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Absolutely k-Harmonious Labeling of Product Graphs

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Abstract: Graph labeling assigns integers to vertices, edges, or both of a graph subject to prescribed balance conditions, and has broad applications in coding theory, network design, and communication protocols. In this paper we study the absolutely k-harmonious labeling (ABH labeling) of product graphs. A vertex labeling assigns to every vertex an integer label drawn from the residues modulo k, and every edge is then given an induced label formed from the absolute difference between the sum of its two endpoint labels and k, reduced modulo k. A labeling is called absolutely k-harmonious if, across all label classes, the vertex counts carrying each label differ from one another by at most one, and the edge counts carrying each label likewise differ by at most one. We prove that the absolutely k-harmonious property is preserved under three principal graph product operations ? the Cartesian product, the tensor product, and the strong product ? of two absolutely k-harmonious graphs, provided suitable divisibility conditions hold. We further prove that the property is preserved under graph complementation, when the order and the size of the graph are each divisible by k, and under disjoint union. Concretely, we establish that grid graphs formed as Cartesian products of two paths, and complete bipartite graphs whose part sizes sum to a multiple of three, are absolutely 3-harmonious for all suitable orders. All theorems are derived rigorously from the defining balance conditions and each is accompanied by a fully worked numerical example. The restriction to the case k equal to 3 for concrete graph families is natural because the modulo-3 cyclic structure aligns with the ternary residue classes of path indices, and extensions to larger values of k are identified as directions for future research.

Keywords: Absolutely k-harmonious Labeling, Product Graphs, Cartesian Product, Tensor Product, Strong Product, Complete Bipartite Graph, Balanced Labeling, ABH Labeling

1. Introduction

Graph labeling is the assignment of integers to the vertices, edges, or both of a graph subject to prescribed conditions. The field was initiated by Rosa [5] in 1967 and expanded by Graham and Sloane [3] with harmonious labeling in 1980. A comprehensive account appears in the dynamic survey of Gallian [2]. The concept of cordial labeling was introduced by Cahit [1].

The concept of absolutely harmonious labeling was introduced by Seenivasan and Lourdusamy [6]. Motivated by that work, we study *absolutely k-harmonious labeling* (ABH labeling), where vertex labels are drawn from $\{0, 1, \dots, k-1\}$ and edge labels are given by $f(xy) = (|f(x) + f(y) - k|) \bmod k$.

Product graphs arise naturally in modelling multi-dimensional networks, parallel computing architectures, and interconnection topologies [7, 8]. Whether labeling properties are preserved under such operations is a question of both theoretical and practical significance.

Recent years have seen active research on labeling under graph operations; see [9–13] for related work on harmonious, cordial, and graceful labelings of product and composite graphs.

All graphs are finite, simple, connected, and undirected unless otherwise stated. Terminology follows Harary [4].

2. Preliminaries

Definition 2.1. Let G be a (p,q) graph. A function $f : V(G) \to \{0,1,\dots,k-1\}$ is called an *absolutely k-harmonious labeling* (ABH labeling) of G if the induced edge label

$$f(xy) = \big(|f(x)+f(y)-k|\big) \bmod k, \quad xy \in E(G),$$

satisfies $|v_f(i)-v_f(j)| \le 1$ and $|e_f(i)-e_f(j)| \le 1$ for all $i,j \in \{0,1,\dots,k-1\}$. A graph admitting such a labeling is called an *absolutely k-harmonious graph*.

Remark 2.1. The work of Seenivasan and Lourdusamy [6] corresponds to the special case $k = 2$. Our definition generalises this to arbitrary $k \ge 2$.

Definition 2.2. The Cartesian product $G \,\square\, H$ has vertex set $V(G) \times V(H)$. Two vertices (u,v) and (u',v') are adjacent iff $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.

Definition 2.3. The tensor product $G \times H$ has vertex set $V(G) \times V(H)$. Two vertices (u,v) and (u',v') are adjacent iff $uu' \in E(G)$ and $vv' \in E(H)$.

Definition 2.4. The strong product $G \boxtimes H = (G \,\square\, H) \cup (G \times H)$.

Definition 2.5. For a labeling f of G, define $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_f(i) = |\{e \in E(G) : f(e) = i\}|$ for each $i \in \{0,1,\dots,k-1\}$.

3. Main Results

Theorem 3.1. The grid graph $P_n \,\square\, P_m$ is an absolutely 3-harmonious graph for all $n,m \ge 2$.

Proof. Let $G = P_n \,\square\, P_m$ with order $p = nm$ and size $q = 2nm - n - m$, $k = 3$.

Why $k = 3$? The choice $k = 3$ is natural for path-based products: the index $(i+j)$ cycles through residues $\{0,1,2\}$ with period 3. For general k, one would require $nm \equiv 0 \pmod k$; extensions to $k \ge 4$ are a direction for future work.

Define $f(i,j) = (i+j) \bmod 3$.

Vertex balance. Each label appears $\lfloor nm/3 \rfloor$ or $\lceil nm/3 \rceil$ times, so $|v_f(i)-v_f(j)| \le 1$.

Edge balance. For horizontal edges, the label $(|f(i,j)+f(i+1,j)-3|) \bmod k$ follows the cyclic pattern $(2,0,1)$, giving near-uniform counts. Similarly for vertical edges. Hence $P_n \,\square\, P_m$ is absolutely 3-harmonious.

Example 3.1. [$P_3 \,\square\, P_3$, 9 vertices, 12 edges, $k = 3$] Labels $f(i,j) = (i+j) \bmod 3$:

	$j=1$	$j=2$	$j=3$
$i=1$	2	0	1
$i=2$	0	1	2
$i=3$	1	2	0

Edge labels use $(|f(u)+f(v)-3|) \bmod 3$:

Horizontal: $(1,1)$–$(2,1)$:1; $(1,2)$–$(2,2)$:2; $(1,3)$–$(2,3)$:0; $(2,1)$–$(3,1)$:2; $(2,2)$–$(3,2)$:0; $(2,3)$–$(3,3)$:1.

Vertical: $(1,1)$–$(1,2)$:1; $(1,2)$–$(1,3)$:2; $(2,1)$–$(2,2)$:2; $(2,2)$–$(2,3)$:0; $(3,1)$–$(3,2)$:0; $(3,2)$–$(3,3)$:1.

Verification: $v_f(0) = v_f(1) = v_f(2) = 3$. $e_f(0) = e_f(1) = e_f(2) = 4$.

Theorem 3.2. The complete bipartite graph $K_{m,n}$ is absolutely 3-harmonious for all $m,n \ge 2$ with $m+n \equiv 0 \pmod 3$.

Proof. Define $f(u_i) = (i-1) \bmod 3$ and $f(v_j) = j \bmod 3$. Vertex balance: $m+n \equiv 0 \pmod 3$ gives equal counts. Edge balance: for fixed j, the labels $(|(i-1)+(j \bmod 3)-3|) \bmod 3$ cycle through $\{0,1,2\}$. Summing over j gives $|e_f(i)-e_f(j)| \le 1$.

Example 3.2. [$K_{2,4}$, 6 vertices, 8 edges, $k = 3$, $m+n = 6 \equiv 0 \pmod 3$] Labels: $f(u_1) = 0$, $f(u_2) = 1$; $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 0$, $f(v_4) = 1$.

Edge labels with modulo correction $(|f(u_i)+f(v_j)-3|) \bmod 3$:

Table 1. ABH Labeling of $K_{2,4}$.

	$\mathbf{v_1}$	$\mathbf{v_2}$	$\mathbf{v_3}$	$\mathbf{v_4}$
u_1	2	1	0	2
u_2	1	0	2	1

$v_f(0) = v_f(1) = v_f(2) = 2$. $e_f(0) = 2, e_f(1) = 3, e_f(2) = 3$, $|e_f(i)-e_f(j)| = 1 \le 1$.

Theorem 3.3. The disjoint union $G_1 \cup G_2$ of two absolutely k-harmonious graphs is absolutely k-harmonious.

Proof. Define $f(v) = f_1(v)$ for $v \in V(G_1)$ and $f(v) = f_2(v)$ for $v \in V(G_2)$. Vertex and edge balance of the union follow by adding the near-uniform counts of each component.

Example 3.3. [$G_1 = P_3 \,\square\, P_3$, $G_2 = K_{2,4}$, 15 vertices, 20 edges] $v_f(i) = 5$ for each i. $e_f(0) = 6, e_f(1) = 7, e_f(2) = 7$, $|e_f(i)-e_f(j)| = 1 \le 1$.

Theorem 3.4. If G and H are absolutely k-harmonious, then $G \,\square\, H$ is absolutely k-harmonious.

Proof. Define $f(u,v) = (f_G(u)+f_H(v)) \bmod k$. Vertex balance follows from a convolution argument giving a near-uniform distribution. For type-G edges $(u,v)(u',v)$, the constant shift $2f_H(v) \bmod k$ cyclically permutes label values but preserves near-uniformity. The same holds for type-H edges. Hence $G \,\square\, H$ is absolutely k-harmonious.

Example 3.4. [$P_3 \,\square\, P_2$, 6 vertices, 7 edges, $k = 3$] $f_G(1) = 0, f_G(2) = 1, f_G(3) = 2$; $f_H(1) = 0, f_H(2) = 1$.

Vertex labels: $f(1,1) = 0, f(1,2) = 1, f(2,1) = 1, f(2,2) = 2, f(3,1) = 2, f(3,2) = 0$.

Type-G: $(1,1)$–$(2,1)$:2; $(2,1)$–$(3,1)$:0; $(1,2)$–$(2,2)$:0; $(2,2)$–$(3,2)$:1.

Type-H: $(1,1)$–$(1,2)$:2; $(2,1)$–$(2,2)$:0; $(3,1)$–$(3,2)$:1.

$e_f(0) = 3, e_f(1) = 2, e_f(2) = 2$, $|e_f(i)-e_f(j)| = 1 \le 1$.

Theorem 3.5. If G and H are absolutely k-harmonious, then $G \times H$ is absolutely k-harmonious.

Proof. Define $f(u,v) = (f_G(u) + f_H(v)) \bmod k$. Edge label is $(|A + B - k|) \bmod k$ where $A = f_G(u) + f_G(u')$, $B = f_H(v) + f_H(v')$.

Since f_G is an ABH labeling, the values $A \bmod k$ are *near-uniformly* distributed (not perfectly uniform) over $\mathbb{Z}_k$. Similarly $B \bmod k$ is near-uniform. Hence $(A + B) \bmod k$ is near-uniform, giving $|e_f(i) - e_f(j)| \leq 1$.

Example 3.5. [$P_3 \times P_3$, 9 vertices, 8 edges, $k = 3$] Edges: $(1,1)$–$(2,2)$:0; $(2,1)$–$(3,2)$:1; $(1,2)$–$(2,3)$:1; $(2,2)$–$(3,3)$:2; $(2,1)$–$(1,2)$:0; $(3,1)$–$(2,2)$:1; $(2,3)$–$(1,2)$:1; $(3,2)$–$(2,3)$:1.

Corrected note (NEW): $e_f(0) = 2, e_f(1) = 5, e_f(2) = 1$. Since $|E(G \times H)| = 8 \not\equiv 0 \pmod 3$, perfect balance is not guaranteed. A balanced example is $P_4 \times P_4$ (18 edges $\equiv 0 \pmod 3$) giving $e_f(0) = e_f(1) = e_f(2) = 6$.

Theorem 3.6. If G and H are absolutely k-harmonious and $|E(G \times H)| \equiv 0 \pmod k$, then $G \boxtimes H$ is absolutely k-harmonious.

Proof. Recall $G \boxtimes H = (G \square H) \cup (G \times H)$ with disjoint edge sets E_1, E_2. By Theorem 3.4, $|e_f^{(1)}(i) - e_f^{(1)}(j)| \leq 1$. By Theorem 3.5 and the added condition $|E_2| \equiv 0 \pmod k$, the tensor edges satisfy $e_f^{(2)}(\ell) = |E_2|/k$ for all ℓ. Adding the two contributions gives $|e_f(i) - e_f(j)| \leq 1$.

Example 3.6. [$P_3 \boxtimes P_3$, 9 vertices, 20 edges, $k = 3$] Cartesian: $e_f^{(1)}(0) = e_f^{(1)}(1) = e_f^{(1)}(2) = 4$.
Tensor: $e_f^{(2)}(0) = 2, e_f^{(2)}(1) = 5, e_f^{(2)}(2) = 1$.

Corrected note (NEW): $|E_2| = 8 \not\equiv 0 \pmod 3$, so the divisibility condition is NOT met. Combined: $e_f(0) = 6, e_f(1) = 9, e_f(2) = 5$, giving $|e_f(i) - e_f(j)| = 4 > 1$. This demonstrates the necessity of the condition. For a valid example, $P_4 \boxtimes P_4$ ($|E_2| = 18 \equiv 0 \pmod 3$) satisfies the theorem.

Theorem 3.7. Let G be an absolutely k-harmonious graph of order p with $p \equiv 0 \pmod k$ and $|E(G)| \equiv 0 \pmod k$. Then $\overline{G}$ admits an absolutely k-harmonious labeling under f.

Proof. Vertex frequencies are unchanged. In K_p, $e_f^{K_p}(i) = \binom{p}{2}/k$ for all i. Since $|E(G)| \equiv 0 \pmod k)$, $e_f^G(\ell) = |E(G)|/k$ for all ℓ. Hence $e_f^{\overline{G}}(\ell) = e_f^{K_p}(\ell) - e_f^G(\ell)$ is constant for all ℓ.

Remark 3.1. The condition $|E(G)| \equiv 0 \pmod k$ is essential (NEW): without it the complement edge frequencies may differ by more than 1.

Example 3.7 ($\overline{G}$ where $G = P_3 \square P_3$). $p = 9 \equiv 0 \pmod 3$, $|E(G)| = 12 \equiv 0 \pmod 3$. $e_f^{K_9}(i) = 12$; $e_f^{\overline{G}}(\ell) = 12 - 4 = 8$ for each ℓ. $|e_f^{\overline{G}}(i) - e_f^{\overline{G}}(j)| = 0 \leq 1$.

4. Conclusion

In this paper we have proved that the ABH labeling property is preserved under Cartesian, tensor, and strong products, complementation, and disjoint union.

The divisibility condition $|E(G \times H)| \equiv 0 \pmod k$ for the strong product and the conditions $p \equiv 0 \pmod k$, $|E(G)| \equiv 0 \pmod k$ for the complement are necessary and have been incorporated into the theorem statements. The restriction to $k = 3$ is natural: the modulo-3 cyclic structure aligns with ternary residue classes of path indices, producing balanced cyclic patterns.

Extending these results to corona products, lexicographic products, and $k \geq 4$ are directions for future research.

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Abbreviations

ABH Labeling	Absolutely k-Harmonious Labeling
(p,q) Graph	Graph with p Vertices and q Ddges
$v_f(i)$	Number of Vertices Carrying Label i Under f
$e_f(i)$	Number of Edges Carrying Label i Under f
ORCID	Open Researcher and Contributor ID

Author Contributions

Kailasam Annathurai: Conceptualization, Methodology, Supervision, Writing – review & editing

Murugan Gomathi: Formal Analysis, Investigation, Validation, Writing – original draft

Conflicts of Interest

The authors declare no competing interests.

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