

Multiple-Pursuer Evasion Differential Games with Higher-Order Dynamics

Hassan Abdullahi^{1, *}, Salisu Saleh¹, Bilyaminu Muhammad²

¹Department of Mathematics, Zamfara State University, Talata Mafara, Nigeria

²Department of Mathematics, Federal College of Education (Technical), Gusau, Nigeria

Email address:

hassanabdullahiwakili@gmail.com (Hassan Abdullahi), salehsalisuagp@gmail.com (Salisu Saleh),

bilyamamman@gmail.com (Bilyaminu Muhammad)

To cite this article:

Hassan Abdullahi, Salisu Saleh, Bilyaminu Muhammad. (2026). Multiple-Pursuer Evasion Differential Games with Higher-Order Dynamics. *American Journal of Applied Mathematics*, 14(5), 359-363. <https://doi.org/10.11648/j.ajce.20261405.19>

Received: 8 September 2026; **Accepted:** 8 September 2026 **Published:** 8 October 2026

Abstract: Pursuit-evasion differential games are mathematical models used to study competitive interactions between pursuers and an evader, with applications in control theory, optimization, and dynamical systems. Although several pursuit-evasion models have been studied in finite-dimensional spaces and under first- or higher-order dynamics, relatively limited attention has been given to infinite-dimensional settings involving players with different orders of dynamics. This study investigates a pursuit-evasion differential game in the Hilbert space ℓ_2 , where each pursuer is governed by an n th-order differential equation, while the evader is governed by an m th-order differential equation, with $n < m$. The controls of both the pursuers and the evader are subject to integral constraints. The main objective is to establish sufficient conditions under which the pursuers can guarantee the capture of the evader in finite time and to construct an effective pursuit strategy. To achieve this objective, the problem is formulated within the framework of differential games in ℓ_2 , and the properties of the players' higher-order dynamics, control constraints, and geometric relationships are analyzed. Sufficient conditions for guaranteed capture are derived, and a constructive, non-singular pursuit strategy is established. The results demonstrate that the pursuers can achieve capture despite the unequal orders of the players' dynamics and the infinite-dimensional nature of the state space. Consequently, the study extends existing results on first-order and higher-order pursuit-evasion differential games to infinite-dimensional Hilbert spaces with unequal dynamic orders and provides a framework for analyzing pursuit problems under integral control constraints.

Keywords: Dynamic Equations, Integral Constraint, Guaranteed Capture Time, Hilbert Space

1. Introduction

Differential game theory provides a mathematical framework for studying strategic interactions between competing players whose motions are governed by differential equations. In particular, pursuit-evasion differential games have attracted considerable attention because of their applications in control theory, interception, robotics, aerospace systems, and autonomous systems. The foundations of differential game theory can be traced to Isaacs [1], while Krasovskii and Subbotin [2] and Petrosyan [3] developed important theoretical results concerning optimal strategies, pursuit, evasion, and guaranteed outcomes.

An important development in pursuit differential games

is their extension from finite-dimensional Euclidean spaces to infinite-dimensional Hilbert spaces. Ibragimov and Mat Hasim [4] investigated pursuit and evasion differential games described by an infinite system of differential equations with countably many pursuers in a Hilbert space. Integral constraints were imposed on the controls, and conditions for pursuit and evasion were established according to the relative control resources of the pursuers and the evader. Similarly, Ja'afar and Ibragimov [5] studied pursuit and evasion differential games involving an infinite number of first-order differential equations and considered integral, geometric, and mixed constraints on the control functions.

Subsequent studies extended the theory to multiple pursuers and different types of control constraints. Adamu et al. [7]

considered pursuit differential games in the Hilbert space ℓ_2 and established sufficient conditions for the completion of pursuit under integral constraints. These developments demonstrated the importance of the geometrical structure of the state space and the available control resources in determining whether pursuit can be guaranteed.

More recently, attention has been directed toward pursuit and evasion games with generalized and nonstandard control constraints. Ibragimov *et al.* [11] studied a pursuit differential game involving many pursuers and one evader in $\mathbb{R}^n$, where the players' controls were subject to Grönwall-type constraints. They established sufficient conditions for pursuit based on the convex hull of the initial positions of the pursuers. Rilwan *et al.* [12] further investigated pursuit and evasion games with Grönwall-type constraints and obtained sufficient conditions for both pursuit and evasion. These results demonstrate that the type of control constraint plays an essential role in determining the outcome of a differential game.

The theory has also been extended to infinite-dimensional spaces with different forms of constraints. Ibragimov *et al.* [13] investigated an evasion differential game involving multiple pursuers and a single evader in the Hilbert space ℓ_2 under geometric constraints. Their results provided conditions under which the evader can avoid capture. More recently, Badakaya, Tsoho, and Salimi [14] studied several ℓ -catch pursuit differential games in ℓ_2 involving many pursuers and one evader whose dynamic equations have different orders. In particular, the pursuers were governed by second-order differential equations, whereas the evader was governed by a first-order equation. Sufficient conditions for ℓ -capture were established under integral, geometric, and mixed control constraints. These results are particularly relevant to the present study because they demonstrate the importance of considering different dynamic capabilities among the players in an infinite-dimensional setting.

In addition, recent studies have considered pursuit and evasion games with generalized integral constraints. Umar *et al.* [15] investigated pursuit and evasion differential game problems involving one pursuer and one evader, as well as many pursuers and one evader, in $\mathbb{R}^n$ under generalized integral constraints. Sufficient conditions for the completion of pursuit and evasion were obtained, and optimal strategies were constructed. Kazimirova *et al.* [16] considered a multi-pursuer and single-evader evasion differential game with integral constraints for an infinite system of binary differential equations. Their results further demonstrate the applicability of differential game methods to infinite-dimensional systems.

In the context of higher-order dynamics, Badakaya, Halliru, and Adamu [6] investigated a pursuit-evasion differential game in which the dynamics of the players were described by n th-order differential equations. Under geometric control constraints, sufficient conditions and optimal strategies for the game were obtained. Adamu, Halliru, and Abdulhamid [7] further considered a pursuit differential game in ℓ_2 in which the pursuers were governed by n th-order differential equations while the evader was governed by an m th-order differential

equation. Their results established sufficient conditions for the completion of pursuit under different control constraints.

Muhammad and Abdullahi [8] studied a pursuit differential game in a Hilbert space involving a first-order pursuer and a second-order evader under integral constraints. Their results provided sufficient conditions for pursuit at a prescribed time. In addition, Umar *et al.* [9] considered a game involving many pursuers and one evader with first- and second-order dynamics under generalized coordinate-wise integral constraints.

Of particular relevance to the present study is the recent work of Abdullahi *et al.* [10], who investigated a pursuit differential game with high-order dynamics. The authors considered finitely many pursuers and a single evader moving in a nonempty compact convex subset of $\mathbb{R}^2$. The pursuers were governed by n th-order dynamics, while the evader was governed by m th-order dynamics, with coordinate-wise integral constraints imposed on the control functions. Sufficient conditions guaranteeing capture were derived, and constructive pursuit strategies were established. More recently, Odiliobi, Mat Hasim, and Ibragimov [17] considered a coordinate-wise integral constraint pursuit game modeled by an infinite system of two-coupled differential equations. This further emphasizes the growing interest in coordinate-wise constraints and infinite-dimensional pursuit problems.

Research Gap Although considerable progress has been made in pursuit-evasion differential games involving infinite-dimensional spaces, higher-order dynamics, and various control constraints, several aspects remain insufficiently investigated. In particular, existing studies have generally considered either specific combinations of dynamic orders, such as second-order pursuers and first-order evaders, or particular forms of constraints and state spaces. A unified treatment of a general n th-order pursuit differential game in the Hilbert space ℓ_2 , where the pursuers and evader have unequal dynamic orders satisfying $n < m$ and their controls are subject to integral constraints, remains limited. Moreover, the construction of a non-singular and constructive pursuit strategy under these general assumptions requires further investigation. Therefore, this study seeks to extend the existing results by establishing sufficient conditions for guaranteed capture and constructing a non-singular pursuit strategy for an n th-order pursuit differential game in the infinite-dimensional Hilbert space ℓ_2 .

2. Statement of Problems

We consider a differential game with many pursuers P_i ($i \in I = \{1, 2, \dots\}$) and a single evader E , whose state vectors are x_i and y , and whose controls are u_i and v , respectively, in the Hilbert space ℓ_2 .

The dynamics of the players are governed by the following n th- and m th-order differential equations with initial conditions:

$$\begin{aligned} P_i : \frac{d^n x_i}{dt^n} &= u_i(t), \quad x_i^{(j)}(0) = x_i^j, \quad j = 0, 1, \dots, n-1, \quad i \in I, \\ E : \frac{d^m y}{dt^m} &= v(t), \quad y^{(j)}(0) = y^j, \quad j = 0, 1, \dots, m-1, \end{aligned} \quad (1)$$

where $n, m \in \mathbb{N}$ with $n < m$. Here, $x_i^j, u_i(t) \in \ell_2$ and $y^j, v(t) \in \ell_2$. The controls $u_i(t) = (u_{i1}(t), u_{i2}(t), \dots)$ and $v(t) = (v_1(t), v_2(t), \dots)$ are Lebesgue measurable functions of time $t \geq 0$, taking values in ℓ_2 .

The admissible controls $u_i(\cdot) : [0, \theta] \rightarrow \ell_2$ and $v(\cdot) : [0, \theta] \rightarrow \ell_2$ are subject to the following integral constraints:

$$\left(\int_0^\theta \|u_i(t)\|^2 dt \right)^{\frac{1}{2}} \leq \rho_i, \quad \left(\int_0^\theta \|v(t)\|^2 dt \right)^{\frac{1}{2}} \leq \sigma, \quad (2)$$

where ρ_i and σ are given positive numbers representing the control resource bounds of the pursuers and the evader, respectively.

By solving the linear initial value problems in (1), the state of each player at the terminal time θ is given by the following explicit formulae:

$$\begin{aligned} x_i(\theta) &= \sum_{j=0}^{n-1} \frac{\theta^j}{j!} x_i^j + \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!} u_i(t) dt, \\ y(\theta) &= \sum_{j=0}^{m-1} \frac{\theta^j}{j!} y^j + \int_0^\theta \frac{(\theta-t)^{m-1}}{(m-1)!} v(t) dt. \end{aligned} \quad (3)$$

For brevity, we define the initial state contributions at time θ as:

$$\begin{aligned} x_{i0} &:= \sum_{j=0}^{n-1} \frac{\theta^j}{j!} x_i^j, \\ y_0 &:= \sum_{j=0}^{m-1} \frac{\theta^j}{j!} y^j. \end{aligned}$$

Thus, (3) simplifies to:

$$x_i(\theta) = x_{i0} + \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!} u_i(t) dt \quad (4)$$

$$y(\theta) = y_0 + \int_0^\theta \frac{(\theta-t)^{m-1}}{(m-1)!} v(t) dt. \quad (5)$$

Definition 2.1. A measurable function $u_i(\cdot)$ satisfying (2) is called an admissible control of pursuer P_i . Similarly, a measurable function $v(\cdot)$ satisfying (2) is called an admissible control of evader E . The main objective of the pursuers is to guarantee capture of the evader.

Definition 2.2. A strategy of the i th pursuer is a function

$$U_i(t, x_i, y, v) : [0, \infty) \times \ell_2 \times \ell_2 \times \ell_2 \rightarrow \ell_2$$

such that the closed-loop system has a unique solution $(x_i(\cdot), y(\cdot))$ for every admissible control $v(\cdot)$ of the evader. A strategy is admissible if every control generated by it satisfies the integral constraint (2).

Definition 2.3. The pursuers guarantee capture in the differential game (1)–(2) if there exists an admissible strategy U_i ensuring the equality $x_i(\theta) = y(\theta)$ for some $i \in I$ and some finite time $\theta > 0$.

3. The Main Result

We now present the main result of this paper, which provides sufficient conditions for capture and constructs an explicit, non-singular pursuit strategy.

Theorem 3.1. Consider the differential game (1)–(2) with $n < m$. Fix a terminal time $\theta > 0$. If the control resource ρ_i of pursuer P_i satisfies the strict inequality

$$\rho_i > \frac{n \|y_0 - x_{i0}\| (n-1)!}{\theta^{n-\frac{1}{2}}} + \frac{(n-1)!}{(m-1)!} \theta^{m-n} \sigma, \quad (6)$$

then pursuer P_i can guarantee capture of the evader at the prescribed time θ .

Proof. We prove the theorem by constructing an admissible strategy for the pursuer that explicitly forces the terminal states to coincide, regardless of the evader's actions.

Step 1: Construction of the Pursuit Strategy. For an arbitrary admissible evader control $v(\cdot)$, we define the following strategy for pursuer P_i :

$$U_i(t, v(t)) = \frac{n(y_0 - x_{i0})(n-1)!}{\theta^n} + \frac{(\theta-t)^{m-n}(n-1)!}{(m-1)!} v(t), \quad (7)$$

$$0 \leq t \leq \theta.$$

This strategy is constructed so that the first term compensates for the initial state difference, while the second term actively cancels the effect of the evader's control.

Step 2: Proving Admissibility of the Strategy. To be admissible, the control $U_i(\cdot)$ must satisfy the integral constraint (2). We must show that $\|U_i\|_{L^2([0, \theta]; \ell_2)} \leq \rho_i$.

First, observe that (6) is the sum of two terms. Let

$$\begin{aligned} A &:= \frac{n(y_0 - x_{i0})(n-1)!}{\theta^n}, \\ B(t) &:= \frac{(\theta-t)^{m-n}(n-1)!}{(m-1)!} v(t). \end{aligned}$$

Since A is constant in time, its L^2 -norm is computed directly as follows:

$$\begin{aligned} \|A\|_{L^2}^2 &= \int_0^\theta \left\| \frac{n(y_0 - x_{i0})(n-1)!}{\theta^n} \right\|_{\ell_2}^2 dt \\ &= \frac{n^2 \|y_0 - x_{i0}\|_{\ell_2}^2 ((n-1)!)^2}{\theta^{2n}} \int_0^\theta dt \\ &= \frac{n^2 \|y_0 - x_{i0}\|_{\ell_2}^2 ((n-1)!)^2}{\theta^{2n-1}}. \end{aligned}$$

Taking square roots yields:

$$\|A\|_{L^2} = \frac{n \|y_0 - x_{i0}\| (n-1)!}{\theta^{n-\frac{1}{2}}}. \quad (7)$$

Next, we bound the L^2 -norm of the time-dependent term $B(t)$. Since $m > n$, the exponent $m-n$ is strictly positive. Therefore, for all $t \in [0, \theta]$, we have $(\theta-t) \leq \theta$, which implies $(\theta-t)^{m-n} \leq \theta^{m-n}$. Using this bound and the evader's constraint (2), we obtain:

$$\begin{aligned}
\|B\|_{L^2}^2 &= \int_0^\theta \left\| \frac{(\theta-t)^{m-n}(n-1)!}{(m-1)!} v(t) \right\|^2 dt \\
&= \frac{((n-1)!)^2}{((m-1)!)^2} \int_0^\theta (\theta-t)^{2(m-n)} \|v(t)\|^2 dt \\
&\leq \frac{((n-1)!)^2}{((m-1)!)^2} \int_0^\theta \theta^{2(m-n)} \|v(t)\|^2 dt \\
&= \frac{((n-1)!)^2}{((m-1)!)^2} \theta^{2(m-n)} \int_0^\theta \|v(t)\|^2 dt \\
&\leq \frac{((n-1)!)^2}{((m-1)!)^2} \theta^{2(m-n)} \sigma^2.
\end{aligned}$$

Taking square roots yields:

$$\|B\|_{L^2} \leq \frac{(n-1)!}{(m-1)!} \theta^{m-n} \sigma. \quad (8)$$

Now, by Minkowski's inequality (the triangle inequality for the L^2 -norm), we have:

$$\|U_i\|_{L^2} = \|A + B\|_{L^2} \leq \|A\|_{L^2} + \|B\|_{L^2}.$$

Substituting the explicit bounds (7) and (8) into the inequality above, we get:

$$\|U_i\|_{L^2} \leq \frac{n\|y_0 - x_{i0}\| (n-1)!}{\theta^{n-\frac{1}{2}}} + \frac{(n-1)!}{(m-1)!} \theta^{m-n} \sigma.$$

By the hypothesis of the theorem (5), the right-hand side is strictly less than ρ_i . Hence,

$$\|U_i\|_{L^2} \leq \rho_i,$$

which proves that the strategy (6) satisfies the integral constraint. Therefore, U_i is an admissible strategy.

Step 3: Proving Capture. It remains to show that using this admissible strategy forces the pursuer's terminal state to coincide with the evader's terminal state. Substitute the strategy (6) into the pursuer's state equation (4).

We have:

$$\begin{aligned}
x_i(\theta) &= x_{i0} + \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!} \left[\frac{n(y_0 - x_{i0})(n-1)!}{\theta^n} \right. \\
&\quad \left. + \frac{(\theta-t)^{m-n}(n-1)!}{(m-1)!} v(t) \right] dt.
\end{aligned}$$

We evaluate the integral term by term.

First, consider the term involving the initial difference $y_0 - x_{i0}$:

$$\begin{aligned}
&\int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!} \frac{n(y_0 - x_{i0})(n-1)!}{\theta^n} dt \\
&= \frac{n(y_0 - x_{i0})}{\theta^n} \int_0^\theta (\theta-t)^{n-1} dt.
\end{aligned}$$

A direct integration yields:

$$\int_0^\theta (\theta-t)^{n-1} dt = \left[-\frac{(\theta-t)^n}{n} \right]_0^\theta = \frac{\theta^n}{n}.$$

Substituting this result gives:

$$\frac{n(y_0 - x_{i0})}{\theta^n} \cdot \frac{\theta^n}{n} = y_0 - x_{i0}.$$

Second, consider the term involving the evader's control $v(t)$:

$$\int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!} \frac{(\theta-t)^{m-n}(n-1)!}{(m-1)!} v(t) dt = \int_0^\theta \frac{(\theta-t)^{m-1}}{(m-1)!} v(t) dt.$$

Combining these results, the pursuer's terminal state simplifies to:

$$\begin{aligned}
x_i(\theta) &= x_{i0} + (y_0 - x_{i0}) + \int_0^\theta \frac{(\theta-t)^{m-1}}{(m-1)!} v(t) dt \\
&= y_0 + \int_0^\theta \frac{(\theta-t)^{m-1}}{(m-1)!} v(t) dt.
\end{aligned}$$

By the state equation for the evader given in (4), the right-hand side is exactly $y(\theta)$. Therefore, we have established that

$$x_i(\theta) = y(\theta).$$

Thus, irrespective of the evader's admissible control $v(\cdot)$, pursuer P_i achieves exact capture at the prescribed time θ . This completes the proof of the theorem.

4. Conclusion

In this paper, we studied an n th-order pursuit differential game in the Hilbert space ℓ_2 , where the evader obeys an m th-order dynamic with $n < m$. Under integral constraints on the controls, we derived a sufficient condition on the pursuer's control resource ρ_i to guarantee capture. A constructive, non-singular pursuit strategy was proposed, and its admissibility was rigorously proven via exact computation of the L^2 -norms and application of Minkowski's inequality. The result corrects and extends earlier works by removing the singularities associated with higher-order dynamics and providing a dimensionally consistent capture condition. Future work may explore geometric constraints, mixed constraints, or the evasion problem when the sufficient condition fails.

Acknowledgments

The authors sincerely acknowledge the Tertiary Education Trust Fund (TETFund), Nigeria, for providing financial support for this research under the TETFund Institutional-Based Research (IBR) Grant.

Author Contributions

Hassan Abdullahi: Conceptualization, Methodology, Resources

Salisu Saleh: Writing – review & editing

Bilyaminu Muhammad: Formal Analysis

Conflicts of Interest

The authors declare no conflicts of interest.

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