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A Comprehensive Examination of CVaR and bPOE in Common Probability Distributions: Applications in Portfolio Optimization and Density Estimation

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Abstract: This study examines portfolio risk assessment in finance using advanced quantile and superquantile techniques. The portfolio analyzed consists of widely recognized stocks, including Apple (AAPL), Microsoft (MSFT), Alphabet (GOOGL), and Tesla (TSLA). The primary objective is to enhance the accuracy and robustness of financial risk evaluation for such a portfolio. To achieve this, we developed innovative methods for computing quantiles and superquantiles, leveraging various probability distributions. In particular, we explored the Exponential, Gumbel, Frchet, and α-stable distributions to model the returns of the selected equities. The parameters of these distributions were estimated based on historical financial data over a defined time horizon. By doing so, we aimed to better capture the statistical characteristics of asset returns and their tail behavior, which is crucial for effective risk management. The findings reveal that these advanced quantile and superquantile approaches provide deeper insights into the potential risks associated with the portfolio. Compared to traditional risk metrics, such as Value at Risk (VaR) and Expected Shortfall (ES), the proposed methodologies offer a more refined evaluation of extreme losses and downside risks. Additionally, the study highlights how different distributional assumptions impact risk estimates, demonstrating the importance of selecting appropriate models for financial data analysis. Furthermore, we illustrate how these improved risk assessment techniques can assist investors and portfolio managers in making more informed decisions regarding risk exposure. By integrating these methods into risk management frameworks, financial professionals can enhance their ability to anticipate and mitigate adverse market conditions. Ultimately, this research contributes to the ongoing efforts to refine quantitative finance tools, ensuring more reliable and data-driven decision-making in portfolio management.

Keywords: Superquantil, Quantile, Portfolio Risk Assessment, Density Estimation, Financial, Probability Distributions, Portfolio Optimization, Stable Distribution

1. Introduction

Financial risk assessment and management remain at the forefront of concerns for investors, portfolio managers, and finance researchers. Diversifying assets within a portfolio is one of the most common strategies to mitigate risks. However, precise risk measurement continues to be an ongoing challenge, especially in an ever-evolving financial landscape.

When confronted with randomness and uncertainty, many of the prevailing techniques used to address such unpredictability follow a parametric approach. In the case of a real-valued random variable X, the analysis becomes significantly more straightforward by assuming that X can be categorized within a specific parametric distribution family. Consider, for instance, the Method of Moments, a straightforward and widely adopted approach for parametric density estimation. Nonetheless, these techniques often necessitate that specific attributes of the distribution family can be adequately represented through a simple mathematical expression, ideally in a closed form. For example, the traditional Method of Moments relies on having closed-form expressions for the moments of the parametric distribution family. Similarly, the Matching of Quantiles procedure, as delineated by [11, 29], hinges on the utilization of expressions for the quantile function. In the realm of portfolio optimization, the availability of simple expressions for the mean and variance of portfolio returns simplifies the formulation of a Markowitz portfolio optimization problem. To address various challenges, the application of parametric methods is contingent upon the existence of a closed-form expression for a specific characteristic of the parametric distribution family.

Fortunately, closed-form expressions are accessible for certain frequently used characteristics of various distributions. These encompass features like moments, quantiles, and the cumulative distribution function (CDF). Nevertheless, in the past two decades, novel fundamental characteristics, such as the superquantile, have surfaced from the field of quantitative risk management, finding extensive applications in disciplines like financial engineering, civil engineering, and environmental engineering (as evidenced in works like [6, 25, 27]). Additionally, closed-form expressions for these features, designed to fit a broad range of commonly used parametric distribution families, have not been widely disseminated. While they initially emerged from specific engineering applications, many of these features exhibit a high degree of generality and can be viewed as fundamental properties of a random variable, much like the mean or quantile. Consequently, integrating these features into parametric methods is a logical proposition. However, in order to make their application more practical, we need to formulate closed-form expressions.

Our primary objective is to formulate expressions for the quantile and superquantile across a range of distribution families. Recent advancements in financial risk theory have placed a strong emphasis on measuring tail risk. The concept of a coherent risk measure was introduced by [2], and the superquantile, also known as Conditional Value at Risk (CVaR) in financial literature, was introduced by [25]. This measure has come to be regarded as a preferred characterization of tail risk when compared to the quantile or Value-at-Risk (VaR). While some closed-form expressions exist for utilizing the superquantile in parametric procedures (as seen in works such as [1, 15, 25]), the range of distributions considered within each of these sources is restricted.

In this context, our study aims to push the boundaries of portfolio risk measurement by proposing innovative approaches to assess quantiles and superquantiles. We consider a portfolio consisting of stocks from four technology giants: Apple Inc. (AAPL), Microsoft Corporation (MSFT), Alphabet Inc. (GOOGL), and Tesla, Inc. (TSLA). These stocks represent major assets in the technology and automotive sectors, and their performances significantly impact the portfolios of investors worldwide.

The fundamental innovation of our approach lies in the use of specific probability distributions to model the returns of these stocks. We explore Exponential, Gumbel, Frechet, and α-stable distributions for this modeling. By applying these new methodologies, we seek to obtain a more accurate assessment of the financial risks associated with such a dynamic asset portfolio.
Our approach is based on the idea that understanding quantiles and superquantiles is essential for anticipating risk scenarios and making informed portfolio management decisions. The new approaches we present allow for a better grasp of extreme situations and potential shocks to which the portfolio may be exposed.

In the remainder of this article, we will describe these new methodologies in detail, highlighting their advantages over conventional approaches. We will then present the results of our application to the AAPL, MSFT, GOOGL, and TSLA portfolio, illustrating how these approaches enhance financial risk management. This study opens new perspectives for investment decision-making and portfolio management in a complex and ever-evolving financial environment.

2. Preliminaries and Notations

2.1. Value-at-risk and Conditional Value at Risk

When dealing with the optimization of tail probabilities, it is common to encounter constraints or objectives related to the probability of exceeding (POE) a certain threshold, $\mathbb{P}_x^{(X)} = \mathbb{P}(X > x)$, or its associated quantile of order ψ (Value-at-risk $VaR_\psi(X)$ or $q_\psi^{(X)}$):

$$VaR_\psi(X) = q_\psi^{(X)} = \min\{x \in \mathbb{R} \mid \mathbb{P}(X \leq x) \geq \psi\} \tag{1}$$

with $0 < \psi < 1$ which designates the probability level.

A notable breakthrough occurred in the work of [25, 26] when they introduced a new concept known as the superquantile or Conditional Value at Risk (CVaR). The CVaR serves as a measure of uncertainty akin to the VaR, albeit possessing superior mathematical characteristics. In precise terms, the Conditional value-at-risk (CVaR) or superquantile for a variable X is defined as,

$$\begin{aligned} CVaR_\psi(X) &= \tilde{q}_\psi^{(X)} = E[X \mid X > q_\psi^{(X)}] \\ &= \frac{1}{1-\psi}\int_{q_\psi^X}^{+\infty} x f(x)dx \\ &= \frac{1}{1-\psi}\int_{\psi}^{1} q_p^{(X)} dp. \end{aligned} \tag{2}$$

Analog to $q_\psi^{(X)}$, the superquantile (CVaR) can be employed to evaluate the distribution's tail. However, the CVaR is notably more manageable within optimization scenarios. Additionally, it possesses a crucial attribute by accounting for the magnitude of events within the tail.

[27] has introduced the concept of buffered probability

in the context of structural design and optimization, this is often referred to as the Buffered Probability of Failure (bPOF). Specifically, for a continuously distributed variable X, bPOE at the threshold x is defined as follows, where supX represents the essential supremum of X and threshold $x \in [E[X]; supX]$.

$$\tilde{\mathbb{P}}_x^{(X)} = \{1-\psi \mid \tilde{q}_\psi^{(X)} = x\}. \tag{3}$$

Much like the CVaR, bPOE serves as a more resilient metric for assessing tail risk since it takes into account not only the likelihood of losses/events surpassing the threshold x but also the magnitude of these prospective events. Furthermore, similar to the CVaR, bPOE can be expressed as the singular minimum point of a one-dimensional convex optimization problem, as described by [22] and [16] in the following formulas, where $[A]^+ = \max\{0, A\}$.

$$\tilde{\mathbb{P}}_x^{(X)} = \min_{\lambda \geq 0} E[\lambda(X-x)+1]^+ = \min_{\tau < x} \frac{E[X-\tau]^+}{x-\tau} \tag{4}$$

$$\tilde{q}_\psi^{(X)} = \min_{\tau} \tau + \frac{E[X-\tau]^+}{1-\psi}. \tag{5}$$

It's important to emphasize that these formulas are applicable to random variables with real values in a general sense, not exclusively to those with continuous distributions. Additionally, it's worth mentioning that the argmin value obtained from both the optimization formulas for bPOE and superquantile corresponds to the quantile. In the formula used to calculate bPOE, the argmin is given by $\tau^* = q_\psi^{(X)}$ where $\psi = 1 - \tilde{\mathbb{P}}_x^{(X)}$ and $\lambda^* = \frac{1}{1-\tau^*}$ for the other representation. In the formula used to compute the superquantile, the argmin corresponds to $\tau^* = q_\psi^{(X)}$ where ψ was the desired probability level for calculating the CVaR.

2.2. Characterization of a α -Stable Distribution

Definition 2.1. Let X be a random variable, X is called to be a stable law or $\alpha-$stable distribution random variable if $\forall(a,b) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, $\exists c > 0$ and $k \in \mathbb{R}$ such that:

$$aX_1 + bX_2 \overset{d}{=} cX + k. \tag{6}$$

With X_1 and X_2 are two random independent variable copies of X;

$\overset{d}{=}$: designates convergence in distribution.

If $k = 0$ then the distribution is strictly stable.

Definition 2.2. A random variable X is said to have a α-stable distribution if and only if, for any integer $n \geq 1$ and for any family $X_1, X_2, \cdots, X_n$ of i.i.d random variables of the same law as X, $\exists(a_n, b_n) \in \mathbb{R}_+^* \times \mathbb{R}$ such that:

$$\frac{(X_1 + X_2 + \cdots + X_n) - b_n}{a_n} \overset{d}{=} X. \tag{7}$$

Variables with a Levy-stable distribution have the disadvantage of not having (except in three cases) explicit forms for the probability density and the distribution function.

Definition 2.3. A random variable X with a α -stable law is typically described by its characteristic function Δ_X defined on $\mathbb{R}$ by:

$$\Delta_X(t) = E[\exp(itx)] = \exp(i\mu t - g_{\alpha,\beta,\sigma}(t)) \tag{8}$$

where:

$$g_{\alpha,\beta,\sigma}(t) = \begin{cases} \sigma^\alpha |t|^\alpha [1 - i\beta sign(t) \tan(\frac{\pi\alpha}{2})] & \text{if } \alpha \neq 1 \\ \sigma|t|[1 + \frac{2}{\pi} i\beta sign(t)(\log|t|] & \text{if } \alpha = 1 \end{cases}$$

$$\text{with} \quad sign(t) = \frac{t}{|t|} = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \end{cases}$$

and having several representations according to the different parameterizations of the stable laws. The most famous of these representations is given in [28] and [19].

The $\alpha-$stable law is thus characterized by four real parameters $\Psi = (\alpha, \beta, \mu, \sigma)$. The parameter α, called characteristic exponent or stability index, is an indicator of the degree of thickness of the tails of the distribution: the smaller it is, the thicker the tails are which corresponds to very large fluctuations. It is the most important parameter, it is between 0 and 2 ($0 < \alpha \leq 2$). Its maximum value $\alpha = 2$, corresponds to a particular stable law: the Gaussian law or normal law. β is the parameter of dissymmetry, it varies between -1 and 1 ($-1 \leq \beta \leq 1$) and when it is null, the distribution is symmetrical with respect to μ. When α approaches 2, β loses its effect leading to a trend towards the normal distribution.The parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ represent the usual characteristics of position and scale respectively with the remark that for the Gaussian distribution, the standard deviation is $\sigma\sqrt{2}$. Finally, a random variable X of stable distribution will be noted, according to [28], by: $X \sim S_\alpha(\beta, \mu, \sigma)$.The three exceptions mentioned above are the very famous Gaussian law $S_2(0, \mu, \sigma)$. and the less known Cauchy's law $S_1(0, \mu, \sigma)$. and Levy's law $S_{\frac{1}{2}}(1, \mu, \sigma)$.The stable law has an additivity property according to which the sum of two independent stable random variables of the same stability index α is still stable with the same characteristic exponent α. This very interesting property is used in finance to study portfolios where two assets with the same value for α can be considered together. One of the particularities of the stable distribution is that it has infinite variance as soon as α is strictly less than 2. In fact, the moments of order $p \in \mathbb{N}$ of $X \sim S_\alpha(\beta, \mu, \sigma)$ are such that for $\alpha = 2$, $E|X|^p < +\infty, \forall p$.

$$E|X|^p = \begin{cases} < \infty & \text{if } p < \alpha > 0 \\ = \infty & \text{if } p \geq \alpha. \end{cases}$$

More precisely, it is shown that (see for example [28])

$$\lim_{t\to\infty} t^\alpha \mathbb{P}(X > t) = C_\alpha \frac{1+\beta}{2} \sigma^\alpha;$$

$$\lim_{t\to\infty} t^\alpha \mathbb{P}(X < -t) = C_\alpha \frac{1-\beta}{2} \sigma^\alpha$$

where C_α is a constant given by:

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x dx \right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)} & \text{if } \alpha \neq 1, \\ \frac{2}{\pi} & \text{if } \alpha = 1. \end{cases}$$

with $\Gamma(\theta)$ is the Euler gamma function defined for $\theta > 0$, by:

$$\Gamma(\theta) = \int_0^{+\infty} x^{\theta-1} e^{-x} dx. \tag{9}$$

We can thus see that the stable law takes into account the distribution tails which are often carriers of essential information, whereas the Gaussian law neglects these tails thus leading to an error which can be fatal for the investor. The disadvantage of the characteristic function 8 is that it is not continuous if $\alpha = 1$ which makes it not adapted to numerical calculations and for these reasons [31] proposed another parameterization called S_α^0 which is usable for numerical calculations. simulate stable laws, there is an algorithm developed by [3]. This one allows to generate a law $S_\alpha(\beta, 0, 1)$. To obtain a law $S_\alpha(\beta, \mu, \sigma)$, with $\alpha \in]0, 2]$ and $\beta \in [-1, 1]$. The parameters α and σ for this generator are very well estimated by the method of [17]. The parameters μ and β are correctly estimated by the method of [17] for small values of β, which is often the case for stock exchange chronicles. A bibliography of methods for estimating the parameters of a α-stable law has been compiled by [4, 5, 13, 14, 24, 30] etc...

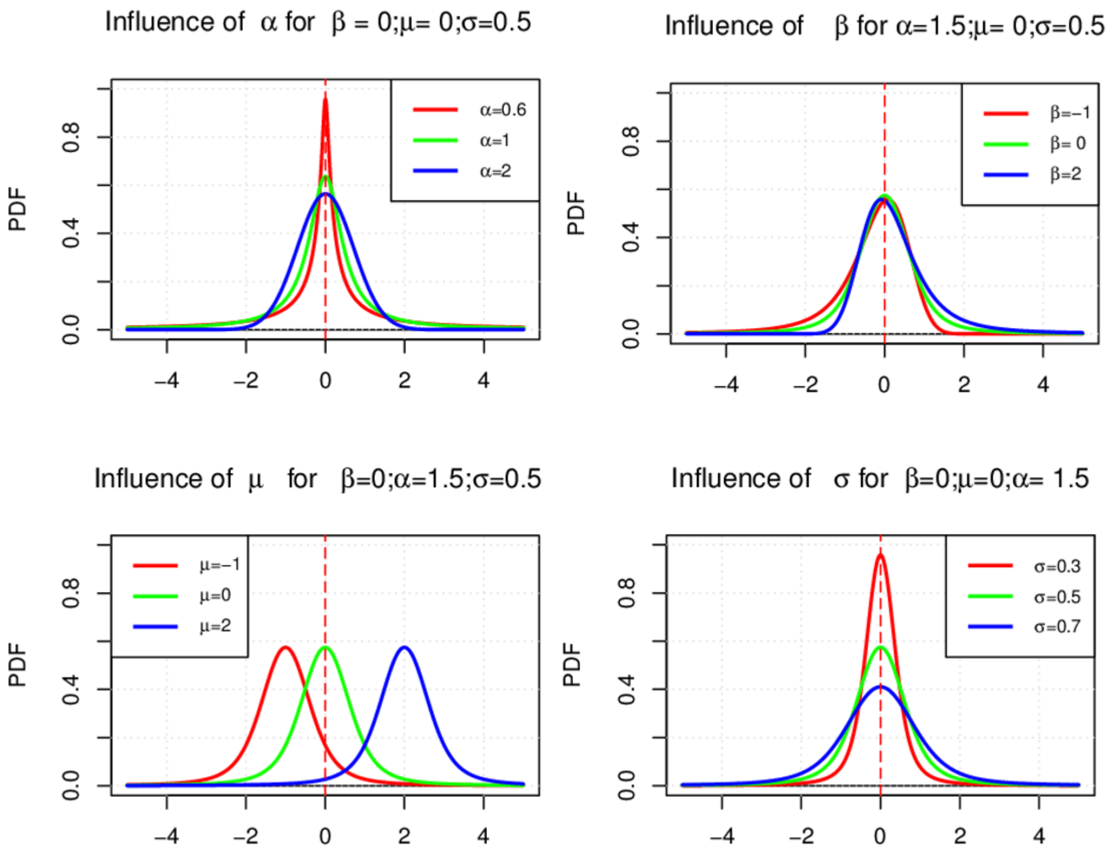

Figure 1. *Influences of the parameters of the α-stable distribution on its PDF.*

The PDF of a standard random variable α-stable law in the S^0 representation (see [18] for more details) i.e. $X \sim S_\alpha^0(1, \beta, 0)$.

The figure 1 illustrates the influence of each parameter of the α-stable distribution on its probability density function (PDF).

3. Main Results and Numerical Simulation

In this section we present the key findings of our in-depth study on portfolio risk measurement. We have developed innovative approaches to assess quantiles (VaR) and superquantiles (CVaR) using various probability distributions, including the Frechet distribution, exponential distribution, Gumbel distribution, Logistic distribution and the α-stable distribution.

Frechet's distribution

Recall that $X \sim Fr(\epsilon)$ then its CDF, PDF, expectation and variance are given by respectively:

$$F(x) = \begin{cases} e^{-x^{-\epsilon}}, & \text{for } x \geq 0 \\ 0, & \text{otherwise} \end{cases};$$

$$f(x) = \begin{cases} -\frac{\epsilon}{x^{1+\epsilon}} e^{-x^{-\epsilon}}, & \text{for } x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} E[X] &= \Gamma(1-\frac{1}{\epsilon}) \text{ for } \epsilon > 1; \\ \sigma^2[X] &= \Gamma(1-\frac{2}{\epsilon}) - (\Gamma(1-\frac{1}{\epsilon}))^2 \text{ for } \epsilon > 2. \end{aligned}$$

Proposition 3.1. Let X be a random variable such that $X \sim Fr(\epsilon)$ Frechet's law of parameter $\epsilon > 0$ then the quantile $q_\psi^{(X)}$ and the superquantile $\tilde{q}_\psi^{(X)}$ of the random variable X are:

$$q_\psi^{(X)} = (-\ln\psi)^{-\frac{1}{\epsilon}}; \quad \tilde{q}_\psi^{(X)} = \frac{-1}{1-\psi}\Gamma_L(\frac{-1}{\epsilon}+1, -\ln\psi),$$

with $\Gamma_L(u,v) = \int_0^v p^{u-1}e^{-p}dp$: the lower incomplete gamma function.

Proof. We have:

$$\begin{aligned} q_\psi^{(X)} &= \min\{x \in \mathbb{R} \mid F(x) \geq \psi\} \\ &= \min\{x \in \mathbb{R} \mid e^{-x^{-\epsilon}} \geq \psi\} \\ &= \min\{x \in \mathbb{R} \mid x \geq (-\ln\psi)^{-\frac{1}{\epsilon}}\} \\ &= (-\ln\psi)^{-\frac{1}{\epsilon}} \quad \text{and} \\ \tilde{q}_\psi^{(X)} &= \frac{1}{1-\psi}\int_\psi^1 q_p^X dp \\ &= \frac{1}{1-\psi}\int_\psi^1 (-\ln(p))^{-\frac{1}{\epsilon}} dp. \end{aligned} \tag{10}$$

Let: $y^{-\epsilon} = \ln(p) \Longrightarrow -\epsilon y^{-1-\epsilon}dy = \frac{1}{p}dp$
$\Longrightarrow dp = -p\epsilon y^{-1-\epsilon}dy = -\epsilon e^{y^{-\epsilon}} y^{-1-\epsilon}dy$

For $p = 1$ then $y = +\infty$ and for $p = \psi$ then $y = (\ln(\psi))^{\frac{-1}{\epsilon}}$. Then we have:

$$\tilde{q}_\psi^{(X)} = \frac{\epsilon}{1-\psi}\int_{(\ln(\psi))^{\frac{-1}{\epsilon}}}^{+\infty} y^{-\epsilon}e^{y^{-\epsilon}}dy. \tag{11}$$

Now let: $t = -y^{-\epsilon} \Longrightarrow dy = \frac{1}{\epsilon}t^{-1-\frac{1}{\epsilon}}dt$. From equation 11 we have:

$$\begin{aligned} \tilde{q}_\psi^{(X)} &= \frac{-1}{1-\psi}\int_0^{-\ln\psi} t^{-\frac{1}{\epsilon}}e^{-t}dt \\ &= \frac{-1}{1-\psi}\Gamma_L(\frac{-1}{\epsilon}+1, -\ln\psi). \end{aligned} \tag{12}$$

Proposition 3.2. If X is a random variable such that $X \sim FrG(s,\epsilon,m)$ is a generalized Frechet distribution, then its quantile and superquantile are:

$$\begin{aligned} q_\psi^{(X)} &= \frac{(-\ln\psi)^{-\frac{1}{\epsilon}}}{s} + m; \\ \tilde{q}_\psi^{(X)} &= \frac{-1}{s(1-\psi)}\Gamma_L(\frac{-1}{\epsilon}+1, -\ln\psi) + m. \end{aligned}$$

Proof. The distribution function of $X \sim FrG(s,\epsilon,m)$ is given by:

$$F(x) = \begin{cases} e^{-(\frac{x-m}{s})^{-\epsilon}}, & \text{for } x > m \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

We have:

$$\begin{aligned} q_\psi^{(X)} &= \min\{x \in \mathbb{R} \mid F(x) \geq \psi\} \\ &= \min\{x \in \mathbb{R} \mid e^{-(\frac{x-m}{s})^{-\epsilon}} \geq \psi\} \\ &= \min\{x \in \mathbb{R} \mid x \geq \frac{(-\ln\psi)^{-\frac{1}{\epsilon}}}{s} + m\} \\ &= \frac{(-\ln\psi)^{-\frac{1}{\epsilon}}}{s} + m. \end{aligned}$$

And we find:

$$\begin{aligned} \tilde{q}_\psi^{(X)} &= \frac{1}{1-\psi}\int_\psi^1 q_p^X dp = \frac{1}{1-\psi}\int_\psi^1 (\frac{(-\ln\psi)^{-\frac{1}{\epsilon}}}{s} + m)dp \\ &= m + \frac{1}{1-\psi}\int_\psi^1 (-\ln(p))^{-\frac{1}{\epsilon}} dp. \end{aligned} \tag{14}$$

Using the same reasoning as for solving the equation 10, we find that the equation 14 is:

$$\tilde{q}_\psi^{(X)} = m + \frac{-1}{s(1-\psi)}\Gamma_L(\frac{-1}{\epsilon}+1, -\ln\psi). \tag{15}$$

Exponential distribution
If $X \sim \exp(\phi)$ with $E(X) = \sigma(X) = \frac{1}{\phi}$ then its PDF and CDF are respectively given by:

$$f(x) = \begin{cases} \phi e^{-\phi x}; & \text{if } x \geq 0 \\ 0, & \text{otherwise,} \end{cases}$$

$$F(x) = \begin{cases} 1 - e^{-\phi x}; & \text{if } x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 3.3. Let X a random variable such that $X \sim \exp(\phi)$ with $\phi > 0$ then the quantile $q_\psi(X)$, the superquantile $\tilde{q}_\psi(X)$ and Buffered Probability of Exceedance (bPOE) $\tilde{P}_x(X)$ of X are given respectively by:

$$\begin{aligned} q_\psi(X) &= -\frac{\ln(1-\psi)}{\phi}; \\ \tilde{q}_\psi(X) &= \frac{-\ln(1-\psi)+1}{\phi}; \\ \tilde{P}_x(X) &= e^{1-\phi x}. \end{aligned}$$

Proof. We have:

$$\begin{aligned} q_\psi(X) &= \min\{x \in \mathbb{R} \mid \mathbb{P}(X \leq x) \geq \psi\} \\ &= \min\{x \in \mathbb{R} \mid F(x) \geq \psi\} \\ &= \min\{x \in \mathbb{R} \mid 1 - e^{-\phi x} \geq \psi\} \\ &= \min\{x \in \mathbb{R} \mid x \geq -\frac{\ln(1-\psi)}{\phi}\} \\ &= -\frac{\ln(1-\psi)}{\phi}. \end{aligned}$$

Similarly,

$$\begin{aligned} \tilde{q}_\psi(X) &= \frac{1}{1-\psi}\int_\psi^1 q_p(X)dp \\ &= \frac{-1}{\phi(1-\psi)}\int_\psi^1 \ln(1-p)dp \\ &= \frac{-1}{\phi(1-\psi)}\int_0^{1-\psi} \ln(y)dy \\ &= \frac{-\ln(1-\psi)+1}{\phi}. \end{aligned} \tag{16}$$

Because we have assumed that: $y = 1 - p$ and using the fact that:

$$\int y dy = y\ln(y) - y + c.$$

We can then see that:

$$\begin{aligned} \tilde{P}_x(X) &= \{1-\psi \mid \tilde{q}_\psi(X) = x\} \\ &= \{1-\psi \mid \frac{-\ln(1-\psi)+1}{\phi} = x\} \\ &= \{1-\psi \mid \ln(1-\psi) = 1 - \phi x\} \\ &= \{1-\psi \mid 1-\psi = e^{1-\phi x}\} \\ &= e^{1-\phi x}. \end{aligned}$$

Next, we establish a link between Buffered Probability of Exceedance (bPOE) and Probability of Exceedance (POE), as well as between the superquantile and the quantile.

Proposition 3.4. Let $X \sim \exp(\phi)$, with mean $\mu = \frac{1}{\phi}$. Then:

$$\tilde{P}_x(X) = \mathbb{P}(X > x - \mu); \quad \text{and} \quad \tilde{q}_\psi(X) = q_\psi(X) + \mu.$$

Proof. We know that if $X \sim \exp(\phi)$ then its cumulative distribution function is given by: $\mathbb{P}(X \leq x) = 1 - e^{-\phi x}$. From the proposition 3.3, we know that:

$$\tilde{P}_x(X) = e^{1-\phi x} = e^{-\phi(\frac{-1}{\phi}+x)}.$$

Then, as $\mu = \frac{1}{\phi}$ it follows that:

$$\tilde{P}_x(X) = e^{-\phi(x-\mu)} = 1 - \mathbb{P}(X \leq x - \mu) = \mathbb{P}(X > x - \mu).$$

From the proposition 3.3, the equality of CVaR is obtained:

$$\tilde{q}_\psi(X) = \frac{-\ln(1-\psi)+1}{\phi} = \frac{-\ln(1-\psi)}{\phi} + \frac{1}{\phi} = q_\psi(X) + \mu.$$

Gumbel's distribution

Since $X \sim G(a, b)$ then recall that its CDF and PDF are:

$$\begin{aligned} F(x) &= \exp(-\exp(-\frac{x-a}{b})); \\ f(x) &= \frac{1}{b}\exp(-\exp(-\frac{x-a}{b}))\exp(-\frac{x-a}{b}) \end{aligned}$$

$E(X) = a + b\gamma$ and $\sigma^2(X) = \frac{\pi^3 b^2}{6}$ with $\gamma \approxeq 0.5772$: Euler's constant.

Proposition 3.5. Let $X \sim G(a, b)$ Gumbel's law of parameters a and b then the quantile $q_\psi^{(X)}$, the superquantile $\tilde{q}_\psi^{(X)}$ and the probability of overshoot $\tilde{P}_x^{(X)}$ of the random variable are given by respectively:

$$q_\psi^{(X)} = -\frac{\ln(-\ln(\psi))}{b} + a; \quad \tilde{q}_\psi^{(X)} = a; \quad \tilde{P}_\psi^{(X)} = \frac{1}{e}.$$

Proof. We have:

$$\begin{aligned} q_\psi^{(X)} &= \min\{x \in \mathbb{R} \mid F(x) \geq \psi\} \\ &= \min\{x \in \mathbb{R} \mid \exp(-\exp(-\frac{x-a}{b})) \geq \psi\} \\ &= \min\{x \in \mathbb{R} \mid -\exp(-\frac{x-a}{b}) \geq \ln(\psi)\} \\ &= \min\{x \in \mathbb{R} \mid x \leq -\frac{\ln(-\ln(\psi))}{b} + a\} \\ &= -\frac{\ln(-\ln(\psi))}{b} + a. \end{aligned}$$

Then

$$\begin{aligned} \tilde{q}_\psi^{(X)} &= \frac{1}{1-\psi}\int_\psi^1 q_p^X \, dp \\ &= \frac{-1}{b(1-\psi)}\int_\psi^1 \ln(-\ln(\psi))dp + a. \end{aligned}$$

Let $y = -\ln p \Longrightarrow p = e^{-y} \Longrightarrow dp = -pdy = e^{-y}dy$

For $p = 1 \Longrightarrow y = 0$. and for $p = \psi \Longrightarrow y = -\ln\psi$. Then we have:

$$\tilde{q}_\psi^{(X)} = a + \frac{-1}{b(1-\psi)}\int_{-\ln\psi}^0 e^{-y}\ln(y)dp. \tag{17}$$

From equation 17 using the technique of integration by parts and remembering the primitive of ln:

$$\int \ln(t) = t\ln(t) - t + C$$

then we find that:

$$\tilde{q}_{\psi}^{(X)} = a.$$

$$\tilde{P}_{(x)}^{X} = \{1-\psi \mid \tilde{q}_{\psi}^{X} = x\} = \{1-\psi \mid a = x\} = F(a) = \frac{1}{e}.$$

Logistic Distribution If $X \sim \text{Logistic}(\theta, r)$ with parameters $\theta \in \mathbb{R}, r > 0$, where $E[X] = \theta$ and $\sigma^2(X) = \frac{r^2\pi^2}{3}$ and its cumulative distribution function (CDF) as well as its probability density function (PDF) are defined by:

$$F(x) = \frac{1}{1+e^{-\frac{x-\theta}{r}}}, \quad f(x) = \frac{e^{-\frac{x-\theta}{r}}}{r\left(1+e^{-\frac{x-\theta}{r}}\right)^2}.$$

Here, we derive an implicit expression for the superquantile of the logistic distribution and formulate a root-finding problem to compute the bPOE. Additionally, we observe a correspondence between these quantities and the binary entropy function.

Proposition 3.6. Let $X \sim \text{Logistic}(\theta, r)$, then

$$q_{\psi}(X) = \theta + r\ln\left(\frac{\psi}{1-\psi}\right) \tag{18}$$

$$\tilde{q}_{\psi}(X) = \theta + \frac{rH(\psi)}{1-\psi}. \tag{19}$$

Where $H(\psi)$ is the binary entropy function

$$H(\psi) = -\psi\ln(\psi) - (1-\psi)\ln(1-\psi).$$

Furthermore, for all $x \geq \theta$, if ψ is a solution of the equation,

$$\frac{H(\psi)}{1-\psi} = \frac{x-\theta}{r}$$

then $\tilde{p}_x(X) = 1-\psi$. Furthermore, $\tilde{p}_x(X) = 1-\psi$ if ψ is the solution to the transformed system,

$$(1-\psi)\psi^{\frac{\psi}{1-\psi}} = e^{-\left(\frac{x-\theta}{r}\right)}.$$

Take note of both functions $\frac{H(\psi)}{1-\psi}$ and $(1-\psi)\psi^{\frac{\psi}{1-\psi}}$ are one-dimensional, convex, and monotonic in $\psi \in [0,1]$, consequently, unique solutions exist and can be readily identified using root-finding methods.

Proof. To establish the expression of the quantile poses no problem, one simply applies the definition. As for the superquantile, we have:

$$\begin{aligned}
\tilde{q}_{\psi}(X) &= \frac{1}{1-\psi}\int_{\psi}^{1} q_p(X)dp \\
&= \frac{1}{1-\psi}\int_{\psi}^{1} \theta + r\ln\left(\frac{\psi}{1-\psi}\right)dp \\
&= \theta + \frac{r}{1-\psi}\int_{\psi}^{1} \ln(p) - \ln(1-p)dp \\
&= \theta + \frac{r}{1-\psi}\left(\int_{\psi}^{1} \ln(p)dp + \int_{\psi}^{1} -\ln(1-p)dp\right).
\end{aligned}$$

By utilizing the fact that $\int \ln(y)dy = y\ln(y) - y + C$, we find:

$$\begin{aligned}\tilde{q}_\psi(X) &= \theta + \frac{r}{1-\psi}(-1-\psi\ln\psi+\psi-(1-\psi)\ln(1-\psi)+(1-\psi)) \\ &= \theta + \frac{r}{1-\psi}(-\psi\ln\psi-(1-\psi)\ln(1-\psi)) \\ &= \theta + \frac{r}{1-\psi}H(\psi).\end{aligned}$$

In deriving the bPOE, we adhere to its definition, aiming to find ψ as a solution to $\theta + \frac{r}{1-\psi}H(\psi) = x$. The transformed system emerges from the amalgamation of logarithms in the superquantile formula and the application of exponential transformations. Moreover, the minimization formula can be employed for bPOE computation. This approach offers the extra benefit of concurrently computing the quantile $q_{1-\tilde{p}_x(X)}(X)$.

Proposition 3.7. If $X \sim \text{Logistic}(\theta, r)$, then

$$\tilde{p}_x(X) = \min_{\tau<x} \frac{r\ln\left(1+e^{-\left(\frac{\tau-\theta}{r}\right)}\right)}{x-\tau},$$

which is a convex optimization problem over $\tau \in (-\infty, x)$. Furthermore, the minimum occurs at τ such that,

$$\frac{r\ln\left(1+e^{-\left(\frac{\tau-\theta}{r}\right)}\right)}{x-\tau} = 1 - F(\tau).$$

Proof. This stems from the expression $E[X-\tau]^+ = \int_\tau^\infty (1-F(t))dt$. Upon evaluating this integral for $X \sim \text{Logistic}(\theta, r)$, we obtain $E[X-\tau]^+ = r\ln\left(1+e^{-\left(\frac{\tau-\theta}{r}\right)}\right)$, which can be inserted into the bPOE minimization formula. The second part of the proposition is deduced from the gradient of the objective function with respect to τ :

$$\frac{r\ln\left(1+e^{-\left(\frac{\tau-\theta}{r}\right)}\right)}{(x-\tau)^2} - \frac{e^{-\left(\frac{\tau-\theta}{r}\right)}}{(x-\tau)\left(1+e^{-\left(\frac{\tau-\theta}{r}\right)}\right)}.$$

Setting this gradient to zero and simplifying, we obtain the indicated optimality condition.

α-stable distribution

Proposition 3.8. Let $X \sim S^1_\alpha(\beta,\mu,\sigma)$ 1-parameterization then the quantile and superquantile are given by:

1. For $0 < \alpha < 2$ and $-1 < \beta \le 1$

$$q_\psi^{(X)} = \exp\left(-\alpha\ln\left(\frac{2\psi}{(1+\beta)C_\alpha\sigma^\alpha}\right)\right); \tilde{q}_\psi^X = \frac{2}{\alpha(1-\psi)}\left(\frac{(1+\beta)C_\alpha\sigma^\alpha}{2\psi}\right)^\alpha.$$

2. For $0 < \alpha < 2$ and $-1 \le \beta < 1$

$$q_\psi^{(X)} = \exp\left(-\alpha\ln\left(\frac{2\psi}{(1-\beta)C_\alpha\sigma^\alpha}\right)\right); \tilde{q}_\psi^X = \frac{2}{\alpha(1-\psi)}\left(\frac{(1-\beta)C_\alpha\sigma^\alpha}{2\psi}\right)^\alpha.$$

Proof. In [28] (page 18 Property 1.2.15) and [19] (page 13 Theorem 1.2) it is shown that:
For $0 < \alpha < 2$ and $-1 < \beta \le 1$ we have:

$$\lim_{x\to\infty} x^\alpha \mathbb{P}(X > x) = C_\alpha \frac{(1+\beta)}{2}\sigma^\alpha. \tag{20}$$

For $0 < \alpha < 2$ and $-1 \le \beta < 1$ we have:

$$\lim_{x\to\infty} x^\alpha \mathbb{P}(X < -x) = C_\alpha \tfrac{(1-\beta)}{2}\sigma^\alpha. \tag{21}$$

It should be noted that there is not yet an explicit expression for the cumulative distribution function of the stable distribution, but there are approximations available in some works, such as for example [19, 28]. Then from equations 20 and 21 one can approximate the cumulative distribution function with:

For $0 < \alpha < 2$; $-1 < \beta \le 1$

$$F(x,\alpha,\beta,\mu,\sigma) \approxeq C_\alpha \frac{(1+\beta)}{2}\sigma^\alpha x^{-\alpha}; \tag{22}$$

For $0 < \alpha < 2$; $-1 \le \beta < 1$

$$F(x,\alpha,\beta,\mu,\sigma) \approxeq C_\alpha \frac{(1-\beta)}{2}\sigma^\alpha x^{-\alpha}. \tag{23}$$

As a result for $0 < \alpha < 2$ and $-1 < \beta \leq 1$ we have:

$$\begin{aligned}
q_{\psi}^{(X)} &= \min\{x \in \mathbb{R} \mid F(x, \alpha, \beta, \mu, \sigma) \geq \psi\} \\
&= \min\left\{x \in \mathbb{R} \mid C_\alpha \frac{(1+\beta)}{2} \sigma^\alpha x^{-\alpha} \geq \psi\right\} \\
&= \min\left\{x \in \mathbb{R} \mid x \leq \exp\left(-\alpha \ln\left(\frac{2\psi}{(1+\beta)C_\alpha \sigma^\alpha}\right)\right)\right\} \\
&= \exp\left(-\alpha \ln\left(\frac{2\psi}{(1+\beta)C_\alpha \sigma^\alpha}\right)\right).
\end{aligned}$$

Since F is symmetrical about β so by analogy for $0 < \alpha < 2$ and $-1 \leq \beta < 1$ the quantile is:

$$q_{\psi}^{(X)} = \exp\left(-\alpha \ln\left(\frac{2\psi}{(1-\beta)C_\alpha \sigma^\alpha}\right)\right).$$

Now we need to calculate the superquantiles. Let's start with the case for $0 < \alpha < 2$ and $-1 < \beta \leq 1$ then we'll deduce the case if $0 < \alpha < 2$ and $-1 \leq \beta < 1$.

$$\begin{aligned}
\tilde{q}_{\psi}^{X} &= \frac{1}{1-\psi} \int_{\psi}^{1} q_p^{(X)}\, dp \\
&= \frac{1}{1-\psi} \int_{\psi}^{1} \exp\left(-\alpha \ln\left(\frac{2p}{(1+\beta)C_\alpha \sigma^\alpha}\right)\right) dp \\
&= \frac{((1+\beta)C_\alpha \sigma^\alpha)^\alpha}{1-\psi} \int_{\psi}^{1} \exp\left(-\alpha \ln(2p)\right) dp \\
&= \frac{2}{\alpha(1-\psi)} \left(\frac{(1+\beta)C_\alpha \sigma^\alpha}{2\psi}\right)^\alpha
\end{aligned}$$

so by analogy for $0 < \alpha < 2$ and $-1 \leq \beta < 1$ the superquantile is:

$$\tilde{q}_{\psi}^{X} = \frac{2}{\alpha(1-\psi)} \left(\frac{(1-\beta)C_\alpha \sigma^\alpha}{2\psi}\right)^\alpha.$$

4. An Approach to Portfolio Optimization Technique

Frequently, a parametric approach to optimizing portfolios involves assuming a specific distribution for portfolio returns. Particularly in a risk-averse context, having analytical expressions for the superquantile and bPOE based on the specified distribution facilitates the formulation of a practical portfolio optimization problem. In this section, we illustrate that our derived formulas for the superquantile and bPOE unveil crucial properties concerning portfolio optimization problems formulated under specific distributional assumptions for portfolio returns.

While portfolio optimization with the superquantile is widespread, we start by elucidating which analytical expressions of the superquantile allow for the formulation of feasible portfolio optimization problems. Conversely, portfolio optimization with the bPOE is uncommon, and we demonstrate its potential advantages over the superquantile approach. Specifically, optimizing with the superquantile necessitates fixing the probability level ψ. It becomes apparent that, for a fixed ψ, the optimal superquantile portfolio may vary depending on the chosen distribution to model returns. We establish that assuming portfolio returns follow a Logistic or Gumbel distribution results in identical minimal bPOE portfolios for a fixed threshold x, irrespective of the selected distribution. This implies a singular optimal portfolio in x-bPOE for various distributional choices.

It is noteworthy that in this section, we consider asset returns C, as is customary in financial problems, where the loss is the opposite of the return:

$$X = -C \quad \text{and } q_\psi(X) = -q_{1-\psi}(C).$$

The portfolio optimization problem aims to determine a vector of asset weights $C \in \mathbb{R}^n$ for a set of n assets with unknown random returns

$C = [C_1, C_2, \ldots, C_n]$, solving the following optimization problem,

$$\begin{aligned}
\max_{\delta \in \mathbb{R}^n} \quad & H(\delta, C) \\
\text{s.t.} \quad & f_{1i}(\delta, C) \leq 0, \quad i = 1, \ldots, I \\
& f_{2j}(\delta, C) = 0, \quad j = 1, \ldots, J \\
& \delta^T 1 = 1 \\
& l \leq \delta \leq u \quad .
\end{aligned} \tag{24}$$

where:

1. $H(\delta, C)$ is a function to maximize;
2. The functions $f_{2j}(\delta, C)$ and $f_{1i}(\delta, C)$ impose inequality and equality constraints, respectively;
3. And the vectors l, u impose upper and lower bounds on the individual weights of the assets.

Consider a classic example of the Markowitz optimization problem, aiming to maximize expected utility. This involves a weighted blend of expected return and its variance, incorporating a positive compromise parameter $\lambda \geq 0$:

$$\begin{aligned}
\max_{\delta \in \mathbb{R}^n} \quad & \delta^T \eta - \lambda \delta^T \Sigma \delta \\
\text{s.t.} \quad & \delta^T 1 = 1 \\
& l \leq \delta \leq u.
\end{aligned} \tag{25}$$

A crucial aspect of the stochastic portfolio return $\delta^T C$, evident in the Markowitz problem and later utilized in this section, is that the expected value $E\left[\delta^T C\right]$ and the variance $\sigma^2\left(\delta^T C\right)$ can be expressed as $\delta^T\eta$ and $\delta^T\Sigma\delta$ respectively. Here, $\eta \in \mathbb{R}^n$ denotes the vector of expected returns for the n assets, and $\Sigma \in \mathbb{R}^{n\times n}$ is the covariance matrix for the n assets. This enables the representation of the expected value and variance of the portfolio return as functions of δ, consequently allowing the formulation of an optimization problem with the decision vector δ.

4.1. Superquantile and bPOE Optimization with Qualified Distributions

As we focus on asset returns instead of losses, it is necessary to establish the superquantile using this notation. The superquantile denotes the anticipated loss above the quantile, signifying the conditional expected value of losses in the right tail. Consequently, in the context of returns, it mirrors the conditional expected value of returns in the left tail, delineated by the left superquantile:

$$-\tilde{q}_{1-\psi}(C) = \frac{1}{1-\psi}\int_0^{1-\psi} q_p(C)dp.$$

Alternatively minimized when considering the negative counterpart.

Utilizing the analytical formulas for the right superquantile $\bar{q}_\psi(C)$ derived in the previous sections, we can compute the left superquantile $\tilde{q}_\psi(C)$ as follows:

$$\psi\tilde{q}_\psi(C) + (1-\psi)\bar{q}_\psi(C) = \int_0^1 q_p(C)dp = E[C] \Longrightarrow -\tilde{q}_{1-\psi}(C) = -\frac{1}{1-\psi}\left(E[C] - \psi\bar{q}_{1-\psi}(C)\right).$$

Since $-\tilde{q}_{1-\psi}(C) = \bar{q}_\psi(X)$, the bPOE is defined as

$$\bar{p}_x(X) = \left\{1-\psi \mid \bar{q}_\psi(X) = x\right\} = \left\{1-\psi \mid \tilde{q}_{1-\psi}(C) = -x\right\}.$$

4.1.1. Qualified Distributions for Portfolio Optimization

The optimization problems in portfolio management, employing superquantile or bPOE, characterize their objective function or constraints in terms of $\tilde{q}_{1-\psi}\left(\delta^T C\right)$ or $\bar{p}_x\left(\delta^T C\right)$. To formulate such problems using a specified distribution, we initiate the process by defining a set of qualified distributions under consideration. These qualified distributions adhere to a predefined set of conditions, ensuring their coherence within portfolio theory. Additionally, they support a superquantile/bPOE expression, facilitating their representation with respect to the decision variable δ:

Definition 4.1. *(Qualified Distribution)* Qualified Distribution $\mathcal{B}$ meets the following conditions:

1. (C1) $\delta^T C \sim \mathcal{B} \Longrightarrow \tilde{q}_{1-\psi}\left(\delta^T C\right) = \delta^T\eta - \sqrt{\delta^T\Sigma\delta}\,\zeta(\psi,\Theta)$, where $\zeta(\psi,\Theta)$ is a function that exclusively relies on ψ and potentially on a set of constant parameters Θ that do not vary with δ. In this context, η represents the vector of anticipated asset returns, and Σ is the covariance matrix of asset returns.
2. (C1) The statistical parameters characterizing the distribution $\mathcal{B}$ should be consistent with the descriptive statistics of the real returns of the assets.
3. (C1) The shape of the Probability Density Function (PDF) for the specified distribution $\mathcal{B}$ must match the form of the empirical PDF of typical real asset returns.

Why should we establish these preconditions? Condition (C1) ensures that the superquantile can be formulated in terms of w. This is essential for formulating the superquantile optimization problem.

Proposition 4.1. If we assume that $\delta^T C \sim \text{Logistic}(\mu, s)$ then we have:

$$\tilde{q}_{1-\psi}(C) = \delta^T\eta - \sqrt{\delta^T\Sigma\delta}\frac{\sqrt{3}(-\psi\ln(\psi) - (1-\psi)\ln(1-\psi))}{\pi(1-\psi)}. \tag{26}$$

Proof. Since $\mu = E\left[\delta^T C\right] = \delta^T\eta$ and $\delta^T\Sigma\delta = \sigma^2\left(\delta^T C\right) = \frac{s^2\pi^2}{3} \Longrightarrow s = \frac{\sqrt{\delta^T\Sigma\delta}\sqrt{3}}{\pi}$.

Recall according to the proposition 3.6 we have shown that:

$$\begin{aligned}\tilde{q}_\psi(C) &= \mu + \frac{sH(\psi)}{1-\psi} \Longrightarrow \tilde{q}_{1-\psi}(C) = \mu + \frac{sH(1-\psi)}{\psi}\\ &= \mu + \frac{s((1-\psi)\ln(1-\psi) - (\psi)\ln(\psi))}{1-\psi}.\end{aligned}$$

where $H(\psi)$ is the binary entropy function:

$$H(\psi) = -\psi\ln(\psi) - (1-\psi)\ln(1-\psi).$$

Then we have:

$$\begin{aligned}\tilde{q}_{1-\psi}(C) &= \frac{1}{1-\psi}\left(E[C]-\psi\bar{q}_{1-\psi}(C)\right)=\frac{1}{1-\psi}\left(\mu-\psi\left[\mu+\frac{s}{\psi}(-(1-\psi)\ln(1-\psi)-\psi\ln(\psi))\right]\right)\\ &= \mu-\frac{s}{1-\psi}(-\psi\ln(\psi)-(1-\psi)\ln(1-\psi))=\delta^T\eta-\sqrt{\delta^T\Sigma\delta}\frac{\sqrt{3}(-\psi\ln(\psi)-(1-\psi)\ln(1-\psi))}{\pi(1-\psi)}. \quad (27)\end{aligned}$$

Proposition 4.2. If we assume that $\delta^T C \sim \exp(\phi)$ then we have:

$$\tilde{q}_{1-\psi}(C)=\delta^T\eta-\sqrt{\delta^T\Sigma\delta}\frac{(-\psi\ln(\psi))}{(1-\psi)}. \quad (28)$$

Proof. According to the proposition 3.3 we have shown that:

$$\tilde{q}_{\psi}(C)=\frac{-\ln(1-\psi)+1}{\phi}\Longrightarrow\tilde{q}_{1-\psi}(C)=\frac{-\ln(\psi)+1}{\phi}.$$

Since $\frac{1}{\phi}=E\left[\delta^T C\right]=\delta^T\eta$ and $\delta^T\Sigma\delta=\sigma^2\left(\delta^T C\right)=\frac{1}{\phi^2}\Longrightarrow\phi=\frac{1}{\sqrt{\delta^T\Sigma\delta}}$. Therefore:

$$\begin{aligned}\tilde{q}_{1-\psi}(C) &= \frac{1}{1-\psi}\left(E[C]-\psi\bar{q}_{1-\psi}(C)\right)\\ &= \frac{1}{1-\psi}\left(\frac{1}{\phi}-\psi\left[\frac{-\ln(\psi)+1}{\phi}\right]\right)\\ &= \frac{1}{\phi}-\frac{1}{\phi}.\frac{(-\psi\ln(\psi))}{(1-\psi)}\\ &= \delta^T\eta-\sqrt{\delta^T\Sigma\delta}\frac{(-\psi\ln(\psi))}{(1-\psi)}.\end{aligned}$$

Remark 4.1. This satisfies (C1). Other examples that meet this condition include the Gumbel and Frechet distributions.

Conditions (C2) and (C3) are straightforward checks of our model assumptions. For instance, in the case of the exponential distribution, where $E[C]=\frac{1}{\phi}=\sigma(C)$, however, for real asset returns, the sample mean is typically not equal to the sample standard deviation.

4.1.2. Superquantile and bPOE Optimization

Another approach to the Markowitz problem is to identify the portfolio with the lowest superquantile (29) or the minimal bPOE (30).

$$\begin{aligned}\min_{\delta\in\mathbb{R}^n}\quad & -\tilde{q}_{1-\psi}\left(\delta^T C\right)\\ \text{s.t.}\quad & \delta^T 1=1 \qquad (29)\\ & l\le\delta\le u.\end{aligned}$$

$$\begin{aligned}\min_{\delta\in\mathbb{R}^n}\quad & \bar{p}_x\left(\delta^T C\right)\\ \text{s.t.}\quad & \delta^T 1=1 \qquad (30)\\ & l\le\delta\le u.\end{aligned}$$

For qualified distributions, these problems can be significantly simplified. Firstly, we observe that 29 simplifies to 31:

$$\begin{aligned}\max_{\delta\in\mathbb{R}^n}\quad & \delta^T\eta-\sqrt{\delta^T\Sigma\delta}\zeta(\psi,\Theta)\\ \text{s.t.}\quad & \delta^T 1=1 \qquad (31)\\ & l\le\delta\le u.\end{aligned}$$

[12] establishes that the optimal solution for 31 aligns with the optimal solution for a Markowitz optimization problem 25 when $\lambda=\frac{\zeta(\psi,\Theta)}{2\sigma(\delta^T C)}$. Consequently, the optimal superquantile portfolio is concurrently optimal in terms of mean-variance following Markowitz principles. Moving on to bPOE, we discover that the scenario is notably simpler. In particular, Proposition 4.3 outlines the following.

Proposition 4.3. If we assume that $\delta^T C\sim\mathcal{D}$ and that $\mathcal{D}$ is a qualified distribution, then (30) reduces to (32).

$$\begin{aligned}\max_{\delta\in\mathbb{R}^n}\quad & \frac{\delta^T\eta+x}{\sqrt{\delta^T\Sigma\delta}}\\ \text{s.t.}\quad & \delta^T 1=1 \qquad (32)\\ & l\le\delta\le u.\end{aligned}$$

Proof. To begin with, it's important to highlight that, following the superquantile definition, $\zeta(\psi,\Theta)$ is expected to exhibit an increasing trend concerning $\psi\in[0,1]$.

Secondly, given that:

$$\begin{aligned}\bar{p}_x(X) &= \{1-\psi\mid\tilde{q}_{1-\psi}(C)=-x\}; \quad \text{and}\\ \tilde{q}_{1-\psi}\left(\delta^T C\right) &= \delta^T\eta-\sqrt{\delta^T\Sigma\delta}\zeta(\psi,\Theta)\end{aligned}$$

for qualified distributions, problem (30) can be reformulated as follows:

$$\begin{aligned}\min_{\delta\in\mathbb{R}^n}\quad & 1-\psi\\ \text{s.t.}\quad & -\delta^T\eta+\sqrt{\delta^T\Sigma\delta}\zeta(\psi,\Theta)=x \qquad (33)\\ & \delta^T 1=1\\ & l\le\delta\le u.\end{aligned}$$

which can then be written as follows:

$$\begin{aligned} &\max_{\delta \in \mathbb{R}^n} \psi \\ &\text{s.t.} \quad \zeta(\psi, \Theta) = \frac{\delta^T \eta + x}{\sqrt{\delta^T \Sigma \delta}} \\ &\qquad \delta^T 1 = 1 \\ &\qquad l \leq \delta \leq u. \end{aligned} \tag{34}$$

Finally, considering that $\zeta(\psi, \Theta)$ is a monotonically increasing function with respect to ψ, and Θ is independent of δ, we observe that we can express the maximization as (32) without altering the argmin.

This proposition holds a crucial implication for portfolio theory: the optimal bPOE portfolio for the qualified distribution remains unaffected by the specific distribution. Irrespective of the chosen distribution, the same portfolio attains the minimum bPOE. The independence of bPOE optimization from distribution assumptions enhances its preference over superquantile optimization.

4.2. Numerical Simulation

This section presents the financial data of Apple (AAPL), Microsoft (MSFT), Google (GOOGL), and TSLA stocks, which will serve as the basis for our portfolio risk analysis. We focus on the period from January 1, 2020, to October 25, 2023, to assess volatility and potential losses. Stock price data was collected from [32] for the period from January 1, 2020, to October 25, 2023. These data are used to model stock returns. The primary objective of this analysis is to measure the potential risks of the portfolio consisting of AAPL, MSFT, GOOGL, and TSLA stocks using simulation methods based on the probability distributions we have developed in this work. Figure 2 and Table 1 provide an overview of descriptive statistics for the financial data of each stock during the study period.

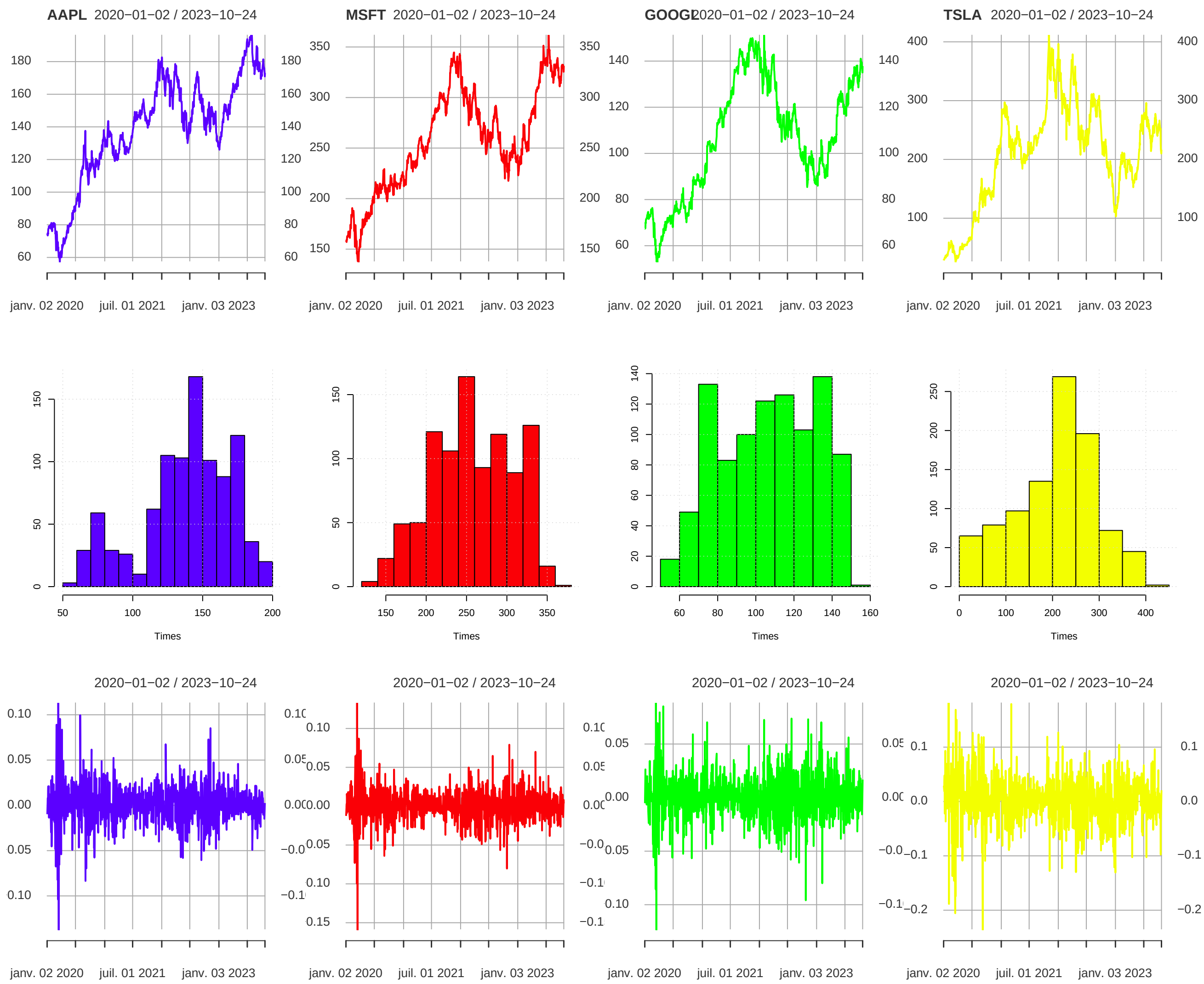

Figure 2. Evolution, histogram and Daily rates of return of AAPL, MSFT, GOOGL and TSLA from january 01, 2020, to october 25, 2023.

Table 1. *Descriptive statistics of Stock price the period from January 01, 2020, to october 25, 2023.*

	Values						
Data	**Min.**	**1st Qu.**	**Median**	**Mean**	**3rd Qu.**	**Max**	**S.d**
AAPL	57.02	122.51	144.00	138.43	162.60	196.24	32.38971
MSFT	137.0	218.3	256.6	257.7	298.9	361.8	50.83856
GOOGL	52.82	87.28	106.81	106.59	128.67	151.25	25.10797
TSLA	24.98	149.81	222.69	207.98	266.32	411.47	87.77104

Table 2 contains a detailed list of parameters for the stable distribution that were computed from the logarithmic return data. These parameters were obtained using the empirical characteristic function parameter estimation method developed by [24]. An important observation is that the data reveals a slight asymmetry; however, this subtle characteristic is not accounted for in the fitting with other distributions (Gumbel, Exponential, Frechet, Normal). In other words, these distributions fail to fully capture the asymmetric nature of the data.

Table 2. *α-Stable law parameters extracted from the log-returns data.*

	α -Stable law parameters			
Data	α	β	σ	μ
AAPL	1.583008914	-0.120136168	0.011551209	0.001105412
MSFT	1.6650635549	-0.1234907731	0.0112564190	0.0009459332
GOOGL	1.510423207	-0.028762124	0.011193913	0.001139006
TSLA	1.509520261	0.023024622	0.022976713	0.001835709

The results of our study demonstrate that the use of different probability distributions, such as Exponential, Gumbel, Frechet, and α-stable, in the calculation of VaR and CVaR has a significant impact on portfolio risk assessment. By comparing confidence levels of 1%, 5%, and 10%, we were able to observe substantial variations in risk measures for portfolios consisting of stocks from Apple Inc. (AAPL) (Table 3), Microsoft Corporation (MSFT) (Table 4), Alphabet Inc. (GOOGL) (Table 5), and Tesla, Inc. (TSLA) (Table 6). Traditional distributions used for VaR and CVaR calculations provide classical risk estimates, but our novel approaches have revealed significant nuances, especially in the distribution tails where rare and extreme events reside. The results highlighted that the parameters of these distributions have a substantial impact on the measured risk levels. Our analysis demonstrated that the α-stable approach, in particular, offers a better account of heavy tails, which is crucial for assessing risk during unexpected market shocks. This innovative approach provides a better understanding of the risk scenarios to which a portfolio could be exposed. By using different confidence levels (1%, 5%, and 10%), we obtained a comprehensive view of the risks associated with these portfolios. Higher confidence levels correspond to more extreme risk scenarios, and our new methodologies provided more precise estimates for these situations.

Table 3. *Parametric Density Estimation of quantile (VaR) and Superquantile (CVaR) of AAPL stock price.*

	confidence levels ψ		
AAPL	**10%**	**5%**	**1%**
Exponential VaR	0.002044284	0.002659674	0.004088567
Exponential CVaR	0.00799039	0.01686860	0.08789429
Normal VaR	0.03633901	0.04313052	0.05640414
Normal CVaR	0.0013350400	0.0003718334	0.0005761733
Gumbel VaR	0.01820652	0.02262876	0.02985461
Gumbel CVaR	0.01490864	0.01484083	0.01542274
Frechet VaR	0.004190899	0.004190899	0.004190899
Frechet CVaR	0.004190899	0.004190899	0.004190899
α-Stable VaR	0.001361574	0.0001313906	0.0007930745
α-Stable CVaR	0.002498214	0.00027120428	0.00048269686

Table 4. *Parametric Density Estimation of quantile (VaR) and Superquantile (CVaR) of MSFT stock price.*

	confidence levels ψ		
MSFT	**10%**	**5%**	**1%**
Exponential VaR	0.001743903	0.002268870	0.003487807
Exponential CVaR	0.006816308	0.014389984	0.074979390
Normal VaR	0.03633901	0.04313052	0.05640414
Normal CVaR	0.001396619	0.000463257	0.000455376
Gumbel VaR	0.01774534	0.02203057	0.02903252
Gumbel CVaR	0.01454964	0.01448394	0.01504782
Frechet VaR	0.00416399	0.00416399	0.00416399
Frechet CVaR	0.00416399	0.00416399	0.00416399
α-Stable VaR	0.0002174864	0.0003076158	0.0012780612
α-Stable CVaR	0.0004447874	0.0006518319	0.0015555238

Table 5. *Parametric Density Estimation of quantile (VaR) and Superquantile (CVaR) of GOOGL stock price.*

	confidence levels ψ		
GOOGL	**10%**	**5%**	**1%**
Exponential VaR	0.001684959	0.002192183	0.003369919
Exponential CVaR	0.006816308	0.014389984	0.074979390
Normal VaR	0.03562591	0.04231070	0.05537575
Normal CVaR	0.0014561644	0.0005080928	0.0004250176
Gumbel VaR	0.01806254	0.02241530	0.02952760
Gumbel CVaR	0.01481648	0.01474974	0.01532250
Frechet VaR	0.004267148	0.004267148	0.004267148
Frechet CVaR	0.004267148	0.004267148	0.004267148
α-Stable VaR	0.0002164164	0.0003356518	0.0001620433
α-Stable CVaR	0.0005582853	0.0010540750	0.00018664069

Table 6. *Parametric Density Estimation of quantile (VaR) and Superquantile (CVaR) of TSLA stock price.*

	confidence levels ψ		
TSLA	**10%**	**5%**	**1%**
Exponential VaR	0.004908138	0.006385635	0.009816276
Exponential CVaR	0.01918420	0.04049997	0.21102616
Normal VaR	0.07383731	0.08757422	0.11442225
Normal CVaR	0.0023645164	0.0004162763	0.0015012195
Gumbel VaR	0.03648979	0.04543449	0.06004993
Gumbel CVaR	0.02981930	0.02968215	0.03085915
Frechet VaR	0.00814095	0.00814095	0.00814095
Frechet CVaR	0.00814095	0.00814095	0.00814095
α-Stable VaR	0.0011543708	0.00010684108	0.00016234935
α-Stable CVaR	0.001268730	0.0001574391	0.0003396768

5. Conclusion

In conclusion, our study falls within the realm of portfolio risk measurement, shedding light on innovative approaches to assess quantiles and superquantiles. Our primary objective was to enhance the understanding of financial risks associated with a portfolio consisting of technology giants such as Apple Inc. (AAPL), Microsoft Corporation (MSFT), Alphabet Inc. (GOOGL), and Tesla, Inc. (TSLA).

Through our research, we adopted a fresh perspective by utilizing specific probability distributions, namely Exponential, Gumbel, Frechet, and α-stable distributions, to model the returns of these major assets. The results obtained reveal that this innovative approach allows for a more accurate evaluation of financial risks, particularly in a swiftly evolving financial landscape.

By examining quantiles and superquantiles, we were able to anticipate more intricate risk scenarios, thereby providing investors and portfolio managers with vital insights for informed decision-making. This profound understanding of extreme situations and potential shocks is crucial for devising robust portfolio management strategies.

Our findings underscore the significance of innovation in the field of risk measurement. Traditional portfolio management tools can benefit from these novel methodologies for better alignment with the realities of today's financial market.

Ultimately, our study opens up new horizons for decision-making in investment and portfolio management. In a complex and ever-changing financial environment, the ability to assess and anticipate risks is a valuable asset for investors and portfolio managers. We hope that our work will inspire further research in this domain, contributing to an improved understanding and management of financial risks.

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Abbreviations

VaR	Value at Risk
CVaR	Conditional Value at Risk
bPOE	Buffered Probability of Failure
POE	Probabiolity of Exceeding
MSFT	Microsoft
i.i.d	Independent and Identically Distributed
PDF	Probability Density Function
CDF	Cumulative Distribution Function

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Author Contributions

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Conflicts of Interest

The author declares no conflicts of interest.

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