \Gamma_{4,2}\right] + \frac{1}{4}S_{(\Gamma_{4,2},4)}\right\} \quad (97)$$

$$M_{\{\Gamma_{4,3}\}} = \left\{-\frac{1}{4}\left[\frac{4\Gamma_{4,3}}{V} + \Gamma_{4,3}\right] + \frac{1}{4}S_{(\Gamma_{4,3},1)}, -\frac{1}{4}\left[\frac{4\Gamma_{4,3}}{V} + \Gamma_{4,3}\right] + \frac{1}{4}S_{(\Gamma_{4,3},2)}, -\frac{1}{4}\left[\frac{4\Gamma_{4,3}}{V} + \Gamma_{4,3}\right] + \frac{1}{4}S_{(\Gamma_{4,3},3)}, -\frac{1}{4}\left[\frac{4\Gamma_{4,3}}{V} + \Gamma_{4,3}\right] + \frac{1}{4}S_{(\Gamma_{4,3},4)}\right\} \quad (98)$$

We suppose that $\dot{S}_{(\Gamma_{4,i},j)} = \left(-\frac{1}{4}\left[\frac{4\Gamma_{4,i}}{V} + \Gamma_{4,i}\right] + \frac{1}{4}S_{(\Gamma_{4,i},j)}\right)$ where $1 \leq i \leq 3$ and $1 \leq j \leq 4$, in order to simplify the expressed values in (eq.96), (eq.97) and (eq.98). Thereby, we have three groups of values as potential solutions for sixth degree polynomial equation shown in (eq.48). These three groups are as shown in (eq.99), (eq.100) and (eq.101) where $\alpha_{(1,\Gamma_{4,i})}$ is as follow:

$$\alpha_{(1,\Gamma_{4,i})} = \frac{\Gamma_{4,i}^4 + \frac{32\Gamma_{4,i}^4}{V^2b^2} - \frac{8\Gamma_{4,i}^4}{Vb} + \frac{12d\Gamma_{4,i}^2}{b^2} - \frac{8c\Gamma_{4,i}^2}{b} - \frac{V^2\left(f-\frac{d^2}{4b}\right)}{b}}{4\Gamma_{4,i}^2}$$

The expression of $\alpha_{(1,\Gamma_{4,i})}$ is an extending of the shown expression of α_1 in (eq.82) by changing the value of Γ_4, where $\Gamma_{4,i}$ belong to the group $\{\Gamma_{4,1}, \Gamma_{4,2}, \Gamma_{4,3}\}$.

$$N_{\{\Gamma_{4,1}\}} = \left\{\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},1)} - \alpha_{(1,\Gamma_{4,1})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},2)} - \alpha_{(1,\Gamma_{4,1})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},3)} - \alpha_{(1,\Gamma_{4,1})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},4)} - \alpha_{(1,\Gamma_{4,1})}\right]\right\} \quad (99)$$

$$N_{\{\Gamma_{4,2}\}} = \left\{\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,2},1)} - \alpha_{(1,\Gamma_{4,2})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,2},2)} - \alpha_{(1,\Gamma_{4,2})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,2},3)} - \alpha_{(1,\Gamma_{4,2})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,2},4)} - \alpha_{(1,\Gamma_{4,2})}\right]\right\} \quad (100)$$

$$N_{\{\Gamma_{4,3}\}} = \left\{\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,3},1)} - \alpha_{(1,\Gamma_{4,3})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,3},2)} - \alpha_{(1,\Gamma_{4,3})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,3},3)} - \alpha_{(1,\Gamma_{4,3})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,3},4)} - \alpha_{(1,\Gamma_{4,3})}\right]\right\} \quad (101)$$

We have twelve values as potential solutions for the sixth degree polynomial equation shown in (eq.48). However, many of them are only redundancies of others and there are only six official solutions to determine. The variables $\{\Gamma_{4,1}, \Gamma_{4,2}, \Gamma_{4,3}\}$ are the responsible of solution redundancies from one group to other.

In order to avoid the complications of calculating the twelve values from the groups $N_{\{\Gamma_{4,1}\}}$, $N_{\{\Gamma_{4,2}\}}$and $N_{\{\Gamma_{4,3}\}}$, and then determining the six solutions for sixth degree polynomial equation shown in (eq.48), we propose the expressed values in (eq.102), (eq.103), (eq.104), (eq.105), (eq.106) and (eq.107) as the six official solutions for sixth degree polynomial equation shown in (eq.48).

The first four proposed values as solutions for sixth degree polynomial equation shown in (eq.48) are from the group $N_{\{\Gamma_{4,1}\}}$, whereas the fifth and sixth values are expressed by deduction using the expressions of quadratic roots. The useless redundancies of solutions are from one group to other; therefore, we choose the first four solutions from the same group $N_{\{\Gamma_{4,1}\}}$.

In our six proposed solutions, we use the values $\dot{S}_{(\Gamma_{4,1},j)} = -\frac{1}{4}\ \Gamma_{4,1} + \frac{1}{4}S_{(\Gamma_{4,1},j)}$ where $1 \leq\ j \leq\ 4$ and $S_{(\Gamma_{4,1},j)}$ from $K_{\{\Gamma_{4,1}\}}$. The variable $\alpha_{(1,\Gamma_{4,i})}$ is expressed as follow:

$$\alpha_{(1,\Gamma_{4,i})} = \frac{\Gamma_{4,i}^4 + \frac{32\Gamma_{4,i}^4}{V^2b^2} - \frac{8\Gamma_{4,i}^4}{Vb} + \frac{12d\Gamma_{4,i}^2}{b^2} - \frac{8c\Gamma_{4,i}^2}{b} - \frac{V^2\left(f-\frac{d^2}{4b}\right)}{b}}{4\Gamma_{4,i}^2}$$

$\Gamma_{4,1}$ is from the group $G_{\{\Gamma_4\}}$ shown in (eq.88) which contains the solutions for polynomial equation (eq.52). The values of $\dot{S}_{(\Gamma_{4,1},1)}$, where $1 \leq\ j \leq\ 4$, are the solutions for quartic equation (eq.50) and they are determined by using Theorem 1.

$$S_1 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},1)}\ - \alpha_{(1,\Gamma_{4,1})}\right] \quad (102)$$

$$S_2 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},2)}\ - \alpha_{(1,\Gamma_{4,1})}\right] \quad (103)$$

$$S_3 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},3)}\ - \alpha_{(1,\Gamma_{4,1})}\right] \quad (104)$$

$$S_4 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},4)}\ - \alpha_{(1,\Gamma_{4,1})}\right] \quad (105)$$

$$S_5 = -\frac{b+S_1+S_2+S_3+S_4}{2} - \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \quad (106)$$

$$S_6 = -\frac{b+S_1+S_2+S_3+S_4}{2} + \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \quad (107)$$

4.3. Third Proposed Theorem

In the second proposed theorem in this paper, we rely on reducing sixth degree polynomial equation to fourth degree in order to find proper roots for concerned equation. However, the fourth degree part of resulted quartic equation by reduction is dependent on the fifth degree part of concerned sixth degree polynomial. Thereby, the equation $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0\ with\ A \neq\ 0$ where the coefficient of fifth degree part is equal zero imposes a problem of reduction to fourth degree.

Therefore, in this subsection, we present a third theorem to propose six formulary solutions for sixth degree polynomial equation in general form $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0\ with\ A \neq\ 0$, where coefficients belong to the group of numbers ℝ and the coefficient of fifth degree part equal zero.

This third proposed theorem is based on the same logic and calculations of Theorem 2. However, it is distinguished by treating sixth degree polynomial where the fifth degree part is absent. First, we pass from equation expression $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0\ with\ A \neq\ 0$ to the expression $w^6 + \frac{Cw^4}{A} + \frac{Dw^3}{A} + \frac{Ew^2}{A} + \frac{Fw}{A} + \frac{G}{A} = 0$, and then we use the expression $w = \sqrt{\frac{-C}{15A}} + x$ to induce a fifth degree part whereas eliminating the fourth degree part of concerned sixth degree polynomial.

Inducing the fifth degree part in polynomial equation shown in (eq.108), whereas eliminating the fourth degree part; help to reduce the amount of calculations comparing to the situation of having a fourth degree element to treat during the process of reduction from sixth degree polynomial equation to quartic equation.

The shown equation in (eq.108) is resulted by replacing w with $w = \sqrt{\frac{-C}{15A}}\ + x$. The coefficients of equation (eq.108) are as expressed in (eq.109), (eq.110), (eq.111), (eq.112) and (eq.113).

$$x^6 + bx^5 + dx^3 + ex^2 + fx + g = 0 \quad (108)$$

$$b = 6\sqrt{\frac{-C}{15A}} \quad (109)$$

$$d = \frac{8C}{3A}\sqrt{\frac{-C}{15A}}\ + \frac{D}{A} \quad (110)$$

$$e = \frac{-C^2}{3A^2}\ + \frac{3D}{A}\sqrt{\frac{-C}{15A}}\ + \frac{E}{A} \quad (111)$$

$$f = -\frac{18C^2}{5A^2}\sqrt{\frac{-C}{15A}}\ - \frac{DC}{5A^2}\ + \frac{2E}{A}\sqrt{\frac{-C}{15A}}\ + \frac{F}{A} \quad (112)$$

$$g = \frac{-16C^3}{3375A^3} - \frac{DC}{15A^2}\sqrt{\frac{-C}{15A}}\ - \frac{EC}{15A^2}\ + \frac{F}{A}\sqrt{\frac{-C}{15A}}\ + \frac{G}{A} \quad (113)$$

Theorem 3

In order to reduce the sixth degree polynomial equation $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0\ with\ A \neq\ 0$ to the quartic equation shown in (eq.114), where coefficients belong to the group of numbers ℝ we first replace w with $w = \sqrt{\frac{-C}{15A}} + x$. Then, the reduction from sixth degree to fourth degree is conducted by supposing $x = (x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3)$, whereas supposing $z = (x_0 +\ \ x_1 + x_2 +\ \ x_3)$ is the solution for fourth degree polynomial equation in (eq.114) by using Theorem 1 and relying on the expression $x_3 =\ -\frac{\Upsilon_3}{4}$. The variable Υ_3 is defined as shown in (eq.115) where α_3 is presented in (eq.119) and Υ_4 is the solution for the polynomial equation (eq.120), which relies on the coefficients (eq.121), (eq.122), (eq.123) and (eq.124). The shown coefficients in (eq.121), (eq.122), (eq.123) and (eq.124) are expressed by using the constant V, which is defined in (eq.125). The coefficients Υ_3, Υ_2, Υ_1 and Υ_0 of quartic equation (eq.114) are determined by using calculated value of Υ_4 and using the shown expressions in (eq.115), (eq.116), (eq.117) and (eq.118). The six proposed solutions for polynomial equation $Aw^6 + Cw^4 + Dw^3 +$

$Ew^2 + Fw + G = 0 \;\; with \;\; A \neq \;\; 0$ are as shown in (eq.136), (eq.137), (eq.138), (eq.139), (140) and (eq.141).

$$z^4 + \Upsilon_3 z^3 + \Upsilon_2 z^2 + \Upsilon_1 z + \Upsilon_0 = 0 \tag{114}$$

$$\Upsilon_3 = \frac{4\alpha_3}{b} \; + \Upsilon_4 \tag{115}$$

$$\Upsilon_2 = \frac{8\Upsilon_4^2}{Vb} - \frac{6d}{b^2} + \frac{\left(f - \frac{d^2}{4b}\right)V^2}{2b\Upsilon_4^2} - \frac{8\Upsilon_4^2}{V^2 b^2} \tag{116}$$

$$\Upsilon_1 = \frac{5\Gamma_4^3}{Vb} + \frac{3Vd^2}{4b^3\Upsilon_4} - \frac{6d\Upsilon_4}{b^2} + \frac{eV}{b\Upsilon_4} - \frac{\Upsilon_4^3}{4} - \frac{8\Upsilon_4^3}{V^2 b^2} + \frac{f - \frac{d^2}{4b}}{4\Upsilon_4 b} V^2 \tag{117}$$

$$\Upsilon_0 = \frac{\Upsilon_4^4}{2Vb} - \frac{V^2 d^3}{16b^4\Upsilon_4^2} + \frac{3Vd^2}{8b^3} - \frac{3d\Upsilon_4^2}{4b^2} + \frac{eV}{2b} - \frac{eV^2 d}{4b^2\Upsilon_4^2} + \frac{gV^2}{2b\Upsilon_4^2} - \left(\frac{\Upsilon_4^2}{4} + V^2 \frac{f - \frac{d^2}{4b}}{4b\Upsilon_4^2}\right)\left(\frac{\Upsilon_4^2}{4} + \frac{8\Upsilon_4^2}{V^2 b^2} - \frac{2\Upsilon_4^2}{VB} + \frac{3d}{b^2} - \frac{\left(f - \frac{d^2}{4b}\right)}{4b\Upsilon_4^2} V^2\right) \tag{118}$$

$$\alpha_3 = -\frac{\Upsilon_4 \frac{4\left(f - \frac{d^2}{4b}\right)}{b}}{\frac{32f}{b^2} + \frac{40d^2}{b^3} + \frac{64e}{b}} \tag{119}$$

$$\beta_3 \Upsilon_4^6 \; + \beta_2 \Upsilon_4^4 \; + \beta_1 \Upsilon_4^2 \; + \beta_0 = 0 \tag{120}$$

$$\beta_3 = -\frac{40960}{V^4 b^4} \; + \frac{16384}{V^3 b^3} \; - \frac{1536}{V^2 b^2} \tag{121}$$

$$\beta_2 = -\frac{24576d}{V^2 b^4} \; + \frac{3072d}{Vb^3} \; + \frac{1024}{V} \tag{122}$$

$$\beta_1 = -\frac{512d}{b} \; + \frac{1536f}{b^3} \; + \frac{28V^2 f}{b} \; - \frac{7V^2 d^2}{b^2} + \frac{96Vf}{b^2} \; - \frac{168d^2 V}{b^3} \; - \frac{192Ve}{b} \; - \frac{3456d^2}{b^4} \; - \frac{1024e}{b^2} \tag{123}$$

$$\beta_0 = \; -\frac{64V^2 d^3}{b^4} \; - \frac{64eV^2 d}{b^2} \; + \frac{128V^2 g}{b} \; + \frac{192V^2 df}{b^3} \tag{124}$$

$$V = -\frac{\frac{32f}{b^2} + \frac{40d^2}{b^3} + \frac{64e}{b}}{\frac{4\left(f - \frac{d^2}{4b}\right)}{b}} \tag{125}$$

4.4. Proof of Theorem 3

Considering the sixth-degree polynomial equation $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0$ where the fifth degree part is absent, we divide the polynomial on A and then we use the expression $w = \sqrt{\frac{-C}{15A}} + x$ in order to induce a fifth degree part and eliminate the fourth degree part. Then, by using the expression (eq.62), we reduce the resulted sixth degree polynomial (eq.108) to the quartic polynomial shown in (eq.126).

$$\upsilon_4 z^4 + \upsilon_3 z^3 + \upsilon_2 z^2 + \upsilon_1 z + \upsilon_0 = 0 \tag{126}$$

We rely on the expressions of $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ in (eq.64), x in (eq.65), x^2 in (eq.66), x^3 in (eq.67), x^5 in (eq.69) and x^6 in (eq.70) to express the fourth degree polynomial shown in (eq.126) where the values of coefficients are as follow:

$$\upsilon_4 = 2b\alpha_3^2$$

$$\upsilon_3 = 8\alpha_3^3 + d\alpha_3 + 2b[\alpha_2 + 6\alpha_4]\alpha_3$$

$$\upsilon_2 = 12[\alpha_2 + 6\alpha_4]\alpha_3^2 + \frac{1}{2}b[(\alpha_2 + 6\alpha_4)^2 - 4\alpha_3^2\alpha_1] + \frac{1}{2}d[\alpha_2 + 6\alpha_4] + \frac{1}{2}f$$

$$\upsilon_1 = 6[(\alpha_2 + 6\alpha_4)]^2\alpha_3 - 2b\alpha_3\alpha_1[\alpha_2 + 6\alpha_4] - d\alpha_1 \;\; \alpha_3 + 2e\alpha_3$$

$$\upsilon_0 = [\alpha_2 + 6\alpha_4]^3 - \frac{1}{2}b\alpha_1[\alpha_2 + 6\alpha_4]^2 - \frac{1}{2}d[\alpha_2 + 6\alpha_4]\alpha_1 + e[\alpha_2 + 6\alpha_4] - \frac{1}{2}f\alpha_1 + g$$

We have the shown fourth degree polynomial in (eq.114) after dividing the polynomial (eq.126) on υ_4. The coefficients of polynomial (eq.114) are as follow:

$$\Upsilon_4 = \frac{[\alpha_2+6\alpha_4]+\frac{d}{2b}}{\alpha_3} \Rightarrow \alpha_2 = \alpha_3\Upsilon_4 - \frac{d}{2b} - 6\alpha_4$$

$$\Upsilon_3 = \frac{4\alpha_3}{b} + \Upsilon_4$$

$$\Upsilon_2 = \frac{6\left[\Upsilon_4\alpha_3 - \frac{d}{2b}\right]}{b} + \frac{\Upsilon_4^2}{4} - \alpha_1 + \frac{\left(f - \frac{d^2}{4b}\right)}{4b\alpha_3^2}$$

$$\Upsilon_1 = \frac{3\left[\Upsilon_4\alpha_3 - \frac{d}{2b}\right]^2}{b\alpha_3} - \Upsilon_4\alpha_1 + \frac{e}{b\alpha_3}$$

$$\Upsilon_0 = \frac{\left[\Upsilon_4\alpha_3 - \frac{d}{2b}\right]^3}{2b\alpha_3^2} - \frac{1}{4}\alpha_1\Upsilon_4^2 + \frac{e\left[\alpha_3\Upsilon_4 - \frac{d}{2b}\right]}{2b\alpha_3^2} + \frac{g - \frac{f\alpha_1}{2} + \frac{d^2\alpha_1}{8b}}{2b\alpha_3^2}$$

We have the fourth degree polynomial $y^4 + My^2 + Ny + O = 0$ by replacing z with $\frac{-\Upsilon_3+y}{4}$ in the polynomial (eq.114). The coefficients M, N and O are as expressed in (eq.127), (eq.128) and (eq.129).

$$M = -2\Upsilon_4^2 - \frac{96\alpha_3^2}{b^2} + \frac{48\Upsilon_4\alpha_3}{b} - \frac{48d}{b^2} - 16\alpha_1 + \frac{4\left(f - \frac{d^2}{4b}\right)}{b\alpha_3^2} \tag{127}$$

$$N = \frac{512\alpha_3^3}{b^3} - \frac{384\alpha_3^2\Upsilon_4}{b^2} + \frac{64\alpha_3\Upsilon_4^2}{b} + \frac{384d\alpha_3}{b^3} + \frac{128\alpha_3\alpha_1}{b} - \frac{32f}{b^2\alpha_3} - \frac{96d\Upsilon_4}{b^2} - 32\Upsilon_4\alpha_1 - \frac{8\Upsilon_4\left(f - \frac{d^2}{4b}\right)}{b\alpha_3^2} + \frac{56d^2}{b^3\alpha_3} + \frac{64e}{b\alpha_3} \tag{128}$$

$$O = -\frac{768\alpha_3^4}{b^4} + \Upsilon_4^4 + \frac{768\Upsilon_4\alpha_3^3}{b^3} + \frac{16\Upsilon_4^3\alpha_3}{b} - \frac{224\Upsilon_4^2\alpha_3^2}{b^2} - \frac{768d\alpha_3^2}{b^4} - \frac{256\alpha_3^2\alpha_1}{b^2} + \frac{64f}{b^3} - \frac{48d\Upsilon_4^2}{b^2} - 16\alpha_1\Upsilon_4^2 + \frac{4\left(f - \frac{d^2}{4b}\right)}{b\alpha_3^2}\Upsilon_4^2 + \frac{384d\Upsilon_4\alpha_3}{b^3} + \frac{128\Upsilon_4\alpha_3\alpha_1}{b} + \frac{32f\Upsilon_4}{b^2\alpha_3} + \frac{40d^2\Upsilon_4}{b^3\alpha_3} - \frac{208d^2}{b^4} - \frac{256e}{b^2} + \frac{64e\Upsilon_4}{b\alpha_3} - \frac{16d^3}{b^4\alpha_3^2} - \frac{64ed}{b^2\alpha_3^2} + \frac{128\left(g - \frac{f\alpha_1}{2} + \frac{d^2\alpha_1}{8b}\right)}{b\alpha_3^2} \tag{129}$$

In order to reduce the expression of O in (eq.129), and find a way to determine the value of Υ_4, we suppose that $\left(\frac{4\Upsilon_4^2\left(f - \frac{d^2}{4b}\right)}{b\alpha_3^2} + \frac{32f\Upsilon_4}{b^2\alpha_3} + \frac{40d^2\Upsilon_4}{b^3\alpha_3} + \frac{64e\Upsilon_4}{b\alpha_3}\right) = 0$ where $\frac{\Upsilon_4}{\alpha_3} \neq 0$. As a result, we have the shown expression in (eq.130).

$$\frac{\Upsilon_4}{\alpha_3} = V = -\frac{\left(\frac{32f}{b^2} + \frac{40d^2}{b^3} + \frac{64e}{b}\right)}{\frac{4\left(f - \frac{d^2}{4b}\right)}{b}} \tag{130}$$

We rely on the same used logic and processes of calculation in the proof of Theorem 2, in order to continue the proof of third proposed theorem. Thereby, the variables α_1, α_4 and α_2 are as expressed in (eq.131), (eq.132) and (eq.133) respectively, whereas the resulted polynomial equation to determine the value of Υ_4 is as shown in (eq.134).

$$\alpha_1 = \frac{\Upsilon_4^4 + \frac{32\Upsilon_4^4}{V^2b^2} - \frac{8\Upsilon_4^4}{Vb} + \frac{12d\Upsilon_4^2}{b^2} - \frac{V^2\left(f - \frac{d^2}{4b}\right)}{b}}{4\Upsilon_4^2} \tag{131}$$

$$2048\alpha_4 = \frac{2048\Upsilon^4}{V^4b^4} - \frac{1024\Upsilon^4}{V^3b^3} - \frac{128\Upsilon^4}{V^2b^2} + \frac{1536d\Upsilon^2}{V^2b^4} + \frac{512\Upsilon^2\alpha_1}{V^2b^2} - \frac{128f}{b^3} - \frac{32V\left(f - \frac{d^2}{4b}\right)}{b^2} + \frac{224d^2}{b^4} + \frac{256e}{b^2} + \frac{64\Upsilon^4}{Vb} - \frac{32fV}{b^2} - \frac{96d\Upsilon^2}{b^2} - 32\Upsilon^2\alpha_1 - \frac{8V^2\left(f - \frac{d^2}{4b}\right)}{b} + \frac{56d^2V}{b^3} + \frac{64eV}{b} \tag{132}$$

$$1024\alpha_2 = \frac{1024\Upsilon^2}{V} - \frac{512d}{b} - \frac{6144\Upsilon^4}{V^4b^4} + \frac{3072\Upsilon^4}{V^3b^3} + \frac{384\Upsilon^4}{V^2b^2} - \frac{4608d\Upsilon^2}{V^2b^4} - \frac{1536\Upsilon^2\alpha_1}{V^2b^2} + \frac{384f}{b^3} + \frac{96V\left(f-\frac{d^2}{4b}\right)}{b^2} - \frac{672d^2}{b^4} - \frac{768e}{b^2} - \frac{192\Upsilon^4}{Vb} + \frac{96fV}{b^2} + \frac{288d\Upsilon^2}{b^2} + 96\Upsilon^2\alpha_1 + \frac{24V^2\left(f-\frac{d^2}{4b}\right)}{b} - \frac{168d^2V}{b^3} - \frac{192eV}{b} \tag{133}$$

$$4\Upsilon_4^4 + \frac{1024\Upsilon_4^2}{V} - \frac{512d}{b} - \frac{6144\Upsilon_4^4}{V^4b^4} + \frac{4608\Upsilon_4^4}{V^3b^3} - \frac{128\Upsilon_4^4}{Vb} + \frac{448\Upsilon_4^4}{V^2b^2} - \frac{5376d\Upsilon_4^2}{V^2b^4} - \frac{2304\Upsilon_4^2\alpha_1}{V^2b^2} + \frac{448f}{b^3} + \frac{24V^2\left(f-\frac{d^2}{4b}\right)}{b} + \frac{192fV}{b^2} - \frac{192d^2V}{b^3} - \frac{192eV}{b} + \frac{240d\Upsilon_4^2}{b^2} + 48\alpha_1\Upsilon_4^2 + \frac{384d\Upsilon_4^2}{Vb^3} - \frac{128\Upsilon_4^2\alpha_1}{Vb} - \frac{880d^2}{b^4} - \frac{1024e}{b^2} - \frac{16V^2d^3}{b^4\Upsilon_4^2} - \frac{64edV^2}{b^2\Upsilon_4^2} - 256\alpha_1^2 + \frac{128V^2g}{b\Upsilon_4^2} - \frac{64V^2f\alpha_1}{b\Upsilon_4^2} + \frac{16V^2d^2\alpha_1}{b^2\Upsilon_4^2} = 0 \tag{134}$$

We use the shown value of $\frac{\Upsilon_4}{\alpha_3}$ in (eq.130) and we replace α_1 and α_2 with their shown expressions in (eq.131) and (eq.133), in order to pass from equation (eq.134) to polynomial expression $\beta_3(\Upsilon_4^2)^4 + \beta_2(\Upsilon_4^2)^3 + \beta_1(\Upsilon_4^2)^2 + \beta_0(\Upsilon_4^2) = 0$ where coefficients are as presented in (eq.121), (eq.122), (eq.123) and (eq.124).

Relying on the proof of Theorem 2, we calculate only the expression $\Upsilon_{4,1}$ shown in (eq.135) as a root for expressed equation in (eq.120), and then we determine the roots of quartic equation $y^4 + My^2 + Ny + O = 0$. Therefore, we start by calculating the values of α_1, α_4 and α_2 by replacing the variable Υ_4 with the value of $\Upsilon_{4,1}$, then we calculate the values of M, N and O, and finally we finish by using Theorem 1.

We suppose that $b^{;} = \frac{\beta_2}{\beta_3}$, $c^{;} = \frac{\beta_1}{\beta_3}$ and $d^{;} = \frac{\beta_0}{\beta_3}$, whereas using the expressions (eq.121), (eq.122), (eq.123) and (eq.124). We suppose also that $D^{;} = 27d^{;} + 2b^{;3} - 9c^{;}b^{;}$ and $C^{;} = 9c^{;} - 3b^{;2}$. The solution $\Upsilon_{4,1}$ for shown equation in (eq.120) is as follow:

$$\Upsilon_{4,1}^2 = \frac{-b^{;}}{3} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2} + \sqrt{\left(\frac{D^{;}}{2}\right)^2 + \left(\frac{C^{;}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2} - \sqrt{\left(\frac{D^{;}}{2}\right)^2 + \left(\frac{C^{;}}{3}\right)^3}} \tag{135}$$

As we mentioned in the proof of Theorem 2, the use of other roots of polynomial (eq.120) in the quartic polynomial $y^4 + My^2 + Ny + O = 0$ to calculate the values of M, N and O generates redundancies of roots for the sixth degree polynomial equation $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0$.

We determine the group $K^{;}_{\{\Upsilon_{4,1}\}}$, which contains the four roots for quartic equation $y^4 + My^2 + Ny + O = 0$, by using Theorem 1.

$$K^{;}_{\{\Upsilon_{4,1}\}} = \left\{S_{(\Upsilon_{4,1},1)}, S_{(\Upsilon_{4,1},2)}, S_{(\Upsilon_{4,1},3)}, S_{(\Upsilon_{4,1},4)}\right\}$$

We present the group of roots for quartic equation shown in (eq.114) as $M^{;}_{\{\Upsilon_{4,1}\}}$, which is determined by relying on the group $K^{;}_{\{\Upsilon_{4,1}\}}$.

$$M^{;}_{\{\Upsilon_{4,1}\}} = \left\{-\frac{1}{4}\left[\frac{4\Upsilon_{4,1}}{V} + \Upsilon_{4,1}\right] + \frac{1}{4}S_{(\Upsilon_{4,1},1)}, -\frac{1}{4}\left[\frac{4\Upsilon_{4,1}}{V} + \Upsilon_{4,1}\right] + \frac{1}{4}S_{(\Upsilon_{4,1},2)}, -\frac{1}{4}\left[\frac{4\Upsilon_{4,1}}{V} + \Upsilon_{4,1}\right] + \frac{1}{4}S_{(\Upsilon_{4,1},3)}, -\frac{1}{4}\left[\frac{4\Upsilon_{4,1}}{V} + \Upsilon_{4,1}\right] + \frac{1}{4}S_{(\Upsilon_{4,1},4)}\right\}$$

We present each root for quartic equation shown in (eq.114) as $\xi_{(\Upsilon_{4,1},i)} = -\frac{1}{4}\left[\frac{4\Upsilon_{4,1}}{V} + \Upsilon_{4,1}\right] + \frac{1}{4}S_{(\Upsilon_{4,1},i)}$, whereas $S_{(\Upsilon_{4,1},i)}$ is from the group $K'_{\{\Upsilon_{4,1}\}}$.

The proposed six solutions for sixth degree polynomial equation $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0$ are as expressed in (eq.136), (eq.137), (eq.138), (eq.139), (140) and (eq.141). The expressions $\xi_{(\Upsilon_{4,1},1)}$, $\xi_{(\Upsilon_{4,1},2)}$, $\xi_{(\Upsilon_{4,1},3)}$ and $\xi_{(\Upsilon_{4,1},4)}$ present the calculated roots for quartic equation (eq.114) by using Theorem 1. $\Upsilon_{4,1}$ is calculated by using the shown expression in (eq.135). We use the expression (eq.131) to calculate the value of $\alpha_{(1,\Upsilon_{3,1})}$; thereby, its value is as follow:

$$\alpha_{(1,\Upsilon_{3,1})} = \frac{\Upsilon_{4,1}^4 + \frac{32\Upsilon_{4,1}^4}{V^2b^2} - \frac{8\Upsilon_{4,1}^4}{Vb} + \frac{12d\Upsilon_{4,1}^2}{b^2} - \frac{V^2\left(f-\frac{d^2}{4b}\right)}{b}}{4\Upsilon_{4,1}^2}$$

$$S_1 = \frac{1}{2}[\xi_{(\Upsilon_{4,1},1)}^2 - \alpha_{(1,\Upsilon_{4,1})}] \tag{136}$$

$$S_2 = \frac{1}{2}[\xi_{(\Upsilon_{4,1},2)}^2 - \alpha_{(1,\Upsilon_{4,1})}] \tag{137}$$

$$S_3 = \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},3)} \quad - \alpha_{(1,\Upsilon_{4,1})}] \tag{138}$$

$$S_4 = \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},4)} \quad - \alpha_{(1,\Upsilon_{4,1})}] \tag{139}$$

$$S_5 = -\frac{b+S_1+S_2+S_3+S_4}{2} - \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \tag{140}$$

$$S_6 = -\frac{b+S_1+S_2+S_3+S_4}{2} + \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \tag{141}$$

5. Conclusion

In the second section of this paper, we propose a list of engineered requirements and techniques to solve polynomial equations of nth degrees where we identify specific formulas and expressions that allow solving these polynomial equations in general forms, whereas we present a structured logic of analysis and calculation to be followed during the process of equations solving.

In the first presented theorem in this paper, we propose four formulary solutions for any quartic polynomial equation in general form, which enabled us to develop the formulary structures of six roots for any sixth-degree polynomial equation in general forms.

The proposed expressions as solutions in the first theorem of this paper are enabling to calculate all the four roots of any quartic polynomial equation nearly simultaneously, whereas the proposed solutions in the second theorem and the third theorem are enabling to calculate the six roots of sixth degree polynomial equations in general forms nearly in parallel.

The third proposed theorem is based on the same logic and calculations of Theorem 2 whereas proposing six roots for sixth degree polynomials. However, it is distinguished by treating the specific form $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0 \ with \ A \neq \ 0$ where the fifth-degree part is absent, which is essential to reduce sixth degree polynomial equation to quartic equation. The third theorem is also distinguished by eliminating the fourth-degree part of concerned sixth degree polynomial, in order to reduce the amount of calculations.

The essential criteria of presented theorems in this innovative paper is proposing new radical solutions for quartic equations and sixth degree polynomial equations to enable the calculation of all prospect roots of these equations nearly in parallel, whereas using the radical expressions of cubic roots and quadratic roots as subparts of each proposed solution.

Furthermore, this paper is presenting a specific engineering methodology to solve nth degree polynomial equations in general forms by relying on a structured logic of analysis and calculation according to a list of engineered requirements and techniques which are built on precis formulas and expressions that allow identifying the values of all roots nearly in parallel.

The presented engineering methodology in this paper is niched for further extension by using it to solve higher degrees of polynomial equations. In addition, we are preparing to scale this methodology on the area of numbers theory by projecting it on Collatz conjecture, in order to architect odd numbers according to a distributed structure of terms that may reveal further insights on prime numbers, which will be published in further articles.

Author Contributions

Yassine Larbaoui is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The author declares no conflicts of interest.

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