American Journal of Applied Mathematics
2026; 14(4): 210-219
http://www.sciencepublishinggroup.com/j/ajam
doi: 10.11648/j.ajam.20261404.14

Laplacian Minimum Domination Energy of Some Derived Graphs

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To cite this article:
Jagadeesh Rajanna, Ashwini Ankanahalli Shashidhara. (2026). Laplacian Minimum Domination Energy of Some Derived Graphs. *American Journal of Applied Mathematics, 14*(4), 210-219. https://doi.org/10.11648/j.ajam.20261404.14

Received: 26 March 2026; **Accepted:** 3 June 2026 **Published:** 17 July 2026

Abstract: Graph energy is an important concept in spectral graph theory with applications in mathematics and chemistry. In this paper, we study the Laplacian minimum domination energy of derived graphs of some standard graphs. The main aim is to obtain formulas, properties, and bounds for this energy measure. The study considers derived graphs of star graphs, complete bipartite graphs, friendship graphs, and healthy spider graphs. Using minimum dominating sets, minimum domination adjacency matrices, and Laplacian minimum domination matrices, the eigenvalues of these derived graphs are determined. Based on these eigenvalues, explicit formulas for the Laplacian minimum domination energy are obtained. Further, some basic properties related to eigenvalues are established. Upper and lower bounds for the Laplacian minimum domination energy are also derived using matrix methods and classical inequalities such as the Cauchy-Schwarz inequality. The results extend existing work on graph energy by combining domination concepts, Laplacian matrices, and derived graphs. The formulas, properties, and bounds obtained in this paper provide a better understanding of the spectral behavior of derived graphs and may be useful for further research in graph theory and its applications.

Keywords: Laplacian Minimum Domination Derived Matrix, Laplacian Minimum Domination Derived Eigenvalues, Laplacian Minimum Domination Energy

1. Introduction

Gutman introduced the concept of graph energy in 1978 [7]. Let G be a simple and connected graph with n vertices and m edges and let $A = (a_{ij})$ be the real symmetric adjacency square matrix of order $n \times n$ of the graph G with real eigen values $\lambda_1, \lambda_2, ..., \lambda_n$ and their sum is equal to zero. The energy of a graph G is given by;

$$\xi(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Theories on the mathematical concepts of graph energy can be seen in [2–6, 11, 16] and the references cited there in. For various upper and lower bounds for energy of a graph can be found in papers [4, 6] and it was observed that graph energy has chemical applications in the molecular orbital theory of conjugated molecules [8].

The Laplacian energy was introduced by Zhou and Gutman in 2006 [9]. For G, let n and m denotes the cardinality of vertices set and edges set. Then, Laplacian matrix $L = L_{ij}$ is defined as follows:

$$(L_{ij}) = \begin{cases} -1 \text{ if, } v_i \, and \, v_j \, are \, adjacent. \\ 0 \text{ if,} v_i \, and \, v_j \, are \, not \, adjacent. \\ d_i \text{ if } v_i = v_j. \end{cases}$$

Where d_i is the number of edges incident to each v_i. Consider $\beta_1, \beta_2, \beta_3, ..., \beta_n$ as the characteristic roots of matrix L and G. Then, Laplacian energy $LE(G)$ is given by;

$$LE(G) = \sum_{i=1}^{n} |\beta_i - \frac{2m}{n}|$$

Here q is the number of edges, p be the vertices and $\frac{2m}{n}$ is the average degree of G.

1.1. Derived Graph

In this paper, we consider simple graphs, which are the graphs without directed, multiple,or weighted edges, and without self loops. Let G be such a graph and let its vertex set be $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set be $E(G) = \{e_1, e_2, ..., e_m\}$. The distance between the vertices v_i and v_j is equal to length of a shortest path between v_i and v_j.

Definition 1.1. Let the derived graph is also a simple graph denoted by G_D, is the graph having the same vertex set as G, in which two vertices are adjacent if and only if their distance in G is 2 [17].

Definition 1.2. The adjacency distance matrix of derived graph G_D of order $n \times n$ is defined by, $A(G_D) = (d_{ij})$, where

$$(d_{ij}) = \begin{cases} 1 \text{ if, } d(i,j) = 2, \\ 0 \text{ if Otherwise.} \end{cases}$$

where $d(i, j)$ is distance between i and j.

The characteristic polynomial of $A(G_D)$ is defined as; $f_n(G_D, \lambda) = det(A(G_D) - \alpha I)$.
The characteristic equation is; $det(A(G_D) - \alpha I) = 0$.
Let $\alpha_1, \alpha_2, ..., \alpha_n$ are the derived eigenvalues of $A(G_D)$.
The energy of derived graph G_D is defined as;

$$\xi(G_D) = \sum_{i=1}^{n} |\alpha_i|.$$

1.2. Minimum Domination Energy of Derived Graph

Let G be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and size m with edge set $E(G) = \{e_1, e_2, ..., e_m\}$. A subset D of $V(G)$ is called a dominating set of G if every vertex of $V(G) - D$ is adjacent to some vertex in D. Any dominating set with minimum cardinality is called a minimum dominating set. Let D be a minimum dominating set of a graph G.

Definition 1.3. The minimum domination adjacency matrix of derived graph G_D of order $n \times n$ is defined by;
$A_{MD}(G_D) = (d_{ij})$, where

$$(d_{ij}) = \begin{cases} 1 \text{ if, } v_i \in D, \\ 1 \text{ if, } d(i,j) = 2, \\ 0 \text{ if Otherewise.} \end{cases}$$

where D is minimum dominating set of G and d(i,j) is distance between i and j.

The characteristic polynomial of $A_{MD}(G_D)$ is defined as; $f_n(G_D, \lambda) = det(A_{MD}(G_D) - \lambda I)$.

The characteristic equation is; $det(A_{MD}(G_D) - \lambda I) = 0$.

Let $\lambda_1, \lambda_2, ..., \lambda_n$ are the minimum dominating derived eigenvalues of $A_{MD}(G_D)$.

Then minimum domination energy of derived graph G_D is defined as;

$$\xi_{MD}(G_D) = \sum_{i=1}^{n} |\lambda_i|.$$

1.3. Laplacian Minimum Dominating Energy of Graph

If $D(G)$ is the diagonal matrix of vertex degree of G with respect to minimum dominating set and $A_k(G)$ is the minimum domination adjacency matrix. Then $L_k(G) = D(G) - A_k(G)$. Consider $\beta_1, \beta_2, \beta_3, ..., \beta_n$ to be the eigenvalues of $L_k(G)$. Then the laplacian minimum dominating energy of graph G is;

$$LE_k(G) = \sum_{i=1}^{n} |\beta_i - \frac{2m}{n}|$$

. Here m is the number of edges, n be the vertices and $\frac{2m}{n}$ is the average degree of G.

1.4. Laplacian Minimum Dominating Energy of Derived Graph

If $D(G_D)$ is the diagonal matrix of vertex degree of G with respect to minimum dominating set and $A_{MD}(G_D)$ is the minimum domination adjacency matrix of derived graph. Then $LMD_k(G_D) = D(G) - A_{MD}(G_D)$. Consider $\beta_1, \beta_2, \beta_3, ..., \beta_n$ to be the eigenvalues of $LMD_k(G_D)$. Then the laplacian minimum dominating energy of graph G_D is;

$$LMDE_k(G_D) = \sum_{i=1}^{n} |\beta_i - \frac{2m}{n}|$$

Here m is the number of edges, n be the vertices and $\frac{2m}{n}$ is the average degree of G_D.

2. Laplacian Minimum Domination Energy of Some Standard Derived Graphs

Definition 2.1. A graph G with $\{v_1, v_2, v_3, ..., v_n\}$ as a set of vertices in which $v_2, v_3, ..., v_n$ are connected to a common vertex v_1 which has a degree $n - 1$, denoted by $K_{1,n}$ [10].

Theorem 2.1. The laplacian minimum domination energy of derived star graph, $K_{D(1,n)}$ is $\frac{1}{n+1}[4n^2 - 5n - 1]$.

Proof. Consider $K_{D(1,n)}$ is derived star graph with $\{v_1, v_2, v_3, ..., v_n\}$ is the vertex set and MDS of $K_{D(1,n)}$ is $\{u_1, u_2, u_3, ..., u_r\}$. Then

$$A_{MD}[K_{D(1,n)}] = \begin{bmatrix} 1 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & \dots & \dots & 1 & 1 & 1 \\ 0 & 1 & 0 & \dots & \dots & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & \dots & \dots & \dots & 0 & 1 & 1 \\ 0 & 1 & 1 & \dots & \dots & \dots & 1 & 0 & 1 \\ 0 & 1 & 1 & \dots & \dots & \dots & 1 & 1 & 0 \end{bmatrix}_{(n+1)\times(n+1)}$$

$$D[K_{D(1,n)}] = \begin{bmatrix} n & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & \dots & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 1 \end{bmatrix}_{(n+1)\times(n+1)}$$

$$LMD[K_{D(1,n)}] = D[K_{D(1,n)}] - A_{MD}[K_{D(1,n)}]$$

$$= \begin{bmatrix} n & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & \dots & \dots & \dots & 1 & 1 & 1 \\ 0 & 1 & 1 & \dots & \dots & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & \dots & \dots & \dots & 1 & 1 & 1 \\ 0 & 1 & 1 & \dots & \dots & \dots & 1 & 1 & 1 \\ 0 & 1 & 1 & \dots & \dots & \dots & 1 & 1 & 1 \end{bmatrix}_{(n+1)\times(n+1)}$$

The Characteristic polynomial is: $[\beta - (n-1)][\beta - n][\beta]^{n-1}$.
The Characteristic equation is: $[\beta - (n-1)][\beta - n][\beta]^{n-1} = 0$.
Then,

$$Spec_{MD}[K_{D(1,n)}] = \begin{pmatrix} n-1 & n & 0 \\ 1 & 1 & n-1 \end{pmatrix}.$$

In this graph, number of vertices $= n+1$ and number of edges $= n(= m)$.
Average degree $= \frac{2n}{n+1}$.
Hence the laplacian minimum domination energy of $K_{D(1,n)}$ is;

$$\begin{aligned}
LMDE[K_{D(1,n)}] &= |(n-1) - \frac{2n}{n+1}|.1 + |n - \frac{2n}{n+1}|.1 \\
&+ |0 - \frac{2n}{n+1}|.(n-1) \\
&= |\frac{n^2-1-2n}{n+1}| + |\frac{n^2+n-2n}{n+1}| \\
&+ |\frac{2n(n-1)}{n+1}| \\
&= \frac{1}{n+1}[4n^2 - 5n - 1].
\end{aligned}$$

Definition 2.2. A complete bipartite graph $K_{m,n}$ is a graph in which each vertex in set A is adjacent to each vertex in set B, with $|A| = m$ & $|B| = n$ [10].

Theorem 2.2. The laplacian minimum domination energy of derived complete bipartite graph, $K_{D(m,n)}$ is $\frac{1}{m+n}[m^3 + n^3 - n^2 - 3mn(3 + m + n)]$.

Proof. Consider $K_{D(m,n)}$ is derived star graph with $\{v_1, v_2, v_3, ..., v_n\}$ is the vertex set and MDS of $K_{D(m,n)}$ is $\{u_1, u_2, v_3, ..., u_r\}$. Then

$$A_{MD}[K_{D(m,n)}] = \begin{bmatrix}
1 & 1 & 1 & \dots & \dots & \dots & 0 & 0 & 0 \\
1 & 1 & 1 & \dots & \dots & \dots & 0 & 0 & 0 \\
1 & 1 & 1 & \dots & \dots & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & \dots & \dots & 0 & 1 & 1 \\
0 & 0 & 0 & \dots & \dots & \dots & 1 & 0 & 1 \\
0 & 0 & 0 & \dots & \dots & \dots & 1 & 1 & 0
\end{bmatrix}_{(m+n)\times(m+n)}$$

$$D[K_{D(m,n)}] = \begin{bmatrix}
n & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\
0 & n & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\
0 & 0 & n & \dots & \dots & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & \dots & \dots & m & 0 & 0 \\
0 & 0 & 0 & \dots & \dots & \dots & 0 & m & 0 \\
0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & m
\end{bmatrix}_{(m+n)\times(m+n)}$$

$$LMD[K_{D(m,n)}] = D[K_{D(m,n)}] - A_{MD}[K_{D(m,n)}]$$

$$= \begin{bmatrix}
n-1 & 1 & 1 & \dots & \dots & \dots & 0 & 0 & 0 \\
1 & n-1 & 1 & \dots & \dots & \dots & 0 & 0 & 0 \\
1 & 1 & n-1 & \dots & \dots & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & \dots & \dots & m & 1 & 1 \\
0 & 0 & 0 & \dots & \dots & \dots & 1 & m & 1 \\
0 & 0 & 0 & \dots & \dots & \dots & 1 & 1 & m
\end{bmatrix}_{(m+n)\times(m+n)}$$

The Characteristic polynomials is: $[\beta-(m+n-1)][\beta-(m+n-2)][\beta-(m-1)]^{m-1}[\beta-(n-2)]^{n-1}$.
The Characteristic polynomials is: $[\beta-(m+n-1)][\beta-(m+n-2)][\beta-(m-1)]^{m-1}[\beta-(n-2)]^{n-1}=0$.
Then,

$$Spec_{MD}[K_{D(m,n)}]=\begin{pmatrix} m+n-1 & m+n-2 & m-1 & n-2 \\ 1 & 1 & m-1 & n-1 \end{pmatrix}.$$

Number of vertices $= m+n$ and number of edges $= mn$.
Average degree $= \frac{2mn}{m+n}$.
Hence the laplacian minimum domination energy of $K_{D(m,n)}$ is;

$$\begin{aligned} LMDE[K_{D(m,n)}] &= |(m+n-1)-\frac{2mn}{m+n}|.1 \\ &+ |m+n-2-\frac{2mn}{m+n}|.1 \\ &+ |m-1-\frac{2mn}{m+n}|.(m-1) \\ &+ |n-2-\frac{2mn}{m+n}|.(n-1) \\ &= \frac{|m^3+n^3-3mn-m^2n-n^2m-n^2|}{m+n} \\ &= \frac{[m^3+n^3-n^2-3mn(3+m+n)]}{m+n}. \end{aligned}$$

Definition 2.3. A graph in which any two distinct vertices are adjacent is called complete graph, K_n [10].

Theorem 2.3. Let K_{Dn} be the minimum domination derived graph of complete graph. Then $LMDE_{MD}[K_{Dn}]=1$.

Definition 2.4. A friendship graph F_n is a graph in which n number of triangles are connected in common central vertex [13].

Theorem 2.4. The laplacian minimum domination energy derived friendship graph F_{Dn} is $\frac{2(3n^2-2n+2)}{2n+1}$.

Proof. Consider F_{Dn} is derived star graph with $\{v_1,v_2,v_3,...,v_n\}$ is the vertex set and MDS of F_{Dn} is $\{u_1,u_2,v_3,...,u_r\}$. Then

$A_{MD}[F_{Dn}]=$

$$\begin{bmatrix} 1 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 1 & 1 & 1 \\ 0 & 0 & 0 & \dots & \dots & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 & 1 & 1 \\ 0 & 0 & 0 & \dots & \dots & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 1 & 0 & 0 \end{bmatrix}_{(2n+1)\times(2n+1)}$$

$$D[F_{Dn}]=\begin{bmatrix} 2n & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & 2 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & 0 & 2 & \dots & \dots & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & \dots & 2 & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 & 2 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 2 \end{bmatrix}_{(2n+1)\times(2n+1)}$$

$LMD[F_{Dn}] = D[F_{Dn}] - A_{MD}[F_{Dn}]$

$$= \begin{bmatrix} 2n-1 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & 2 & 0 & \dots & \dots & \dots & 1 & 1 & 1 \\ 0 & 0 & 2 & \dots & \dots & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & \dots & 2 & 1 & 1 \\ 0 & 0 & 0 & \dots & \dots & \dots & 1 & 2 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 1 & 0 & 2 \end{bmatrix}_{(2n+1)\times(2n+1)}$$

The Characteristic polynomial is: $\beta^{n-1}[\beta - 2n][\beta - (2n-1)][\beta - 2]^n$.
The Characteristic equation is: $\beta^{n-1}[\beta - 2n][\beta - (n2-1)][\beta - 2]^n = 0$.
Then,

$$Spec_{MD}[F_{Dn}] = \begin{pmatrix} 0 & 2n & 2n-1 & 2 \\ n-1 & 1 & 1 & n \end{pmatrix}.$$

Number of vertices $= 2n+1$ and number of edges $= 3n$.
Average degree $= \frac{6n}{2n+1}$.
Hence the laplacian minimum domination energy of F_{Dn} is;

$$\begin{aligned} LMDE[F_{Dn}] &= |0 - \frac{6n}{2n+1}|.(n-1) + |2n - \frac{6n}{2n+1}|.1 \\ &+ |2n - 1 - \frac{6n}{2n+1}|.1 + (2 - \frac{6n}{2n+1}).n \\ &= \frac{1}{2n+1}|6n^2 - 4n + 4| \\ &= \frac{1}{2n+1}2[3n^2 - 2n + 2]. \end{aligned}$$

Definition 2.5. Let G be a Healthy spider graph with $2n+1$ vertices and $2n$ edges,denoted by Hs_n [1].

Theorem 2.5. Let Hs_{Dn} be the derived graph of healthy spider graph. Then the laplacian minimum domination energy of Hs_{Dn} is $\frac{n}{2n+1}(2n+5)$.

Proof. Consider Hs_{Dn} is derived star graph with $\{v_1, v_2, v_3, ..., v_n\}$ is the vertex set and MDS of Hs_{Dn} is $\{u_1, u_2, v_3, ..., u_r\}$. Then

$$A_{MD}[Hs_{Dn}] = \begin{bmatrix} 0 & 0 & 0 & \dots & \dots & \dots & 1 & 1 & 1 \\ 0 & 1 & 1 & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & \dots & \dots & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \end{bmatrix}_{(2n+1)\times(2n+1)}$$

$$D[Hs_{Dn}] = \begin{bmatrix} n & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & 2 & 0 & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & 0 & 2 & \dots & \dots & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 1 \end{bmatrix}_{(2n+1)\times(2n+1)}$$

$LMD[Hs_{Dn}] = Hs_{Dn} - A_{MD}[Hs_{Dn}]$

$$= \begin{bmatrix} n & 0 & 0 & \dots & \dots & \dots & 1 & 1 & 1 \\ 0 & 1 & 1 & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & \dots & \dots & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & \dots & \dots & 1 & 0 & 0 \\ 1 & 0 & 0 & \dots & \dots & \dots & 0 & 1 & 0 \\ 1 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & 1 \end{bmatrix}_{(2n+1)\times(2n+1)}$$

The Characteristic polynomial is: $\beta^n[\beta - (n+1)][\beta - n][\beta - 1]^{(n-1)}$.
The Characteristic equation is: $\beta^n[\beta - (n+1)][\beta - n][\beta - 1]^{(n-1)} = 0$.
Then,

$$Spec_{MD}[Hs_{Dn}] = \begin{pmatrix} 0 & n+1 & n & 1 \\ n & 1 & 1 & n-1 \end{pmatrix}.$$

Number of vertices $= 2n+1$ and number of edges $= 2n$.
Average degree $= \frac{4n}{2n+1}$.
Hence the laplacian minimum domination energy of F_{Dn} is;

$$\begin{aligned} LMDE[F_{Dn}] &= |0 - \frac{4n}{2n+1}|.(n-1) + |n+1 - \frac{4n}{2n+1}|.1 \\ &+ |n - \frac{4n}{2n+1}|.1 + (1 - \frac{4n}{2n+1}).n \\ &= \frac{1}{2n+1}|2n^2 + 5n| \\ &= \frac{1}{2n+1}[2n^2 + 5n]. \end{aligned}$$

3. Properties of Minimum Domination Derived Graphs Energy

Theorem 3.1. For a given derived graph G_D vertex set, $V(G_D) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G_D) = \{e_1, e_2, ..., e_m\}$ along with minimal dominating set $K = \{u_1, u_2, ..., u_r\}$. If $\beta_1, \beta_2, ..., \beta_n$ are the eigenvalues of $LMD(G_D)$. Then

1) $\sum_{i=1}^{n} \beta_i = 2|E| - |K|$.

2) $\sum_{i=1}^{n} \beta_i^2 = 2M$,

where $M = \frac{1}{2}\sum_{i=1}^{n}(d_i - k_i)^2 + (n-1)$.

Proof. 1) We know that, the sum of eigenvalues of $LMD(G_D)$ is trace of $LMD(G_D)$. Then
$\therefore$

$$\begin{aligned} \sum_{i=1}^{n} \beta_i &= \sum_{i=1}^{n} L_{ii} \\ &= \sum_{i=1}^{n} d_i - |K| \\ &= 2|E| - |K|. \end{aligned}$$

2) We have,

$$\begin{aligned}\sum_{i=1}^{n}\beta_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}L_{ij}L_{ji}\\ &= \sum_{i=1}^{n}L_{ii}^2+\sum_{i\neq j}L_{ij}^2\\ &= \sum_{i=1}^{n}(d_i-k_i)^2+2\sum_{i\leq j}^{n}(-1)^2\\ &= \sum_{i=1}^{n}(d_i-k_i)^2+2(n-1).\end{aligned}$$

Where $M=\frac{1}{2}\sum_{i=1}^{n}(d_i-k_i)^2+(n-1)$ and $k_i=\begin{cases}1, & v\in D\ ;\\ 0, & v\notin D.\end{cases}$.

Theorem 3.2. Consider a derived graph G_D along with minimum dominating set K. if the addition of modulus of all values of $LMD(G_D)$ eigenvalues is a non-irrational number. Then $LMDE(G_D)\equiv|K|(mod\,2)$.

Proof. Let $\beta_1,\beta_2,...,\beta_n$ are the eigenvalues of $LMD(G_D)$ in which $\beta_1,\beta_2,...,\beta_n$ are positive and the remaining are non-positive, then

$$\sum_{i=1}^{n}|\beta_i|=(\beta_1,\beta_2,...,\beta_r)-(\beta_{r+1},\beta_{r+2},...,\beta_n)$$
$$\sum_{i=1}^{p}|\beta_i|=2(\beta_1,\beta_2,...,\beta_r)-(\beta_1,\beta_2,...,\beta_p)$$
$$LMDE(G_D)=2(\beta_1,\beta_2,...,\beta_r)-\sum_{i=1}^{p}\beta_i$$
$$LMDE(G_D)=2(\beta_1,\beta_2,...,\beta_r)-|K|$$
$$\therefore\ LMDE(G_D)\equiv|K|(mod\,2)$$

4. Bounds for Laplacian Minimum Domination Energy Derived Graphs

Theorem 4.1. Let G_D be a derived graph of order n and size m. If K is it's MDS of a derived graph, then

$$LMDE(G_D)\leq\sqrt{2Mn}+2m.$$

Proof. We have Cauchy Schwarz inequality, which is;

$$(\sum_{i=1}^{n}a_ib_i)^2\leq(\sum_{i=1}^{n}a_i^2)(\sum_{i=1}^{n}b_i^2)$$

Let $a_i=1$ and $b_i=|\beta_i|$

Hence,

$$(\sum_{i=1}^{n}1|\beta_i|)^2\leq\sum_{i=1}^{n}1^2.\sum_{i=1}^{n}|\beta_i|^2$$
$$\sum_{i=1}^{n}|\beta_i|^2\leq n\sum_{i=1}^{n}|\beta_i|^2$$
$$\sum_{i=1}^{n}|\beta_i|\leq\sqrt{2Mn}.$$

By Triangular inequality,

$$|\beta_i-\frac{2m}{n}|\leq|\beta_i|+|\frac{2m}{n}|$$
$$\sum_{i=1}^{n}|\beta_i-\frac{2m}{n}|\leq\sum_{i=1}^{n}|\beta_i|+\sum_{i=1}^{n}|\frac{2m}{n}|$$
$$LMDE(G_D)\leq\sqrt{2Mn}+2m.$$

Theorem 4.2. If β_1 is the largest eigenvalue of $A=LMD(G_D)$, then $\beta_1\geq\frac{2m-|K|-D(G_D)}{n}$.

Proof. Let X be any non-zero vector. Then by [16], We have $\beta_1(A)=\max_{X\neq0}\{\frac{X'AX}{X'X}\}$
$\therefore\ \beta_1\geq\frac{J'AJ}{J'J}=\frac{2m-|K|-D(G_D)}{n}$, where J is a unit matrix $[1,1,1,...,1]$.

Similar to Koolen and Moulton's [11] upper bound for energy of a graph, upper bound for $LMD(G_D)$ is given in the following theorem.

Theorem 4.3. If G_D is a derived graph with n vertices and m edges and $|K|+2(n-1)\geq m$, then $LMDE(G_D)\leq\frac{2M}{p}+\sqrt{(n-1)[2m-(\frac{2M}{n})^2]}$.

Proof. We have Cauchy-Schwarz inequality;

$$(\sum_{i=2}^{n}a_ib_i)^2\leq(\sum_{i=21}^{n}a_i^2)(\sum_{i=2}^{n}b_i^2)$$

Let $a_i=1$ and $b_i=|\beta_i|$

$$(\sum_{i=2}^{n}|\beta_i|)^2\leq\sum_{i=2}^{n}1^2\sum_{i=1}^{n}|\beta_i|^2$$
$$[LMDE(G_D)-\beta_1]^2\leq(n-1)[2M-\beta_1^2]$$
$$LMDE(G_D)\leq\beta_1+\sqrt{(p-1)[2M-\beta_1^2]}.$$

Let $f(x)=x+\sqrt{(n-1)[2M-x^2]}$ For decreasing function $f'(x)\leq0$
$\Rightarrow 1-\frac{(n-1)x}{\sqrt{(n-1)[2M-x^2]}}\leq0$
$\Rightarrow x\geq\sqrt{\frac{2M}{n}}$

Since $2M\geq m$, we have $\sqrt{\frac{2M}{n}}\leq\frac{2M}{n}\leq\beta_1$
$\therefore f(\beta_1)\leq f(\frac{2M}{n})$

That is, $LMDE(G_D) \leq f(\beta_1) \leq f(\frac{2M}{n})$
$\Rightarrow LMDEG_D \leq f(\frac{2M}{n})$
$\Rightarrow LMDE(G_D) \leq \frac{2M}{n} + \sqrt{(n-1)[2M - (\frac{2M}{n})^2]}$.

Theorem 4.4. For any derived graph G_D with n, m are cardinality of vertices and edges and K is cardinality of MDS. Then,

$$LMDE(G_D) \leq \sqrt{2Mn - 4m^2 + 4mK}.$$

Proof. We have Cauchy-Schwarz inequality;

$$(\sum_{i=2}^{n} a_i b_i)^2 \leq (\sum_{i=2}^{n} a_i^2)(\sum_{i=2}^{n} b_i^2)$$

Let $a_i = 1$ and $b_i = |\beta_i - \frac{2m}{n}|$

$$\begin{aligned}(\sum_{i=1}^{n}|\beta_i - \frac{2m}{n}|)^2 &\leq \sum_{i=2}^{n} 1^2 \sum_{i=1}^{n}(|\beta_i - \frac{2m}{n}|)^2\\ &\leq p[\sum_{i=2}^{n}\beta_i^2 + \sum_{i=2}^{n}\frac{4m^2}{n^2} - \frac{4q}{n}\sum_{i=2}^{n}\beta_i]\\ &\leq n[2M + \frac{4m^2}{n^2}n - \frac{4m}{n}(2m-K)]\\ (\sum_{i=1}^{n}|\beta_i - \frac{2m}{n}|)^2 &\leq 2Mn - 4m^2 + 4mK\\ LMDE(G_D) &\leq \sqrt{2Mn - 4m^2 + 4mK}.\end{aligned}$$

Theorem 4.5. For any derived graph G_D with n, m are cardinality of vertices and edges and $D = |det[LMD(G_D)]|$. Then,

$$LMDE(G_D) \geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}} - 2m.$$

Proof. Consider,

$$\begin{aligned}|\sum_{i=1}^{n}|\beta_i||^2 &= (\sum_{i=1}^{n}|\beta_i|)(\sum_{j=1}^{n}|\beta_j|)\\ &= (\sum_{i=1}^{n}|\beta_i|^2) + \sum_{i\neq j}^{.}|\beta_i||\beta_j|.\end{aligned}$$

Therefore,

$$\sum_{i\neq j}^{.}|\beta_i||\beta_j| = (\sum_{i=1}^{n}|\beta_i|)^2 - \sum_{i=1}^{n}|\beta_i|^2$$

Using the relation between arithmetic mean and geometric mean for $(n^2 - n)$ terms, we have

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\beta_i||\beta_j| \geq (\prod_{i\neq j}|\beta_i||\beta_j|)^{\frac{1}{n(n-1)}}$$

$$\begin{aligned}\frac{1}{n(n-1)}\sum_{i\neq j}|\beta_i||\beta_j| &\geq (\prod_{i\neq j}^{n}|\beta_i|^{2(n-1)})^{\frac{1}{n(n-1)}}\\ \sum_{i\neq j}|\beta_i||\beta_j| &\geq n(n-1)\prod_{i\neq j}^{n}(|\beta_i|^{2/n})\\ (\sum_{i=1}^{n}|\beta_i|)^2 - \sum_{i=1}^{n}|\beta_i|^2 &\geq n(n-1)\prod_{i\neq j}^{n}(|\beta_i|^{2/n})\\ (\sum_{i=1}^{n}|\beta_i|)^2 - 2m &\geq n(n-1)\prod_{i\neq j}^{n}(|\beta_i|^{2/n})\\ (\sum_{i=1}^{n}|\beta_i|)^2 &\geq 2M + n(n-1)\prod_{i\neq j}^{n}(|\beta_i|^{2/n})\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n}|\beta_i| \geq \sqrt{2M + n(n-1)\prod_{i\neq j}^{n}D^{2/n}}.$$

Now, we know that

$$\begin{aligned}|\beta_i| - |\frac{2m}{n}| &\leq |\beta_i - \frac{2m}{n}|\\ \sum_{i=1}^{n}|\beta_i| - \sum_{i=1}^{n}|\frac{2m}{n}| &\leq \sum_{i=1}^{n}|\beta_i - \frac{2m}{n}|\\ \sum_{i=1}^{n}|\beta_i| - 2m &\leq LMDE(G_D)\\ LMDE(G_D) &\geq \sum_{i=1}^{n}|\beta_i| - 2m\end{aligned}$$

Therefore

$$LMDE(G_D) \geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}} - 2m.$$

5. Conclusions

In this paper, we studied the Laplacian minimum domination energy of derived graphs. We obtained formulas for this energy for several standard graphs, including star graphs, complete bipartite graphs, friendship graphs, and healthy spider graphs. We also established important properties and derived upper and lower bounds. These results contribute to a better understanding of graph energy and may be useful for further research in graph theory.

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Abbreviations

LE	Laplacian Energy
LMDE	Laplacian Minimum Domination Energy
MDS	Minimum Dominating Set

Author Contributions

Jagadeesh Rajanna: Conceptualization, Writing – original draft.

Ashwini Ankanahalli Shashidhara: Methodology, Writing – original draft, Writing – review & editing

Conflicts of Interest

The authors declare no conflicts of interest.

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