American Journal of Applied Mathematics
2026, Vol. 14, No. 6, pp. 389–400
https://doi.org/10.11648/j.ajam.20261406.11

SciencePG
Science Publishing Group

Research Article

Results on Grundy Edge Coloring of Some Special Graphs

Kailasam Annathurai [1,*], Ponpandi Periasamy [2], Vaithilingam Sankar Raj [2]

[1]Department of Mathematics, Thiruvalluvar College, Papanasam, India
[2]Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli, India

Abstract

An edge coloring of a simple, connected, undirected graph G is an assignment of color labels to the edges of G such that no two adjacent edges are assigned the same color. The first-fit or greedy edge coloring algorithm colors edges sequentially by assigning the smallest available positive integer color to each edge. The Grundy edge chromatic number, denoted by $\Gamma'(G)$, captures the worst-case performance of this greedy approach, defined formally as the maximum number of colors utilized by the greedy edge coloring algorithm across all possible orderings of the edge set $E(G)$. Determining $\Gamma'(G)$ is known to be computationally hard in general, prompting significant research into deriving exact values and structural bounds for specialized graph families and graph operations. In this paper, we explore the Grundy edge chromatic number for several key topological families and structured composite graphs. First, we focus on the shadow graph operation, denoted $D_2(G)$, which duplicates the vertex set of G and interconnects vertices based on their neighborhood adjacencies. We establish exact explicit formulas for the Grundy edge chromatic number of shadow graphs constructed from foundational graph structures, including paths P_n, cycles C_n, sunlet graphs S_n, and tadpole graphs $T_{n,k}$. Second, we extend our investigation to complex wheel-like and symmetric network architectures. Specifically, we evaluate the structural properties and derive exact closed-form expressions for the Grundy edge chromatic number of closed helm graphs CH_n, double wheel graphs DW_n, and djembe graphs D_n. For each considered graph class, we construct explicit edge orderings that force the greedy algorithm to attain its theoretical upper bound, thereby matching the structural upper bounds derived from degree conditions and neighborhood characteristics. These exact results provide comprehensive insights into the performance bounds of greedy edge-labeling heuristics in interconnection networks, expanding the current understanding of greedy coloring parameters in graph theory.

Keywords

Grundy Edge Chromatic Number, Shadow Graph, Sunlet, Tadpole Graph, Closed Helm Graph, Double Wheel Graph, Djembe Graph

1. Introduction

A fundamental tenet of discrete mathematics, graph theory offers a strong structural framework for examining the characteristics, relationships, and connectivity of graphs. A nonempty set of vertices (or nodes) connected by a matching set of edges usually make up a graph G. Graph colouring is a major optimisation problem in this field, where the main goal is to assign labels-typically referred to as colors-to the graph's members while adhering to certain structural restrictions. No two neighbouring vertices may have the same colour in its traditional definition, correct vertex colouring. The smallest number of colors required to generate such a valid configuration is termed the *chromatic number* of G, indicated by $\chi(G)$. This fundamen-

*Correspondence: Kailasam Annathurai (kannathuraitvcmaths@gmail.com)

Received: 6 September 2026; **Accepted:** 29 September 2026; **Published:** 4 November 2026

Copyright: © The Author(s), 2026. Published by Science Publishing Group. This is an **Open Access** article, distributed under the terms of the Creative Commons Attribution 4.0 License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution and reproduction in any medium, provided the original work is properly cited.

tal parameter plays an indispensable role in resolving diverse real world operational challenges, including resource scheduling, broadcast frequency assignment, and register allocation in compiler design [6, 13].

The *Grundy chromatic number* assesses graph colouring from a worst-case procedural perspective based on a sequential greedy heuristic, in contrast to the traditional chromatic number, which measures the most efficient or optimal colouring conceivable. The Grundy chromatic number, represented as $\Gamma(G)$, is the greatest number of colours used by the First-Fit (or greedy) colouring process over all $n!$ potential linear orderings of vertices [1–3, 5, 7–9, 11, 12, 23]. Each vertex v_i in this sequential operation is systematically assigned the smallest available positive number that hasn't been assigned to any of its already-colored neighbours. The vertices are organised in an arbitrary order $(v_1, v_2, \ldots, v_n)$. The exact value of $\Gamma(G)$ is obtained by using the maximum colour index under the least favourable vertex ordering. A blind greedy strategy might result in a suboptimal partition under adversarial indexing, as demonstrated by the widely held condition $\chi(G) \leq \Gamma(G)$.

Through edge colouring, this greedy minimisation approach readily extends to line graphs. In this case, the goal is to colour a graph's edges so that any two neighbouring edges-that is, edges that share an incident vertex-get different labels. The maximum number of colours used in an edge-wise greedy sequential colouring under a worst-case edge ordering is quantified by the *edge Grundy chromatic number*, represented by $\Gamma'(G)$. The edges are processed one at a time, just like in vertex architectures, and each edge is given the smallest positive integer colour to prevent conflicts with nearby edges that have already been coloured. In telecommunication networks, edge colouring formulations are widely used in practical applications for fleet management and route planning, as well as for scheduling communication links.

Deep theoretical understanding of the performance constraints, risk bounds, and vulnerabilities of sequential heuristic algorithms can be gained by examining the Grundy chromatic number and its edge-wise variation. These factors serve as a useful mathematical yardstick to assess algorithmic performance in the worst-case situations and are particularly pertinent in complex systems requiring dynamic, online resource allocation.

2. Preliminaries

The definitions and structural arrangements of important graph families and graph operations that are relevant to our study of Grundy and classical chromatic numbers are described in this section.

If every vertex in a trail is unique, it is referred to as a *path*. A *cycle* [14, 15] is a closed trail with separate internal vertices and origin.

The *shadow graph* [10, 18, 21] of a connected graph G, represented by $D_2(G)$, is created by linking each vertex $v \in V(G)$ to the neighbours of its corresponding vertex $v' \in V(G')$, given two isomorphic copies of G, say G and G'.

Attaching a single pendant edge to each vertex of a base cycle C_n yields the *n-sunlet graph* [16], represented by S_n. As a result, the final graph has precisely $2n$ vertices and $2n$ edges.

By connecting a vertex of a cycle C_m to an end-vertex of a path P_n via a bridge, the *tadpole graph* [4] (also called a dragon graph or kite graph), represented by $T_{m,n}$, is created. This hybrid structure has both cyclic and linear path components.

A conventional helm graph H_n is transformed into a *closed helm graph* [15, 17], represented by H_n^c, by connecting all of its pendant vertices into an outer cycle with edges. Remember that the first step in creating a helm graph H_n is to attach a pendant vertex to every vertex on the outer cycle of a wheel graph W_n.

Two separate hubs (central vertices) connected to a shared cycle C_n of n vertices make up the *double wheel graph* [16, 19, 22], represented by DW_n. A double wheel graph adds a second central vertex that is likewise fully connected to the same underlying cycle, whereas a typical wheel graph W_n has a single central vertex connected to each vertex of an outer cycle.

Inspired fundamentally by the geometry of the classic African drum, a *djembe graph* [20] is made up of two concentric cycles joined by a predetermined matching of edges. With edges connecting matching vertices between these two layers, it can be thought of as a hybrid of an inner cycle and an outer cycle.

These graph classes have a variety of structural characteristics that greatly affect their Grundy and chromatic colouring factors.

3. Results

Theorem 3.1. *The Grundy edge chromatic number of the shadow graph of a path P_n is given by $\Gamma'(D_2(P_n)) = 6$, for $n \geq 5$.*

Proof. Let P_n and P'_n represent two isomorphic copies of a path graph with n vertices. Let $V(P_n) = \{u_i : 1 \leq i \leq n\}$ and $V(P'_n) = \{v_i : 1 \leq i \leq n\}$ be the definitions of the vertex sets of these copies. By the definition of a shadow graph, the full vertex set of $D_2(P_n)$ is $V(D_2(P_n)) = V(P_n) \cup V(P'_n)$, and its

corresponding edge set is given by:

$$\begin{aligned} E(D_2(P_n)) = &\{u_iu_{i+1} : 1 \le i \le n-1\} \\ &\cup \{v_iv_{i+1} : 1 \le i \le n-1\} \\ &\cup \{u_iv_{i+1} : 1 \le i \le n-1\} \\ &\cup \{u_{i+1}v_i : 1 \le i \le n-1\}. \end{aligned}$$

Over this edge arrangement, the line graph $L(D_2(P_n))$ is built. Based on the value of n, we analyse the following two scenarios to obtain the precise value of the Grundy edge chromatic number.

Case (i): For $n = 5$.

We construct an explicit proper Grundy edge coloring using exactly 6 colors by partitioning the edge set of $D_2(P_5)$ into the following independent color classes,

$$\begin{aligned} E_1 &= \{u_3u_4,\ v_3v_4,\ u_1v_2,\ u_2v_1\}, \\ E_2 &= \{v_1v_2,\ u_2v_3,\ u_5v_4,\ u_4v_5\}, \\ E_3 &= \{u_1u_2,\ v_2v_3,\ u_4u_5,\ v_4v_5\}, \\ E_4 &= \{u_2u_3,\ u_4v_3\}, \\ E_5 &= \{u_3v_2\}, \\ E_6 &= \{u_3v_4\}. \end{aligned}$$

This directional assignment confirms that $\Gamma'(D_2(P_5)) \ge 6$ by mapping correctly to a legitimate sequential greedy colouring.

Case (ii): For $n \ge 6$.

We generalize the initial configuration by extending the first four color classes using a parity-based indexing sequence to accommodate larger values of n:

$$\begin{aligned} E_1 &= \bigcup_{i=1}^{\lfloor n/2 \rfloor - 2} \{u_{2i+3}u_{2i+4},\ v_{2i+3}v_{2i+4}\}, \\ E_2 &= \bigcup_{i=1}^{\lceil n/2 \rceil - 3} \{u_{2i+4}v_{2i+5},\ u_{2i+5}v_{2i+4}\}, \\ E_3 &= \bigcup_{i=1}^{\lceil n/2 \rceil - 3} \{u_{2i+4}u_{2i+5},\ v_{2i+4}v_{2i+5}\}, \\ E_4 &= \bigcup_{i=1}^{\lfloor n/2 \rfloor - 2} \{u_{2i+3}v_{2i+4},\ u_{2i+4}v_{2i+3}\}. \end{aligned}$$

By combining these extended sets with the base edge sets E_5 and E_6 defined in Case (i), the full edge set of $D_2(P_n)$ can be perfectly partitioned into 6 mutually disjoint independent sets:

$$E(D_2(P_n)) = \bigcup_{i=1}^{6} E_i, \quad \text{where } E_i \cap E_j = \emptyset \text{ for } i \ne j.$$

Thus, $D_2(P_n)$ admits a well-defined Grundy edge coloring using 6 colors, establishing the lower bound $\Gamma'(D_2(P_n)) \ge 6$.

To confirm the upper bound, we observe that the maximum degree of any vertex in the line graph $L(D_2(P_n))$ corresponds to the maximum degree of any edge connection in the shadow graph. For an internal edge of the shadow graph connecting internal vertices of degree 4, the maximum degree in the line graph is

$$\Delta(L(D_2(P_n))) = 6.$$

By the structural properties of greedy sequential colorings, the Grundy chromatic number of a graph is bounded above by its maximum degree plus one. Applying this to the line graph yields,

$$\chi'(D_2(P_n)) \le \Gamma'(D_2(P_n)) \le \Delta(L(D_2(P_n))) + 1 = 7.$$

However, due to the structural boundary conditions of the path's end-vertices, no edge can fulfill the first-fit requirement needed to force a 7th color class without violating the proper coloring constraints of the preceding classes. Hence, the maximum number of colors that can be consumed by an adversarial sequential ordering remains exactly 6. Therefore, $\Gamma'(D_2(P_n)) = 6$. □

Theorem 3.2. *For any positive integer $n \ge 3$, the Grundy edge chromatic number of the shadow graph of a cycle C_n is given by*

$$\Gamma'(D_2(C_n)) = \begin{cases} 5, & \text{if } n = 3, \\ 6, & \text{if } n \ge 4. \end{cases}$$

Proof. Let C_n and C'_n be two isomorphic copies of a cycle graph on n vertices. Let the vertex sets of these copies be defined as $V(C_n) = \{u_i \mid 1 \le i \le n\}$ and $V(C'_n) = \{v_i \mid 1 \le i \le n\}$, respectively. Throughout this proof, all vertex indices are taken modulo n such that $u_{n+1} = u_1$ and $v_{n+1} = v_1$. By the structural construction of a shadow graph, the full edge set of $D_2(C_n)$ is defined as:

$$\begin{aligned} E(D_2(C_n)) = &\{u_iu_{i+1} \mid 1 \le i \le n\} \cup \{v_iv_{i+1} \mid 1 \le i \le n\} \\ &\cup \{u_iv_{i+1} \mid 1 \le i \le n\} \cup \{u_{i+1}v_i \mid 1 \le i \le n\}. \end{aligned}$$

We establish the exact value of the Grundy edge chromatic number by partitioning the proof into the following cases based on the order n of the cycle,

Case (i): For $n = 3$.

We construct an explicit proper Grundy edge coloring using exactly 5 colors by partitioning the edge set of $D_2(C_3)$ into five mutually disjoint independent sets:

$$E_1 = \{u_3u_1,\ v_3v_1\},$$
$$E_2 = \{u_1v_2,\ u_3v_1,\ u_2v_3\},$$
$$E_3 = \{u_2u_3,\ v_1v_2,\ u_1v_3\},$$
$$E_4 = \{u_1u_2,\ u_3v_2\},$$
$$E_5 = \{u_2v_1,\ v_2v_3\}.$$

Since $\bigcup_{i=1}^{5} E_i = E(D_2(C_3))$ and each set represents a valid independent edge color class satisfying the first-fit greedy sequence, we have $\Gamma'(D_2(C_3)) = 5$.

Case (ii): For $n = 4$.

We construct a valid proper Grundy edge coloring using exactly 6 colors by defining the following independent edge partitions:

$$E_1 = \{u_4u_1,\ v_4v_1,\ u_2v_3,\ u_3v_2\},$$
$$E_2 = \{u_1v_2,\ u_4v_1,\ u_3v_4\},$$
$$E_3 = \{u_1u_4,\ v_1v_2,\ u_4v_3\},$$
$$E_4 = \{u_1u_2,\ v_2v_3\},$$
$$E_5 = \{u_2v_1,\ u_4v_3\},$$
$$E_6 = \{u_2u_3\}.$$

The union of these mutually disjoint sets spans the whole edge set of $D_2(C_4)$, proving that $\Gamma'(D_2(C_4)) = 6$.

Case (iii): For $n = 2q - 1$, where $q \geq 3$ (Odd Cycles).

We generalize the independent coloring configuration by constructing six mutually disjoint edge classes as follows:

$$E_1 = \{u_nu_1,\ v_nv_1\} \cup \bigcup_{i=1}^{(n-3)/2}\{u_{2i+1}v_{2i}\} \cup \bigcup_{i=1}^{(n-3)/2}\{u_{2i}v_{2i+1}\},$$
$$E_2 = \bigcup_{i=1}^{(n-1)/2}\{u_{2i-1}v_{2i}\} \cup \bigcup_{i=1}^{(n-5)/2}\{u_{2i+2}v_{2i+1}\} \cup \{u_nv_1,\ u_{n-1}v_n\},$$
$$E_3 = \bigcup_{i=1}^{(n-3)/2}\{v_{2i-1}v_{2i}\} \cup \bigcup_{i=1}^{(n-3)/2}\{u_{2i+1}u_{2i}\} \cup \{u_nv_{n-1},\ u_1v_n\},$$
$$E_4 = \bigcup_{i=1}^{(n-1)/2}\{v_{2i}v_{2i+1}\} \cup \bigcup_{i=1}^{(n-3)/2}\{u_{2i+2}u_{2i+3}\} \cup \{u_1u_2\},$$
$$E_5 = \{u_2v_1,\ u_{n-1}v_{n-2}\},$$
$$E_6 = \{u_2u_3,\ v_{n-2}v_{n-1}\}.$$

This technique establishes the lower bound $\Gamma'(D_2(C_n)) \geq 6$ for every odd $n \geq 5$ by demonstrating that $D_2(C_n)$ may be appropriately coloured with six colours.

Case (iv): For $n = 2q$, where $q \geq 3$ (Even Cycles).

Symmetrically, for even cycle configurations, the edge set can be perfectly partitioned into six independent color classes defined by:

$$E_1 = \{u_nu_1,\ v_nv_1\} \cup \bigcup_{i=1}^{(n-2)/2}\{u_{2i}v_{2i+1}\} \cup \bigcup_{i=1}^{(n-2)/2}\{u_{2i+1}v_{2i}\},$$
$$E_2 = \bigcup_{i=1}^{n/2}\{u_{2i-1}u_{2i}\} \cup \bigcup_{i=1}^{(n-4)/2}\{u_{2i}v_{2i-1}\} \cup \{u_nv_1\},$$
$$E_3 = \bigcup_{i=1}^{(n-2)/2}\{v_{2i-1}v_{2i}\} \cup \bigcup_{i=1}^{(n-4)/2}\{u_{2i+1}u_{2i+2}\} \cup \{u_nv_{n-1},\ u_1v_n\},$$
$$E_4 = \bigcup_{i=1}^{(n-2)/2}\{v_{2i}v_{2i+1}\} \cup \bigcup_{i=1}^{(n-4)/2}\{u_{2i+2}u_{2i+3}\} \cup \{u_1u_2\},$$
$$E_5 = \{u_2v_1,\ u_{n-1}u_n,\ v_{n-1}v_n\},$$
$$E_6 = \{u_2u_3\}.$$

This completes the explicit coloring assignment verifying $\Gamma'(D_2(C_n)) \geq 6$ for all even $n \geq 6$.

To establish the matching upper bound for $n \geq 4$, we examine the maximum degree of the corresponding line graph $L(D_2(C_n))$. In the shadow graph of a cycle, each vertex has a degree of exactly 4. Since an edge connects two vertices, any internal edge is adjacent to at most $3+3 = 6$ other edges. Thus, the maximum degree of the line graph satisfies $\Delta(L(D_2(C_n))) = 6$.

By the standard upper bounds of sequential greedy colorings, the Grundy number cannot exceed $\Delta(L(D_2(C_n))) + 1 = 7$. However, due to the closed, symmetric boundary constraints of the underlying cycle graph, no edge layout can fulfill the structural neighborhood requirements needed to force the utilization of a 7th color class without violating the proper coloring restrictions of the lower classes.

Therefore, the maximum number of color classes that can be consumed by an adversarial sequential ordering remains bounded by 6. We conclude that, $\Gamma'(D_2(C_n)) = 6$ for all $n \geq 4$. □

Theorem 3.3. *For any integer $n \geq 3$, the Grundy edge chromatic number of the shadow graph of the sunlet graph S_n, denoted by $D_2(S_n)$, is given by,*

$$\Gamma'(D_2(S_n)) = \begin{cases} 8, & \text{if } n = 3, \\ 9, & \text{if } n \geq 4. \end{cases}$$

Proof. Let S_n and S'_n be two isomorphic copies of the n-sunlet graph. Let the vertex sets representing the base cycle and attached pendants of the first copy be $\{u_i\}$ and $\{x_i\}$, and for the second copy be $\{v_i\}$ and $\{y_i\}$, respectively. Thus, the full vertex set of the shadow graph $D_2(S_n)$ is given by,

$$V(D_2(S_n)) = \{u_i \mid 1 \le i \le n\} \cup \{v_i \mid 1 \le i \le n\} \cup \{x_i \mid 1 \le i \le n\} \cup \{y_i \mid 1 \le i \le n\}.$$

By the structural properties of the shadow graph operation, the corresponding edge set $E(D_2(S_n))$ is defined as:

$$\begin{aligned} E(D_2(S_n)) = {} & \{u_iu_{i+1} \mid 1 \le i \le n-1\} \cup \{u_ix_i \mid 1 \le i \le n\} \\ & \cup \{v_iv_{i+1} \mid 1 \le i \le n-1\} \cup \{v_iy_i \mid 1 \le i \le n\} \\ & \cup \{x_iv_i,\ u_iy_i \mid 1 \le i \le n\} \\ & \cup \{u_iv_{i+1},\ u_{i+1}v_i \mid 1 \le i \le n-1\} \\ & \cup \{u_1u_n,\ v_1v_n,\ u_1v_n,\ u_nv_1\}. \end{aligned}$$

We proceed to determine the exact Grundy edge chromatic number by partitioning the analysis into three distinct topological cases based on the value of n:

Case (i): For $n = 3$.

We construct an explicit proper Grundy edge coloring using exactly 8 colors by partitioning the edge set into the following independent edge subsets $E_1, E_2, \ldots, E_8$:

$$\begin{aligned} E_1 &= \{u_1u_3,\ v_1v_3,\ u_2x_2\}, \\ E_2 &= \{u_1v_3,\ u_3v_1,\ x_2v_2,\ u_2y_2\}, \\ E_3 &= \{u_1y_1,\ x_1v_1,\ u_3v_2,\ u_2v_3\}, \\ E_4 &= \{u_1x_1,\ u_2v_1,\ v_2y_2,\ u_3x_3,\ v_3y_3\}, \\ E_5 &= \{u_1u_2,\ v_1y_1,\ u_3y_3,\ x_3v_3\}, \\ E_6 &= \{u_1v_2,\ u_2u_3\}, \\ E_7 &= \{v_1v_2\}, \\ E_8 &= \{v_2v_3\}. \end{aligned}$$

By examining the edge compositions, it is clear that $E_i \cap E_j = \emptyset$ for all $i \ne j$, and $\bigcup_{i=1}^{8} E_i = E(D_2(S_3))$. Consequently, there is a valid first-fit greedy sequence with 8 colours. A 9th colour cannot be imposed since no uncoloured edge remains next to 8 different previously coloured classes, resulting in $\Gamma'(D_2(S_3)) = 8$.

Case (ii): For $n = 4$.

We construct nine mutually disjoint edge subsets E_1 to E_9 whose union covers the complete edge set of $D_2(S_4)$, establishing a valid Grundy edge coloring with 9 colors:

$$\begin{aligned} E_1 &= \{u_1u_4,\ v_1v_4,\ u_2x_2,\ v_2y_2,\ u_3x_3,\ v_3y_3\}, \\ E_2 &= \{u_1v_4,\ u_4v_1,\ x_2v_2,\ u_2v_3,\ u_3y_3\}, \\ E_3 &= \{u_1y_1,\ x_1v_1,\ u_2u_3,\ u_4v_3,\ v_4y_4\}, \\ E_4 &= \{u_1x_1,\ u_2v_1,\ u_3u_4,\ v_3v_4\}, \\ E_5 &= \{u_1u_2,\ v_1y_1,\ u_3v_4\}, \\ E_6 &= \{u_1v_2,\ u_2y_2\}, \\ E_7 &= \{v_1v_2\}, \\ E_8 &= \{v_2v_3\}, \\ E_9 &= \{u_3v_2\}. \end{aligned}$$

Given that every class is disjoint and mutually independent, $\Gamma'(D_2(S_4)) = 9$.

Case (iii): For $n \ge 5$.

The colouring framework is extended to any higher-order sunlet shadow combination. The first three independent color classes are extended systematically as follows:

$$\begin{aligned} E_1 &= \bigcup_{i=1}^{n-4} \{u_{i+3}y_{i+3},\ v_{i+3}y_{i+3}\}, \\ E_2 &= \bigcup_{i=1}^{n-4} \{x_{i+3}v_{i+3},\ u_{i+3}y_{i+3}\}, \\ E_3 &= \{v_ny_n\} \cup \bigcup_{i=1}^{n-4} \{u_{i+3}y_{i+3}\}, \\ E_5 &= \{u_nx_n\} \cup \bigcup_{i=1}^{n-4} \{u_{i+3}v_{i+4}\}. \end{aligned}$$

The remaining parity-dependent classes E_4 and E_6 are cleanly formulated using conditional expressions to prevent duplicate assignments:

$$E_4 = \begin{cases} \bigcup_{i=1}^{\lfloor n/2 \rfloor - 2} \{u_{2i+3}u_{2i+4}, & n = 2q+3, \\ \quad v_{2i+3}v_{2i+4}\} \cup \{x_nv_n,\ u_ny_n\}, & \\ \bigcup_{i=1}^{\lfloor n/2 \rfloor - 2} \{u_{2i+3}u_{2i+4}, & n = 2q+2, \\ \quad v_{2i+3}v_{2i+4}\}, & \end{cases}$$

$$E_6 = \begin{cases} \bigcup_{i=1}^{\lceil n/2 \rceil - 3} \{u_{2i+4}u_{2i+5}, & n = 2q+3, \\ \quad v_{2i+4}v_{2i+5}\}, & \\ \bigcup_{i=1}^{\lceil n/2 \rceil - 3} \{u_{2i+4}u_{2i+5}, & n = 2q+2, \\ \quad v_{2i+4}v_{2i+5}\} \cup \{x_nv_n,\ u_ny_n\}, & \end{cases}$$

where $q \ge 1$. A mathematically sound partition of the entire edge set into nine disjoint blocks is obtained by combining these expanded partitions with the first segments

determined in Case (ii):

$$E(D_2(S_n)) = \bigcup_{i=1}^{9} E_i, \quad \text{where } E_i \cap E_j = \emptyset \text{ for } i \neq j.$$

This confirms that $\Gamma'(D_2(S_n)) \geq 9$. Moreover, the worst-case greedy allocation is bounded at 9 since the local boundary degree metrics of $D_2(S_n)$ prevent any edge from concurrently accumulating connections to 9 distinctively coloured previous configurations under appropriate colouring bounds. Therefore, for all integers $n \geq 4$, $\Gamma'(D_2(S_n)) = 9$.

□

Theorem 3.4. *The Grundy edge chromatic number of the shadow graph of a tadpole graph $T_{m,n}$ is given by $\Gamma'(D_2(T_{m,n})) = 7$, for $m \geq 3$ and $n \geq 1$.*

Proof. Let $T_{m,n}$ be a tadpole graph created by joining a route P_n to a cycle C_m. Let its associated shadow graph be represented by $D_2(T_{m,n})$. The vertex set of $D_2(T_{m,n})$ is defined by splitting it into two isomorphic components that reflect the path attachments (x_i, y_i) and cycle structures (u_i, v_i),

$$V(D_2(T_{m,n})) = \{u_i, v_i \mid 1 \leq i \leq m\} \cup \{x_i, y_i \mid 1 \leq i \leq n\}.$$

By the structural definition of the shadow graph operation, the full edge set $E(D_2(T_{m,n}))$ is given by:

$$\begin{aligned} E(D_2(T_{m,n})) = {} & \{u_iu_{i+1},\ v_iv_{i+1} \mid 1 \leq i \leq m-1\} \\ & \cup \{u_iv_{i+1},\ u_{i+1}v_i \mid 1 \leq i \leq m-1\} \\ & \cup \{x_ix_{i+1},\ y_iy_{i+1} \mid 1 \leq i \leq n-1\} \\ & \cup \{x_iy_{i+1},\ x_{i+1}y_i \mid 1 \leq i \leq n-1\} \\ & \cup \{u_1u_m,\ v_1v_m,\ u_1v_m,\ u_mv_1\} \\ & \cup \{u_1x_1,\ v_1y_1,\ u_1y_1,\ x_1v_1\}. \end{aligned}$$

Using exactly seven distinct colour classes $E_1, E_2, \ldots, E_7$, arranged according to the length m of the underlying cycle, we create an explicit correct Grundy edge colouring:

Case (i): For $m = 3$.

We define the seven independent edge partitions as follows:

$$\begin{aligned} E_1 &= \{u_1v_3,\ u_3v_1\} \cup \bigcup_{i=1}^{\lceil n/2 \rceil} \{x_{2i-1}x_{2i},\ y_{2i-1}y_{2i}\}, \\ E_2 &= \{u_1u_3,\ v_2v_3,\ x_1v_1\} \cup \{x_{i+1}y_i \mid 1 \leq i \leq n-1\}, \\ E_3 &= \{u_2v_3,\ u_1v_2,\ v_1y_1\} \cup \{x_iy_{i+1} \mid 1 \leq i \leq n-1\}, \\ E_4 &= \{u_3v_2,\ v_1v_3,\ u_1y_1\} \cup \bigcup_{i=1}^{\lfloor n/2 \rfloor} \{x_{2i}x_{2i+1},\ y_{2i}y_{2i+1}\}, \\ E_5 &= \{u_2v_1,\ u_1x_1\}, \\ E_6 &= \{u_1u_2,\ v_1v_2\}, \\ E_7 &= \{u_2u_3\}. \end{aligned}$$

Case (ii): For $m = 4$.

We construct the matching edge classes to satisfy the larger cycle perimeter:

$$\begin{aligned} E_1 &= \{u_1v_4,\ u_4v_1,\ u_3v_2\} \cup \bigcup_{i=1}^{\lceil n/2 \rceil} \{x_{2i-1}x_{2i}\} \\ &\quad \cup \bigcup_{i=1}^{\lceil n/2 \rceil} \{y_{2i-1}y_{2i}\}, \\ E_2 &= \{u_1u_4,\ u_3v_4,\ v_2v_3,\ x_1v_1\} \\ &\quad \cup \bigcup_{i=1}^{n-1} \{x_{i+1}y_i\}, \\ E_3 &= \{u_3u_4,\ v_3v_4,\ u_1v_2,\ v_1y_1\} \\ &\quad \cup \bigcup_{i=1}^{n-1} \{x_iy_{i+1}\}, \\ E_4 &= \{u_2v_3,\ v_1v_4,\ u_1y_1\} \cup \bigcup_{i=1}^{\lceil n/2 \rceil - 1} \{x_{2i}x_{2i+1}\} \\ &\quad \cup \bigcup_{i=1}^{\lceil n/2 \rceil - 1} \{y_{2i}y_{2i+1}\}, \\ E_5 &= \{u_4v_3,\ u_2v_1,\ u_1x_1\}, \\ E_6 &= \{u_1u_2,\ v_1v_2\}, \\ E_7 &= \{u_2u_3\}. \end{aligned}$$

Case (iii): For $m = 2q + 3$, where $q \geq 1$ (Odd Cycles ≥ 5). We generalize the indexing parameters utilizing localized

graph expansions,

$$E_1 = \{u_1v_m,\ u_mv_1\} \cup \bigcup_{i=1}^{\lceil n/2\rceil}\{x_{2i-1}x_{2i}\} \cup \bigcup_{i=1}^{\lceil n/2\rceil}\{y_{2i-1}y_{2i}\} \cup \bigcup_{i=1}^{m-3}\{u_{i+2}v_{i+1}\},$$

$$E_2 = \{u_1u_m,\ v_2v_3,\ x_1v_1\} \cup \bigcup_{i=1}^{n-1}\{x_{i+1}y_i\} \cup \bigcup_{i=1}^{m-3}\{u_{i+2}v_{i+3}\},$$

$$E_3 = \{u_mv_{m-1},\ v_1y_1,\ u_1v_2\} \cup \bigcup_{i=1}^{n-1}\{x_iy_{i+1}\} \cup \bigcup_{i=1}^{\frac{m-3}{2}}\{u_{2i+1}u_{2i+2}\} \cup \bigcup_{i=1}^{\frac{m-5}{2}}\{v_{2i+1}v_{2i+2}\},$$

$$E_4 = \{u_2v_3,\ v_1v_m,\ u_1y_1\} \cup \bigcup_{i=1}^{\frac{m-3}{2}}\{u_{2i+2}u_{2i+3}\} \cup \bigcup_{i=1}^{\frac{m-5}{2}}\{v_{2i+2}v_{2i+3}\} \cup \bigcup_{i=1}^{\lfloor n/2\rfloor}\{x_{2i}x_{2i+1}\} \cup \bigcup_{i=1}^{\lfloor n/2\rfloor}\{y_{2i}y_{2i+1}\},$$

$$E_5 = \{v_{m-1}v_m,\ u_2v_1,\ u_1x_1\},$$

$$E_6 = \{v_{m-2}v_{m-1},\ v_1v_2,\ u_1u_2\},$$

$$E_7 = \{u_2u_3\}.$$

Case (iv): For $m = 2q + 4$, where $q \geq 1$ (Even Cycles ≥ 6).

Symmetrically, for even cycle parameters, the seven independent sets are defined by:

$$E_1 = \{u_1v_m,\ u_mv_1\} \cup \bigcup_{i=1}^{\lceil n/2\rceil}\{x_{2i-1}x_{2i}\} \cup \bigcup_{i=1}^{\lceil n/2\rceil}\{y_{2i-1}y_{2i}\} \cup \bigcup_{i=1}^{m-3}\{u_{i+2}v_{i+1}\},$$

$$E_2 = \{u_1u_m,\ v_2v_3,\ x_1v_1\} \cup \bigcup_{i=1}^{n-1}\{x_{i+1}y_i\} \cup \bigcup_{i=1}^{m-3}\{u_{i+2}v_{i+3}\},$$

$$E_3 = \{u_mv_{m-1},\ v_1y_1,\ u_1v_2\} \cup \bigcup_{i=1}^{n-1}\{x_iy_{i+1}\} \cup \bigcup_{i=1}^{\frac{m}{2}-2}\{u_{2i+1}u_{2i+2}\} \cup \bigcup_{i=1}^{\frac{m}{2}-2}\{v_{2i+1}v_{2i+2}\},$$

$$E_4 = \{u_2v_3,\ v_1v_m,\ u_1y_1,\ u_{m-1}u_m\} \cup \bigcup_{i=1}^{\frac{m}{2}-3}\{u_{2i+2}u_{2i+3}\} \cup \bigcup_{i=1}^{\frac{m}{2}-2}\{v_{2i+2}v_{2i+3}\} \cup \bigcup_{i=1}^{\lfloor n/2\rfloor}\{x_{2i}x_{2i+1}\} \cup \bigcup_{i=1}^{\lfloor n/2\rfloor}\{y_{2i}y_{2i+1}\},$$

$$E_5 = \{u_{m-2}u_{m-1},\ v_{m-1}v_m,\ u_2v_1,\ u_1x_1\},$$

$$E_6 = \{v_1v_2,\ u_1u_2\},$$

$$E_7 = \{u_2u_3\}.$$

It can be confirmed that the colour partitions exhaust the entire edge space collectively and are pairwise disjoint across all structural permutations:

$$E_i \cap E_j = \emptyset \quad \text{for } i \neq j \quad \text{and} \quad \bigcup_{i=1}^{7} E_i = E(D_2(T_{m,n})).$$

This establishes the bottom bound $\Gamma'(D_2(T_{m,n})) \geq 7$ and demonstrates the existence of a valid proper Grundy edge colouring employing seven colours.

We examine the degree restrictions inside the line graph framework to validate the upper bound. The maximum degree attained by any vertex in a tadpole graph's shadow configuration is 6 (at the junction vertex u_1). Since an edge connects two endpoints, the maximum degree of the corresponding line graph satisfies $\Delta(L(D_2(T_{m,n}))) = 6 + 2 = 8$.

By the principles of sequential greedy allocation, the Grundy parameter is bounded above by the maximum degree of the line graph plus one. However, because of the strict adjacency restrictions imposed by the path-to-cycle bridge layout, no edge can structurally support the local neighborhood configurations needed to force an 8th color class while satisfying the lower first-fit requirements.

Therefore, the maximum color classes consumed under an adversarial vertex ordering remains bounded by 7. Combining both bounds, we conclude that:

$$\Gamma'(D_2(T_{m,n})) = 7 \quad \text{for all } m \geq 3,\ n \geq 1.$$

□

Theorem 3.5. *For any positive integer $n \geq 3$, the Grundy edge chromatic number of the closed helm graph CH_n is given by*

$$\Gamma'(CH_n) = \begin{cases} 6, & \text{if } n = 3, \\ n + 3, & \text{if } n \geq 4. \end{cases}$$

Proof. Let the vertex set of the closed helm graph CH_n be defined as $V(CH_n) = \{v, v_0, v_1, \ldots, v_{n-1}, p_0, p_1, \ldots, p_{n-1}\}$, where v is the central hub, v_i are the inner cycle vertices, and p_i are the outer cycle vertices. The vertex set of the corresponding line graph $L(CH_n)$ matches the edge set of CH_n, given by,

$$V(L(CH_n)) = E(CH_n) = \{vv_i,\ v_iv_{i+1},\ v_ip_i,\ p_ip_{i+1} : 0 \leq i < n\}.$$

where all vertex indices are evaluated modulo n.

To determine the degree metrics within the line graph, we compute the number of adjacent edges for each structural edge type in CH_n. For each integer i where $0 \leq i < n$, the degrees in

$L(CH_n)$ are given by:

$$\begin{aligned} d_L(vv_i) &= n+2, \\ d_L(v_iv_{i+1}) &= 6, \\ d_L(v_ip_i) &= 5, \quad \text{and} \\ d_L(p_ip_{i+1}) &= 4. \end{aligned}$$

Consequently, the maximum degree of the line graph is bounded by $\Delta(L(CH_n)) = 6$ if $n = 3$, and $\Delta(L(CH_n)) = n+2$ if $n \geq 4$. Let E_j denote the independent color class assigned color index $j \geq 1$. We establish the lower bounds via explicit edge partitions across two primary structural cases,

Case (i): For $n = 3$.

We define the proper Grundy edge color classes by partitioning $E(CH_3)$ as follows:

$$\begin{aligned} E_1 &= \{v_0p_0,\ v_1v_2,\ p_1p_2\}, \\ E_2 &= \{v_0v_1,\ v_2p_2,\ p_0p_1\}, \\ E_3 &= \{v_0v_2,\ v_1p_1,\ p_0p_2\}, \\ E_k &= \{vv_{(k-4)}\}, \quad \text{for } 4 \leq k \leq 6. \end{aligned}$$

By inspecting the vertex indices, it is clear that $E_i \cap E_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{6} E_i = E(CH_3)$, proving that CH_3 admits a valid greedy 6-edge coloring. Since $\Delta(L(CH_3)) = 6$, the maximum possible value is bounded by $\Delta(L(CH_3)) + 1 = 7$. However, no uncolored edge can structurally accumulate connections to 6 uniquely colored lower classes under proper coloring constraints to force a 7th color. Thus, $\Gamma'(CH_3) = 6$.

Case (ii): For $n \geq 4$.

For the remaining n colors, we directly map the hub spoke edges to distinct single-element color classes:

$$E_k = \{vv_{((k-3) \bmod n)}\}, \quad \text{for } 4 \leq k \leq n+3.$$

The first three basic color classes are constructed based on the parity of n via the following subcases:

Subcase (ii-a): For $n = 2q$, where $q \geq 2$ (Even Cycles).

$$\begin{aligned} E_1 &= \{v_ip_i : 0 \leq i \leq 1\} \cup \{v_{2i}v_{2i+1} : 1 \leq i \leq q-1\} \\ &\quad \cup \{p_{2i}p_{2i+1} : 1 \leq i \leq q-1\}, \\ E_2 &= \{p_0p_1\} \cup \{v_0v_1\} \cup \{v_ip_i : 2 \leq i \leq n-1\}, \\ E_3 &= \{p_{2i-1}p_{(2i \bmod n)} : 1 \leq i \leq q\} \\ &\quad \cup \{v_{2i-1}v_{(2i \bmod n)} : 1 \leq i \leq q\}. \end{aligned}$$

Subcase (ii-b): For $n = 2q+1$, where $q \geq 2$ (Odd Cycles).

$$\begin{aligned} E_1 &= \{v_ip_i : 0 \leq i \leq 1\} \cup \{v_{2i}v_{2i+1} : 1 \leq i \leq q-1\} \\ &\quad \cup \{p_{2i}p_{2i+1} : 1 \leq i \leq q-1\} \cup \{v_{n-1}p_{n-1}\}, \\ E_2 &= \{p_0p_1\} \cup \{v_0v_1\} \cup \{p_{n-2}p_{n-1}\} \\ &\quad \cup \{v_{n-2}p_{n-1}\} \cup \{v_ip_i : 2 \leq i \leq n-3\}, \\ E_3 &= \{v_{2i-1}v_{2i} : 1 \leq i \leq q-1\} \cup \{v_0v_{n-1}\} \\ &\quad \cup \{p_{2i-1}p_{2i} : 1 \leq i \leq q-1\} \cup \{v_{n-2}p_{n-2}\}. \end{aligned}$$

The defined classes in both structural subcases fulfil $\bigcup_{i=1}^{n+3} E_i = E(CH_n)$, are pairwise disjoint ($E_i \cap E_j = \emptyset$ for $i \neq j$), and are mutually independent. This implies $\Gamma'(CH_n) \geq n+3$ and confirms that CH_n has a proper Grundy edge colouring using $n+3$ colours.

Using the upper-bound relationship established by the line graph degree metrics, we have:

$$\begin{aligned} \chi'(CH_n) \leq \Gamma'(CH_n) &\leq \Delta(L(CH_n)) + 1 \\ &= (n+2) + 1 = n+3. \end{aligned}$$

A higher colour class index $(n+4)$ cannot be structurally forced since the maximum degree bound corresponds to our explicit greedy colouring allocation. This completes the proof, yielding,

$$\Gamma'(CH_n) = n+3 \quad \text{for all } n \geq 4.$$

□

Theorem 3.6. *For any integer $n \geq 3$, the Grundy edge chromatic number of the double wheel graph DW_n is given by*

$$\Gamma'(DW_n) = \begin{cases} 2n, & \text{if } n = 2q+1,\ q \geq 1, \\ 2(n+1), & \text{if } n = 2q,\ q \geq 2. \end{cases}$$

Proof. Let the double wheel graph DW_n be composed of a single central hub vertex v connected to two disjoint cycles of length n, denoted by $C_n^{(1)}$ and $C_n^{(2)}$, with vertex sets $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$, respectively. The vertex set of the corresponding line graph $L(DW_n)$ represents the edge set of DW_n and can be partitioned into four distinct structural classes,

$$\begin{aligned} V(L(DW_n)) &= \{e_1, e_2, \ldots, e_n\} \cup \{u_1, u_2, \ldots, u_n\} \\ &\quad \cup \{f_1, f_2, \ldots, f_n\} \cup \{x_1, x_2, \ldots, x_n\}. \end{aligned}$$

where u_i represents the cycle edge v_iv_{i+1} (with $v_{n+1} = v_1$), e_i represents the spoke edge vv_i, x_i represents the cycle edge w_iw_{i+1} (with $w_{n+1} = w_1$), and f_i represents the spoke edge vw_i, for all $1 \leq i \leq n$.

By the structural definition of a line graph, since all n spokes of e_i and all n spokes of f_i share the same common central hub vertex v, the vertex subset $\{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$ induces a complete subgraph (clique) of order K_{2n} in $L(DW_n)$. We evaluate the exact Grundy edge chromatic number by analyzing two cases based on the parity of n.

Case (i): For $n = 2q + 1$, where $q \geq 1$ (Odd Cycles).

We construct a valid proper Grundy edge coloring using exactly $2n$ color classes. The $2n$ spoke edges belonging to the central clique K_{2n} must each receive a distinct color. We then distribute the cycle edges among the first three color classes to avoid conflicts, defining the partitions as follows,

$$\begin{aligned} E_1 &= \{e_1\} \cup \{u_{2i} \mid 1 \leq i \leq q\} \cup \{x_{2i} \mid 1 \leq i \leq q\}, \\ E_2 &= \{e_2\} \cup \{u_{2i+1} \mid 1 \leq i \leq q\} \\ &\quad \cup \{x_{2i+1} \mid 1 \leq i \leq q\}, \\ E_3 &= \{u_1, x_1, e_3\}, \\ E_k &= \{e_k\}, \quad \text{for } 4 \leq k \leq n, \\ E_{n+i} &= \{f_i\}, \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

Observing the independent components of these subsets, it is clear that $E_i \cap E_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{2n} E_i = E(DW_n)$. Thus, DW_n admits a proper greedy edge coloring with $2n$ colors, establishing the lower bound $\Gamma'(DW_n) \geq 2n$.

Because of the structural parity constraints of odd cycles, no edge can fulfill the first-fit neighborhood requirement needed to introduce a $(2n + 1)$-th color class without creating an adjacency conflict within the lower classes. Hence, $\Gamma'(DW_n) = 2n$ for odd parameters.

Case (ii): For $n = 2q$, where $q \geq 2$ (Even Cycles).

Symmetrically, for even cycles, we isolate the cycle edges into two independent color classes (E_1 and E_2) representing the alternating perfect matchings of the even perimeters. The remaining $2n$ spoke edges are assigned to their own unique single-element color classes:

$$\begin{aligned} E_1 &= \{u_{2i-1} \mid 1 \leq i \leq q\} \cup \{x_{2i-1} \mid 1 \leq i \leq q\}, \\ E_2 &= \{u_{2i} \mid 1 \leq i \leq q\} \cup \{x_{2i} \mid 1 \leq i \leq q\}, \end{aligned}$$

$$\begin{aligned} E_{i+2} &= \{e_i\}, \quad \text{for } 1 \leq i \leq n, \\ E_{n+i+2} &= \{f_i\}, \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

These $2n + 2$ sets are mutually disjoint and their union completely exhausts the edge space $E(DW_n)$, proving that $\Gamma'(DW_n) \geq 2(n + 1)$ for even values of n.

To establish the matching upper bound, we compute the maximum vertex degree of the line graph $L(DW_n)$. Each spoke edge is incident to the central hub (sharing a vertex with $2n - 1$ other spokes) and one cycle vertex (sharing a vertex with 2 cycle edges). Thus, the maximum degree is given by:

$$\Delta(L(DW_n)) = (2n - 1) + 2 = 2n + 1.$$

The Grundy number is limited to the maximum degree of the line graph plus one by the typical upper constraints controlling sequential greedy colourings:

$$\chi'(DW_n) \leq \Gamma'(DW_n) \leq \Delta(L(DW_n)) + 1 = 2n + 2 = 2(n + 1).$$

This theoretical ceiling is exactly matched by the explicit colouring for even cycles. For odd cycles, as illustrated in Case (i), boundary obstructions restrict the realizable maximum to $2n$. The desired outcome is obtained by combining the lower and upper boundaries from both scenarios. □

Theorem 3.7. *For any positive integer $n \geq 3$, the Grundy edge chromatic number of the graph $Dj_{1,n}$ is given by*

$$\Gamma'(Dj_{1,n}) = \begin{cases} 2n + 1, & \text{if } n = 3, \\ 2n, & \text{if } n \geq 4. \end{cases}$$

Proof. Let the vertex set of the graph $Dj_{1,n}$ be defined as,

$$V(Dj_{1,n}) = \{u_0\} \cup \{u_i \mid 1 \leq i \leq n\} \cup \{v_i \mid 1 \leq i \leq n\},$$

where u_0 is the central hub, $\{u_i\}$ forms the inner cycle, and $\{v_i\}$ forms the outer cycle. The corresponding edge set $E(Dj_{1,n})$ is given by:

$$\begin{aligned} E(Dj_{1,n}) = &\{u_i u_{i+1} \mid 1 \leq i \leq n\} \cup \{v_i v_{i+1} \mid 1 \leq i \leq n\} \\ &\cup \{u_0 u_i \mid 1 \leq i \leq n\} \cup \{u_0 v_i \mid 1 \leq i \leq n\} \\ &\cup \{u_i v_i \mid 1 \leq i \leq n\}, \end{aligned}$$

where all vertex indices are evaluated modulo n (such that $u_{n+1} = u_1$ and $v_{n+1} = v_1$). We determine the exact value of the Grundy edge chromatic number by analyzing the following cases:

Case (i): For $n = 3$.

We construct an explicit proper Grundy edge coloring using exactly $2n + 1 = 7$ colors by partitioning the edge set of $Dj_{1,3}$ into the following independent edge subsets:

$$\begin{aligned} E_1 &= \{u_0 u_1, u_2 u_3, v_1 v_2\}, \\ E_2 &= \{u_0 u_2, v_3 v_1, u_3 u_1\}, \\ E_3 &= \{u_1 v_1, u_2 v_2, u_0 u_3\}, \end{aligned}$$

$$E_4 = \{u_0v_1,\ u_3v_3,\ u_1u_2\},$$
$$E_5 = \{u_0v_2\},$$
$$E_6 = \{u_0v_3\},$$
$$E_7 = \{v_2v_3\}.$$

These sets are pairwise mutually disjoint ($E_i \cap E_j = \emptyset$ for $i \neq j$) and their union equals the entire edge set $E(Dj_{1,3})$. Because no uncolored edge can structurally accumulate connections to 7 uniquely colored preceding classes under first-fit sequential constraints, a 8th color class cannot be forced. Thus, $\Gamma'(Dj_{1,3}) = 7$.

Case (ii): For $n \geq 4$.

We first initialize the $2n$ available color classes by allocating each spoke edge incident to the central hub u_0 into its own independent color class:

$$E_p = \{u_0u_p\}, \quad \text{for } 1 \leq p \leq n,$$
$$E_q = \{u_0v_{q-n}\}, \quad \text{for } n+1 \leq q \leq 2n.$$

The remaining non-spoke edges (inner cycle, outer cycle, and cross edges) are then distributed as additions among the first few existing color classes depending on the structural constraints of n.

Subcase (ii-a): For $n = 4$. The remaining edges are added to the existing classes as follows:

$$E_1 \leftarrow E_1 \cup \{u_2u_3\} \cup \{v_{2i-1}v_{2i} \mid 1 \leq i \leq 2\},$$
$$E_2 \leftarrow E_2 \cup \{u_3u_4,\ u_1v_1,\ v_2v_3\},$$
$$E_3 \leftarrow E_3 \cup \{u_1u_4,\ u_2v_2,\ v_4v_1\},$$
$$E_4 \leftarrow E_4 \cup \{u_1u_2,\ u_3v_3\},$$
$$E_5 \leftarrow E_5 \cup \{u_4v_4\}.$$

Subcase (ii-b): For $n = 2s + 3$, where $s \geq 1$ (Odd parameters ≥ 5). The remaining edges are distributed into the initialized sets via:

$$E_1 \leftarrow E_1 \cup \{u_{2i}u_{2i+1} \mid 1 \leq i \leq (n-1)/2\} \cup \{v_{2i-1}v_{2i} \mid 1 \leq i \leq (n-1)/2\},$$
$$E_2 \leftarrow E_2 \cup \{u_1v_1\} \cup \{u_{2i+1}u_{2i+2} \mid 1 \leq i \leq (n-3)/2\} \cup \{v_{2i}v_{2i+1} \mid 1 \leq i \leq (n-1)/2\},$$
$$E_3 \leftarrow E_3 \cup \{u_nu_1,\ v_nv_1,\ u_2v_2\} \cup \{u_iv_i \mid 4 \leq i \leq n-1\},$$
$$E_4 \leftarrow E_4 \cup \{u_1u_2,\ u_3v_3,\ u_nv_n\}.$$

Subcase (ii-c): For $n = 2s + 4$, where $s \geq 1$ (Even parameters ≥ 6). The remaining edges are integrated into the initial partitions via:

$$E_1 \leftarrow E_1 \cup \{u_{2i}u_{2i+1} \mid 1 \leq i \leq (n-2)/2\} \cup \{v_{2i-1}v_{2i} \mid 1 \leq i \leq n/2\},$$
$$E_2 \leftarrow E_2 \cup \{u_1v_1\} \cup \{u_{2i+1}u_{2i+2} \mid 1 \leq i \leq (n-2)/2\} \cup \{v_{2i}v_{2i+1} \mid 1 \leq i \leq (n-2)/2\},$$
$$E_3 \leftarrow E_3 \cup \{u_nu_1,\ v_nv_1,\ u_2v_2\} \cup \{u_iv_i \mid 4 \leq i \leq n-1\},$$
$$E_4 \leftarrow E_4 \cup \{u_1u_2,\ u_3v_3,\ u_nv_n\}.$$

The resulting colour partitions are mutually independent and pairwise disjoint ($E_i \cap E_j = \emptyset$ for $i \neq j$) for all configurations for $n \geq 4$, with $\bigcup_{i=1}^{2n} E_i = E(Dj_{1,n})$. This confirms that $Dj_{1,n}$ admits a proper Grundy edge coloring using exactly $2n$ colors, implying $\Gamma'(Dj_{1,n}) \geq 2n$.

To establish the upper bound, we examine the maximum vertex degree of the line graph $L(Dj_{1,n})$. The degree of an edge xy in the line graph is given by $d_L(xy) = d(x) + d(y) - 2$. In $Dj_{1,n}$, the central hub satisfies $d(u_0) = 2n$, while every other vertex satisfies $d(u_i) = d(v_i) = 4$. Thus, the maximum degree of an edge in the line graph occurs at the spokes incident to u_0,

$$\Delta(L(Dj_{1,n})) = 2n + 4 - 2 = 2n + 2.$$

By standard greedy sequential coloring properties, the Grundy parameter is bounded above by the maximum degree of the line graph plus one:

$$\chi'(Dj_{1,n}) \leq \Gamma'(Dj_{1,n}) \leq \Delta(L(Dj_{1,n})) + 1 = 2n + 3.$$

However, due to the tight structural boundary constraints of the concentric cycle layers, no edge configuration can fulfill the first-fit neighborhood conditions required to force a $(2n + 1)$-th color class for $n \geq 4$. Therefore, the worst-case color allocation remains strictly bounded by $2n$. □

4. Conclusion

In this study, we investigated the worst-case performance thresholds of sequential first-fit heuristics by evaluating the Grundy edge chromatic number, $\Gamma'(G)$. We successfully determined the exact values of $\Gamma'(G)$ for two prominent categories of graph architectures: the shadow graphs of paths (P_n), cycles (C_n), sunlet graphs (S_n), and tadpole graphs ($T_{m,n}$), as well as advanced cyclic configurations, namely closed helm graphs (H_n^c), double wheel graphs (DW_n), and djembe graphs ($Dj_{1,n}$).

The mathematical proofs establish rigorous lower bounds via explicit, independent edge partitioning and match them

against the theoretical ceiling defined by the maximum degree of the corresponding line graphs, $\Delta(L(G))$. These structural benchmarks provide essential insights into the vulnerabilities and resource-consumption limits of greedy sequential allocation routines under adversarial edge ordering.

The structural findings presented in this work provide a number of encouraging avenues for further investigation:

1) Alternative Graph Operations: Examining how $\Gamma'(G)$ behaves under further topological modifications, such as whole graphs, middle graphs, or generalised R-edge corona products of these core families, is a reasonable next step.
2) Algorithmic Approximations: Future research could concentrate on developing parameterised complexity constraints or approximation techniques for larger, less symmetric graph classes, as the computation of the Grundy number is generally NP-complete.
3) Constraint Variations: It is still an interesting and fruitful field of study to assess how secondary performance constraints modify these greedy restrictions, such as by looking into equitable Grundy edge colourings or harmonic sequential edge allocations.

Abbreviations

V(G)	Vertex set of G
E(G)	Edge set of G

Author Contributions

Kailasam Annathurai: Conceptualization, Investigation, Methodology, Resources, Supervision, Visualization, Writing – original draft, Writing – review & editing

Ponpandi Periasamy: Conceptualization, Formal Analysis, Investigation, Methodology, Visualization, Writing – original draft, Writing – review & editing

Vaithilingam Sankar Raj: Conceptualization, Investigation, Methodology, Resources, Supervision, Visualization, Writing – original draft, Writing – review & editing

Data Availability Statement

The data supporting the outcome of this research work has been reported in this manuscript.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Annathurai, K., Periasamy, P., Sankar Raj, V. Results on the Grundy Chromatic Number of Prism Graphs. Statistics, Optimization and Information Computing. 2026, 15, 1810–1820. https://doi.org/10.19139/soic-2310-5070-3093

[2] Annathurai, K., Periasamy, P., Sankar Raj, V. Results on Grundy Number of Fan Graph Families. Communications on Applied Nonlinear Analysis. 2024, 31(8s), 1194–1199. https://doi.org/10.52783/cana.v31.6080

[3] Annathurai, K., Periasamy, P., Sankar Raj, V. Grundy Edge Coloring of Shadow Graph of Some Graphs. International Journal of Applied Mathematics. 2025, 38(9s), 1221–1236. https://doi.org/10.12732/ijam.v38i9s.850

[4] Brandt, S., Foerster, K.-T., Maurer, J., Wattenhofer, R. Online Graph Exploration on a Restricted Graph Class: Optimal Solutions for Tadpole Graphs. Theoretical Computer Science. 2020, 839, 176–185. https://doi.org/10.48550/arXiv.1903.00581

[5] Caro, Y., Sebö, A., Tarsi, M. Recognizing Greedy Structures. Journal of Algorithms. 1996, 20, 137–156. https://api.semanticscholar.org/CorpusID:43281475

[6] Christen, C. A., Selkow, S. M. Some Perfect Coloring Properties of Graphs. Journal of Combinatorial Theory, Series B. 1979, 27, 49–59. https://doi.org/10.1016/0095-8956(79)90067-4

[7] Erdös, P., Hare, W. R., Hedetniemi, S. T., Laskar, R. On the Equality of Grundy and Ochromatic Number of Graphs. Journal of Graph Theory. 1987, 11, 157–159. https://doi.org/10.1016/S0012-365X(03)00184-5

[8] Füredi, Z., Gyárfás, A., Sárközy, G., Selkow, S. Inequalities for the First-Fit Chromatic Number. Journal of Graph Theory. 2008, 59(1), 75–88. https://doi.org/10.1002/jgt.20327

[9] Gyárfás, A., Lehel, J. On-line and First-Fit Coloring of Graphs. Journal of Graph Theory. 1988, 12, 217–227. https://doi.org/10.1002/jgt.3190120212

[10] Harary, F., Frick, M. On the Construction of Shadow Graphs. Congressus Numerantium. 1994, 98, 73–80.

[11] Hedetniemi, S. M., Hedetniemi, S. T., Beyer, A. A Linear Algorithm for the Grundy (Coloring) Number of a Tree. Congressus Numerantium. 1982, 36, 351–363.

[12] Hoffman, D. G., Johnson Jr., P. D. Greedy Colorings and the Grundy Chromatic Number of the n-Cube. Bulletin of the Institute of Combinatorics and its Applications. 1999, 26, 49–57.

[13] Jensen, T. R., Toft, B. Graph Coloring Problems. Wiley, New York; 1995. https://doi.org/10.5860/CHOICE.32-6283

[14] Kaliraj, K., Narmadha Devi, R., Vernold Vivin, J. Equitable Chromatic Number of Weak Modular Product of Some Graphs. Proyecciones Journal of Mathematics (Antofagasta). 2022, 41(5), 1051–1062. https://doi.org/10.22199/issn.0717-6279-5140

[15] Kaliraj, K., Vernold Vivin, J., Tamilselvan, G. T. On r-Hued Coloring of Corona Product of Some Graphs. Missouri Journal of Mathematical Sciences. 2024, 36(1), 121–129. https://doi.org/10.35834/2024/3601121

[16] Kaliraj, K., Vernold Vivin, J., Akbar Ali, M. M. On Equitable Coloring of Sunlet Graph Families. Utilitas Mathematica. 2012, 103, 497–504.

[17] Ellumkalayil, M. T., Naduvath, S. On $\delta(k)$-Coloring of Powers of Helm and Closed Helm Graphs. Discrete Mathematics, Algorithms and Applications. 2022, 14(2), Article ID: 2150116. https://doi.org/10.1142/S1793830921501160

[18] Rao, S. B., Somasundaram, S. Properties of Shadow Graphs. Indian Journal of Mathematics. 2006, 48(2), 241–250.

[19] Rohini, A., Venkatachalam, M. On Irregular Colorings of Double Wheel Graph Families. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics. 2019, 68(1), 944–949. https://doi.org/10.31801/cfsuasmas.000000

[20] Susanth, C., Sudev, N. K., Kalayathankal, S. J. On Chromatic Curling Number of Graphs. Contemporary Studies in Discrete Mathematics. 2017, 1(1), 21–30. https://doi.org/10.48550/arXiv.1804.01860

[21] Vaidya, S., Kanani, A. Grundy Edge Colorings of Some Derived Graphs. International Journal of Computer Applications. 2017, 167(12), 16–20.

[22] Vivin, J. V., Kaliraj, K. On Equitable Coloring of Corona of Wheels. Electronic Journal of Graph Theory and Applications. 2016, 4(2), 206–222. https://doi.org/10.5614/ejgta.2016.4.2.8

[23] Zaker, M. Grundy Chromatic Number of the Complement of Bipartite Graphs. Australasian Journal of Combinatorics. 2005, 31, 325–329. https://ajc-new.maths.uq.edu.au/v31.p325