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Research Article

Unification of Maxwell Systems, Einstein, and Dirac Equations in Pseudo-Riemannian Space $R^{1,3}$ by Clifford Algebra

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Abstract

This paper presents the unification of Einstein's equations, Maxwell's equation systems, and Dirac's equation for three generations of particles in the $R^{1,3}$ pseudo-Riemannian space with torsion. The use of Dirac matrices as an orthonormal basis (in general, as a canonical basis) on the tangent plane permits the replacement of vectors with second-rank tensors. The symmetric component of the differential form *DA* (*D*•*A*, where *D* is the Dirac operator, *A* is the tensor field, and • is the inner product) represents the deformation of the field, while the antisymmetric one (*D*Λ*A*, Λ is the outer product) denotes the torsion. The differentiation of *DA* (i.e., *DDA*) yields an equation from which both Einstein's equation and the two independent Maxwell systems can be derived. The differentiation of the field deformation *D*(*D*•*A*), that is, the gradient of the field divergence, yields a four-dimensional current. This four-current formulation results in nonlinearity in the inhomogeneous Maxwell's equations. In particular, the four-current *J* is not a constant in the inhomogeneous system of Maxwell's equations, *D*•*F* = *J*. In accordance with this definition, a field singularity is defined as a source of current, or alternatively described as a "hole," which is a necessary component for the existence of the field. The description of field inhomogeneity (*DA*) in the form of biquaternions through complex hyperbolic functions in $R^{1,3}$ permits the decomposition of *DA* into three pairs of spinors–antispinors (spinor bundle). The differentiation of spinors and the subsequent determination of eigenvalues and eigenfunctions yield three pairs of Dirac-type equations that are applicable to both bosons and fermions, which describe the fundamental particles of the three generations. The solution of Dirac-type equations in pseudo-Euclidean space for massless particles (eigenvalues $m = 0$) unifies the photon and three generations of neutrinos (γ, ν_e, ν_μ, ν_τ) into a single entity, namely, a singlet (photon) + a triplet (three generations of neutrinos).

Keywords

Einstein Equations, Maxwell Equations, Dirac Equations, Unified field theory, Pseudo-Riemannian Manifold $R^{1,3}$, Biquaternions

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1. Introduction

The use of new applications of algebra complements and expands the possibilities for describing physical processes. For example, the methods of Clifford algebra [1] unite fundamental fields into a single field in pseudo-Riemannian space.

In this research, we will combine the homogeneous and inhomogeneous systems of equations of Maxwell, Einstein's [2] and Dirac's equations [3] for three generations of particles using the generalized Clifford algebra method in $R^{1,3}$ with torsion. In other words, we will create a single theoretical framework by combining the vector (electromagnetism), tensor (gravity), and spinor (fermions) fields into one field.

This study extends and generalizes previous research on Clifford algebra applications in field theory.

In contrast to the classical Clifford algebra, which is applicable to pseudo-Euclidean space (Minkowski [4]), the generalized Clifford algebra describes objects in non-Euclidean space, $R^{1,3}$ [5]. The incorporation of torsion in the Riemannian manifold $R^{1,3}$ is a necessary condition for the unification of Maxwell's and Einstein's equations.

The representation of the differential form *DA* (*D* is the Dirac operator; *A* is a tensor field) in the form of generalized biquaternions [6] permits the expression of *DA* (inhomogeneity) as the sum of three pairs of independent spinors - antispinors [6]. Subsequently, Dirac-type equations are derived by differentiating the spinors-antispinors.

Using the Clifford product $ab = a\bullet b + a\wedge b$ (outer $a\wedge b = (ab - ba)/2$ and inner $a\bullet b = (ab + ba)/2$) instead of the usual scalar and vector products of vectors allows us to reconsider many previous concepts. For example, the divergence gradient $D(D\bullet A)$ is interpreted as a four-dimensional electromagnetic current, thereby expanding its physical and geometric meaning.

2. Results

2.1. Theoretical Basis

Let be given a vector field A in a pseudo-Riemannian space $R^{1,3}$:

$$A = \sum_{i=0}^{3} e^i A_i \tag{1}$$

Formula (1) is the expansion of A in the basis $\{e^i\}$.

Clarification. In classical Clifford algebra, e^i are called vectors, but in fact, e^i are 4x4 matrices related to the Dirac matrices (γ^0, γ^1, γ^2, γ^3) through the transition functions $X = X_i (q^j)$ by the formula:

$$e_i = \sum_{j=0}^{3} \gamma^j \frac{\partial X_j}{\partial q^i} \equiv \sum_{j=0}^{3} \gamma^j \partial_i X_j \tag{2}$$

The basis $\{\gamma^i\}$ of Dirac matrices is referred to as canonical:

$$\gamma_i \gamma_j + \gamma_j \gamma_i = \pm 2I\delta_{ij} \tag{3}$$

δ_{ij} is the Kronecker delta and *I* is the identity matrix. If $i = 0$ and/or $j = 0$, then the sign "+" is used in (3). In other cases, the sign "-" is used.

Mathematically, e_i are second-rank tensors. The terms "second-rank tensor" and "vector" are equivalent because Dirac matrices are employed as the orthonormal basis, rather than ordinary vectors (unit vectors).

Basis (3) greatly expands the concept of an orthonormal basis because it also includes ordinary (scalar and vector) vector products. This allows us to consider space not only with curvature (deformation) but also with torsion.

The terms "vector, bivector, trivector" accepted in Clifford algebra are quite acceptable, since the internal structure of the Dirac matrices and their products γ^i, $\gamma^i\gamma^j$, $\gamma^i\gamma^j\gamma^k$ does not change, and therefore the name has no principled importance. The main objective is to satisfy Condition (3).

The movable (local) basis $\{e^i\}$ is connected to the canonical basis, which is drawn at the point of contact of the local tangent plane with the pseudo-Riemann surface. The basis vectors e^i depend on the coordinates, that is, they change when moving "parallel to themselves" (in $R^{1,3}$, along geodesic lines). Thus, ordinary differentiation must be replaced by covariant differentiation.

$e^i e^j = e^i\bullet e^j + e^i\wedge e^j$ is the Clifford vector product [1]:

$e^i\bullet e^j = (e^i e^j + e\, e^i)/2$ is the inner product of vectors.

$e^i\wedge e^j = (e^i e^j - e\, e^i)/2$ is the outer product of vectors or a bivector.

In general, $e^i\bullet e^j$ is a symmetric second-rank tensor obtained by the contraction of the dyadic product of two second-rank tensors [7], and then by symmetrization:

$$e^i\bullet e^j = (e\otimes e' + e'\otimes e)/2 = g^{kn} (e_{ik}\, e_{nj} + e_{jn}\, e_{ki})/2$$

In particular, $e^i\bullet e^j = I\, g_{ij}$, where g_{ij} is a metric tensor. In other words, $e^i\bullet e^j$ is the curvature of the metric, that is, a measure of the difference from flat space, a measure of deformation.

In general, $e^i\wedge e^j$ is an antisymmetric tensor of the second rank, obtained by the contraction of the dyadic product of two tensors [7] and then by their antisymmetrization:

$$e^i\wedge e^j = (e\otimes e' - e'\otimes e)/2 = g^{kn} (e_{ik}\, e_{nj} - e_{jn}\, e_{ki})/2$$

$e^i\wedge e^j$ denotes the torsion tensor [8]. Module $|e^i\wedge e^j|$ is the area of the parallelogram constructed from vectors e^i and e^j. It is clear that $e^i\wedge e^j = -e^j\wedge e^i$.

In 3-dimensional space, the geometric meaning of the torsion tensor $e^i\wedge e^j$ is more obvious: $e^i\wedge e^j$ corresponds to the associated accompanying axial vector (dual, i.e., pseudovector) e^k, which, like a screw, changes its direction depending on the replacement of e^i and e^j.

2.2. The Inhomogeneity of a Vector Field

We consider the gradient from A to:

$$DA = e^i e^j D_i A_j \quad (4)$$

In accordance with Clifford's product, Equation (4) can be separated into symmetric and antisymmetric components:

$$DA = ID\bullet A + D\wedge A = Ie^i\bullet e^j D_i A_j + e^i\wedge e^j D_i A_j \quad (5)$$

where $D = e^i D_i$ is the Dirac operator [9] and D_i is the symbol of covariant differentiation [10].

We refer to this (5) as the field inhomogeneity (DA), which comprises the following:

1) field deformations:

$$I\, D\bullet A = Ie^i\bullet e^j D_i A_j = Ig^{ij} D_i A_j = I\, D_i A^i,$$

where $D_i A_j = D_i A^i = Div\ A$ is a 4-dimensional divergence, a true scalar, and the first invariant of the DA [11], because $g^{ij} D_i A_j$ is the contraction of the tensor DA.

2) field torsion:

$$D\wedge A = e^i\wedge e^j D_i A_j = e^i\wedge e^j (D_i A_j - D_j A_i)$$

$D\wedge A$ is an antisymmetric second rank tensor. $F = D\wedge A$ is the electromagnetic field tensor.

We again differentiate Equation (5). The DDA can then be written in two equivalent versions:

$$D(DA) = D(D\bullet A) + D\bullet(D\wedge A) + D\wedge(D\wedge A) \quad (6)$$

$$(DD)A = (D\bullet D)A + (D\wedge D)\bullet A + (D\wedge D)\wedge A \quad (7)$$

Next, we will verify that (6) gives the system of Maxwell's equations and (7) gives Einstein's equation.

2.3. Homogeneous System of Maxwell's Equations

The last term in Equation (6) is equal to zero [12]:

$$D\wedge(D\wedge A) = D\wedge F = 0 \quad (8)$$

(8) is a system of homogeneous Maxwell's equations. We write (8) in coordinate form as

$$D\wedge F = E^{ijkn} D_k F_{ij} = 0 \quad (9)$$

where E^{ijkn} is a contravariant antisymmetric tensor of fourth rank or Levi-Civita symbol [13].

2.4. Four-dimensional Electromagnetic Current

In Equation (6), the first term, $D(D\bullet A)$, is identified using the density of the four-dimensional electromagnetic current:

$$j = D(D\bullet A) \quad (10)$$

Then, the electric charge q and 3-dimensional current $\boldsymbol{j}$ are expressed as

$$q = e^0 e^0 j_0 = g^{00} D_0 (D\bullet A) \quad (11)$$

$$\boldsymbol{j} = e^0\wedge e^\alpha j_\alpha = e^0\wedge e^\alpha D_\alpha (D\bullet A) \quad (12)$$

In a flat space, the charge and current have the following forms:

$$q = \gamma^0\gamma^0 j_0 = \sigma_0 j_0, \qquad \boldsymbol{j} = \gamma^0\gamma^\alpha j_\alpha = \sigma_\alpha j_\alpha,$$

since $\gamma^0\gamma^i = \sigma_i$. where σ_i is a Pauli matrix.

Formula (11) has a clear geometric interpretation: the electric charge can be understood as the time derivative of the divergence of the field $D_0 (D\bullet A)$. Therefore, an electric charge has only two signs: “plus” or source, if $D_0 (D\bullet A) > 0$ and “minus” or drain if $D_0 (D\bullet A) < 0$.

The electric charge (±) can be represented as a "hole" in the field. Gauges, such as Lorentz and Coulomb gauges. [14] can be understood as a method of removing singularities or sources of the field (current) with the aim of simplifying calculations.

2.5. Inhomogeneous System of Maxwell's Equations

The third-rank tensor $D(DA)$ or trivector, or more precisely its part $D\wedge(D\wedge A)$, is dual to the pseudovector [12], which is equal to zero, i.e. $D\wedge(D\wedge A)$ is equivalent to the homogeneous system of Maxwell's equations (9) [12].

The remaining part is equivalent to the vector:

$$D(DA) = \mu T\bullet A \quad (13)$$

The inner product $T\bullet A$ gives a vector, as does the right side of equation (6). T is a second-rank tensor; μ is the proportionality coefficient. The properties of T and μ will be defined later.

Considering (10), (13), and $F = D\wedge A$, we obtain an inhomogeneous system of Maxwell's equations:

$$D\bullet F = \mu T\bullet A - j = J \quad (14)$$

When $\mu T\bullet A = 0$, (14) has the classical form. For values of $\mu T\bullet A \neq 0$, the inhomogeneous system of Maxwell's equations is nonlinear, that is, the effective charge J (right-hand side of (14)) is not constant; it increases or decreases depending on the growth of $\mu T\bullet A$ (charge screening).

2.6. Continuity Equation

The differentiation of (14) gives the continuity equation:

$$\mu\ T\bullet\ (DA)\ -\ D\bullet j\ =\ 0, \tag{15}$$

because $D\bullet(D\bullet F) = 0$ and $(D\bullet T)\bullet A = 0$ [15]. The inner product of the vectors is used in Eq. (15).

Formula (15) in general is a differential form of the law of conservation of energy-momentum in an elementary volume, i.e., the law of conservation of 4-current.

We write Equation (15) in the coordinate form and transform the left side.

$$\mu\, T^k{}_n\, A^n{}_k = 0.5\, \mu\, (g^{nk} g^{ij} T_{ki} \nabla_n A_j + g^{nk} g^{ij} T_{ki}\, D_j\, A_n) =$$

$$= 0.5\, \mu\, g^{nk} g^{ij} T_{ki}\, (D_n A_j + D_j A_n) = 0.5 \mu\, T^{nj}\, \varepsilon_{nj}$$

Then, we obtain the continuity equation in the form [15]:

$$0.5\mu\ T^{nj}\ \varepsilon_{nj}\ -\ D_k\ j^{\ k}\ =\ 0 \tag{16}$$

$\varepsilon_{nj} = 0.5\ (D_n A_j + D_j A_n)$ is the deformation tensor.

The four-current j_k consists of the sum of the positive and negative currents:

$$j_k\ =\ \Sigma j^+{}_k\ +\ \Sigma j^-{}_k \tag{17}$$

Equation (16) represents the law of conservation of four-currents during field deformation. The deformation of electromagnetic and/or gravitational fields gives rise to the creation of pairs of positive and negative electric charges (particles + antiparticles), resulting in a constant total charge.

From (16), it is clear that tensor T (13) is the energy -momentum tensor.

2.7. Law of the Conservation of Eddy Currents or Electromagnetic Induction

By differentiating (14), we obtain the law of conservation of the eddy current [15]:

$$D \wedge J\ =\ 0 \tag{18}$$

$D \wedge (D\bullet\ F) = 0$, because $D\bullet\ F$ is "scalar".

Equation (18) represents the law of conservation of eddy currents, also known as Faraday's law of electromagnetic induction (Faraday's law) [16]. Note that in (18), the outer product of the vectors is employed.

Formula (18) in the usual 3-dimensional form looks like [15]:

$$-\nabla\rho\ +\ \partial_t \boldsymbol{j}\ +\ rot\ \boldsymbol{j}\ =\ 0 \tag{19}$$

where ρ is the electric charge density; $\partial_t \boldsymbol{j}$ is a change in 3-dimensional current over time; $rot\boldsymbol{j}$ is the curl (rotor, vortex) of the 3-dimensional current. Law (19) is explained in Figure 1.

Let the current carriers be positive charges and $\rho_1 < \rho_2$. For simplicity, we let $\nabla\rho$ = const. Then, the change in current over time (in general, $-\nabla\rho + \partial_t \boldsymbol{j}$) is compensated by the current vortex, that is, $\partial_t \boldsymbol{j}$ and $rot\, \boldsymbol{j}$ will be directed opposite to other. Simply put, if $\partial_t \boldsymbol{j} > 0$, then the eddy current $rot\boldsymbol{j}$ decreases, and if $\partial_t \boldsymbol{j} < 0$, $rot\, \boldsymbol{j}$ increases.

Within the framework of generalized Clifford algebra, we combined two independent Maxwell system equations into a single equation: In addition, we derived conservation laws for the four-current.

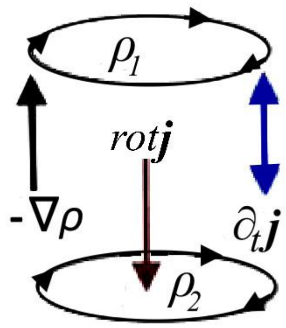

Figure 1. *Eddy currents conservation.*

2.8. Einstein's Equations

We now obtain Einstein's equation from Equation (7). Because

$(D \wedge D) \wedge A = 0$, $(DD)A = \mu T\bullet A$, $(D\bullet D)A = D^2 A = \Box A$,

where $\Box$ is the D'Alembert operator,

$(D \wedge D)\bullet A = -\ Ric\bullet A$,

where Ric is the Ricci tensor, we can write Equation (7) in the following form:

$$\mu T\bullet A\ =\ D\bullet DA\ -\ Ric\bullet A \tag{20}$$

According to the Lichnerowicz formula [17, 18]:

$$\Box A\ =\ \Box_B A\ +\ 0.5\ \mathcal{R}\mathrm{ic}(A), \tag{21}$$

where $\Box A = e^i\bullet e^j e^k\ D_i D_j\ A_k$ is the action of the Lichnerowicz operator on vector A;

$\Box_B\ A \equiv -\ \mathrm{tr}\ (\Box A) = -\ e^i\bullet e^i e^k\ D_i D_i A_k$ is the action of the Bochner Laplacian [18] on A_k (on a scalar). This is the trace of $\Box A$, or the "rough" D'Alembertian;

λ_k are the eigenvalues of each eigenfunction (scalar!) A_k of the operator $\nabla\bullet\nabla$:

$$\mathcal{R}\mathrm{ic}(A) = e^i\bullet e^j e^k R_{ij}\, A_k = g^{ij} e^k R_{ij}\, A_k = e^k A_k\, R;$$

R is the scalar curvature.

Then, considering (21), from (20), we obtain the Einstein equation:

$$Ric\ -\ 0.5\ R\ +\ \lambda\ =\ -\ \mu T \tag{22}$$

λ_k represent the eigenvalues of the Bochner operator:

$$-e^i \bullet e^i D_i D_i A^k = \delta_k^n \lambda_n A^k$$

From (22), we can determine the coefficient μ. $\mu = -8\pi k/c^4$ [2].

Furthermore, it is assumed that all components of the vector A^k (without e_k) are a scalar function, i.e., each scalar function A^k corresponds to its own number λ_k (there are four of them).

Einstein's equation (22) can be derived by differentiating Eq. (20) [19]. The proof is provided in Appendix 1.

Obtaining Einstein equation (22) by direct calculation from the inhomogeneous system of Maxwell equations in curvilinear coordinates proves the validity of the Lichnerowicz formula (21) for the case of a 4-dimensional pseudo -Riemannian space. Thus, from an inhomogeneous system of Maxwell equations, we obtain the Einstein equation.

2.9. Biquaternions, Bispinors

We square the field inhomogeneity (5), which consists of symmetric and antisymmetric parts, as follows:

$$(DA)^2 = (D\bullet A + D\wedge A)^2$$

$$(DA)^2 = (D\bullet A)^2 + 2\,(D\bullet A)\ D\wedge A + (D\wedge A)^2 \tag{23}$$

$$\text{where } D\wedge A = \boldsymbol{E} + \boldsymbol{H} \tag{24}$$

$\boldsymbol{E} = e^\alpha \wedge e^0 F_{\alpha 0}$ is the electric field vector [20];

$\boldsymbol{H} = \gamma\, E_{\lambda\mu\alpha 0}\, e^\alpha \wedge e^0\, F^{\lambda\mu}$ is the magnetic field vector [20];

$\gamma = \gamma^0\gamma^1\gamma^2\gamma^3$ is the matrix analogue of the imaginary unit: $(\gamma)^2 = (\gamma^0\gamma^1\gamma^2\gamma^3)^2 = -I$.

Substituting $\boldsymbol{E}$ and $\boldsymbol{H}$ into (23), opening the brackets, and simplifying, we see that (23) consists of scalar SR, pseudo-scalar SP, bivector VR, and pseudo-bivector VP [6]:

$$(DA)^2 = SR + SP + VR + VP \tag{25}$$

where

$SR = (D\bullet A)^2 I$ is scalar,

$SP = -0.25\,\gamma\, E_{ijkn}\, F^{ij} F^{kn}$ is pseudoscalar;

$VR = e_\alpha \wedge e_0\, F_{\alpha 0}\, (D\bullet A)$ is bivector;

$VP = \gamma\,(e_\alpha \wedge e_0)\, E^{\beta\lambda\alpha 0}\, F_{\beta\lambda}\, (D\bullet A)$ is pseudobivector.

Greek letters take the values 1, 2, 3.

SR, SP, VR and VP are expressed using hyperbolic functions:

$SR = |\tau_{\alpha 0}|\,|\tau_{\beta 0}|\, cosh((\eta_\alpha + \eta_\beta)/2)\, cosh(\gamma\,(\varphi_\alpha + \varphi_\beta)/2)$;

$SP = |\tau_{\alpha 0}||\tau_{\beta 0}|\, sinh((\eta_\alpha + \eta_\beta)/2)\, sinh(\gamma\,(\varphi_\alpha + \varphi_\beta)/2)$;

$VR = \tau_{\alpha 0}\,|\tau_{\beta 0}|(sinh((\eta_\alpha + \eta_\beta)/2)\, cosh(\gamma\,(\varphi_\alpha + \varphi_\beta)/2) - sinh((\eta_\alpha - \eta_\beta)/2)\, cosh(\gamma\,(\varphi_\alpha - \varphi_\beta)/2))$;

$VP = \tau_{\alpha 0}\,|\tau_{\beta 0}|(cosh((\eta_\alpha + \eta_\beta)/2)\, sinh(\gamma\,(\varphi_\alpha + \varphi_\beta)/2) - cosh((\eta_\alpha - \eta_\beta)/2)\, sinh(\gamma\,(\varphi_\alpha - \varphi_\beta)/2))$

where

$\tau_{\alpha 0} = e_\alpha \wedge e_0$, $|\tau_{\alpha 0}| = (g_{0\alpha}\, g_{0\alpha} - g_{00}\, g_{\alpha\alpha})^{0.5}$,

η_α is rapidity or angle of rotation on a hyperplane $x_\alpha 0t$,

φ_α is angle of spatial rotation around an axis x_α.

Substituting the new notations SR, SP, VR, and VP into (25), and simplifying and extracting the square root, we obtain:

$$DA = \sum_{\alpha=1}^{3} R_\alpha = \sum_{\alpha=1}^{3}\left(|\tau_{\alpha 0}| cosh\frac{z_\alpha}{2} + \tau_{\alpha 0} sin\frac{z_\alpha}{2}\right) \tag{26}$$

where $z_\alpha = I\eta_\alpha + \gamma\,\varphi_\alpha$.

In (25), we change the order of multiplication of bivectors $e_\alpha \wedge e_0 = -e_0 \wedge e_\alpha$, in general, for all even Clifford numbers: $e_i \bullet e_j$, $e_i \wedge e_j$, $e_i \wedge e_j \wedge e_k \wedge e_n$ [6]. By repeating the procedure that we have already performed, we obtain:

$$\widetilde{DA} = \sum_{\alpha=1}^{3} \tilde{R}_\alpha = \sum_{\alpha=1}^{3}\left(|\tau_{0\alpha}| cosh\frac{z_\alpha}{2} + \tau_{0\alpha} sin\frac{z_\alpha}{2}\right) \tag{27}$$

R_α and $\tilde{R}_\alpha$ are referred to as generalized biquaternions and antibiquaternions, respectively. They described rotations on Riemann surfaces (on tangent planes $x0y$, $z0y$, $y0z$, $t0x$, $t0y$, $t0z$).

Formula (27) means that the field inhomogeneity DA is the sum of three biquaternions (antibiquaternions). Mathematically, biquaternions are second-rank tensors.

In the Minkowski space, where e^i are replaced by Dirac matrices γ^i, biquaternions assume the form of a matrix exponential [21]:

$$R_\alpha(\tilde{R}_\alpha) = Icosh\frac{z_\alpha}{2} \pm \gamma^\alpha\gamma^0 sin\frac{z_\alpha}{2} = exp\left(\pm\gamma^\alpha\gamma^0\frac{z_\alpha}{2}\right) \tag{28}$$

Thus, it is difficult to overestimate the role of biquaternions in physics. Any transformations of a vector, including those in curvilinear coordinates, are performed by biquaternions [22]:

$$x' = R_\alpha x \tilde{R}_\alpha / |\tau_{0\alpha}|^2 \tag{29}$$

Using Euler's formulas, we write the biquaternion R_α as:

$$R_\alpha = |\tau_{\alpha 0}|\frac{exp\left(\frac{z_\alpha}{2}\right)+exp\left(-\frac{z_\alpha}{2}\right)}{2} + \tau_{\alpha 0}\frac{exp\left(\frac{z_\alpha}{2}\right)-exp\left(-\frac{z_\alpha}{2}\right)}{2} \quad \text{or}$$

$$R_\alpha = (|\tau_{\alpha 0}| + \tau_{\alpha 0})\frac{exp\left(\frac{z_\alpha}{2}\right)}{2} + (|\tau_{\alpha 0}| - \tau_{\alpha 0})\frac{exp\left(-\frac{z_\alpha}{2}\right)}{2} \tag{30}$$

where

$$\Psi_\alpha = (|\tau_{\alpha 0}| + \tau_{\alpha 0})\frac{exp\left(\frac{z_\alpha}{2}\right)}{2} = (|\tau_{\alpha 0}| + \tau_{\alpha 0})\Phi_\alpha \tag{31}$$

$$\tilde{\Psi}_\alpha = (|\tau_{\alpha 0}| - \tau_{\alpha 0})\frac{exp\left(-\frac{z_\alpha}{2}\right)}{2} = (|\tau_{\alpha 0}| - \tau_{\alpha 0})\tilde{\Phi}_\alpha \tag{32}$$

In general

$$\Phi_\alpha(\tilde{\Phi}_\alpha) = \frac{1}{2}exp\left(\pm\frac{z_\alpha}{2}\right) = \frac{1}{2}exp\left(\pm\frac{IX_\alpha+\gamma Y_\alpha}{2}\right) \tag{33}$$

$X_\alpha(q^i)$ and $Y_\alpha(q^i)$ are scalar functions.

Ψ_α and $\tilde{\Psi}_\alpha$ will be called generalized bispinor - antibispinor [6].

According to the strict terminology of algebra, an ideal of a ring K is a subring k for $\forall b \in K$ and $\forall S \in k$ the equality holds [23]:

$$bS = cS$$

where c is a real number. If $c > 0$, then S is a positive ideal, if $c < 0$, then S is a negative ideal. In physics, ideals are associated with spinors: if $c > 0$, then $S = \Psi_\alpha$ is a bispinor, if $c < 0$, then $S = \widetilde{\Psi}_\alpha$ is an antibispinor.

Really,

$$\tau_{\alpha 0} \bullet \Psi_\alpha = ((e_\alpha \wedge e_0)|e_\alpha \wedge e_0| + (e_\alpha \wedge e_0)^2)\Phi_\alpha =$$

$$= |\tau_{\alpha 0}|(\tau_{\alpha 0} + |\tau_{\alpha 0}|\Phi_\alpha) = |\tau_{\alpha 0}|\Psi_\alpha,$$

Since $c = |\tau_{\alpha 0}| > 0$, then Ψ_α is a bispinor. The anti-bispinor can be similarly checked:

$$\tau_{\alpha 0} \bullet \widetilde{\Psi}_\alpha = \left((e_\alpha \wedge e_0)|e_\alpha \wedge e_0| - (e_\alpha \wedge e_0)^2\right)\widetilde{\Phi}_\alpha =$$

$$= |\tau_{\alpha 0}|\left(\tau_{\alpha 0} - |\tau_{\alpha 0}|\widetilde{\Phi}_\alpha\right) = -|\tau_{\alpha 0}|\widetilde{\Psi}_\alpha$$

It can be verified that all three ideals (α = 1, 2, 3) are two-sided (left- or right-hand multiplication of $\tau_{\alpha 0}$ is equal), since $\tau_{\alpha 0}$ commutes with Φ_α and $\widetilde{\Phi}_\alpha$.

Spinors and antispinors are independent [6]. The equality

$$\sum_{\alpha=1}^{3}(\mu_\alpha \Psi_\alpha + \tilde{\mu}_\alpha \widetilde{\Psi}_\alpha) = 0$$

is satisfied if all real numbers $(\mu_\alpha, \tilde{\mu}_\alpha)$ are equal to zero.

From (30), it is evident that a biquaternion is the sum of a spinors and antispinors. In the general case, the field inhomogeneity *DA* (26) consists of three generalized biquaternions R_α, that is, three pairs of independent spinors and antispinors [6]. Because *DA* is a second-rank tensor, we can say that biquaternions, bispinors, and antibispinors are also second-rank tensors ($\Psi_{\alpha 0}$). In simple terms, a spinor field is a spinor bundle [24] of a tensor, that is, a *DA* inhomogeneity.

2.10. Dirac Type Equations

By differentiating bispinors (31) and (32), and finding the eigenvalues and eigenbispinors, we obtain Dirac-type equations in covariant form:

$$D(\Psi_\alpha(\widetilde{\Psi}_\alpha)) = m\Psi_\alpha(\widetilde{\Psi}_\alpha) \tag{34}$$

We write Equation (34) in the coordinate form as:

$$e^i(I|\tau^{\alpha 0}| + e^\alpha \wedge e^0)D_i\Phi_{\alpha 0} = Ig^{j\alpha}m_j\Psi_{\alpha 0}$$

or

$$(|\tau^{\alpha 0}|e^i + e^i \bullet \tau^{\alpha 0} + e^i \wedge \tau^{\alpha 0})D_i\Phi_{\alpha 0} = Ig^{j\alpha}m_j\Psi_{\alpha 0} \tag{35}$$

$$|\tau^{\alpha 0}| = |e^\alpha \wedge e^0|$$

Equation (35) is a Dirac-type equation expressed in the coordinate form. No summation for α, and 0.

In Minkowski space, equation (34) assumes the form of classical Dirac equations:

$$\gamma^i \partial_i \Psi_\alpha(\widetilde{\Psi}_\alpha) = -m_\alpha \Psi_\alpha(\widetilde{\Psi}_\alpha) \tag{36}$$

where

$$\Psi_\alpha(\widetilde{\Psi}_\alpha) = (I \pm \gamma^\alpha\gamma^0)\Phi_\alpha(\widetilde{\Phi}_\alpha) = \frac{I \pm \gamma^\alpha\gamma^0}{2} e^{\pm\frac{IX_\alpha + \gamma Y_\alpha}{2}} \tag{37}$$

3. Calculations

Equation (36) was solved for massless particles [25]. Below, we discuss several important aspects.

Let α=3.

3.1. Solutions for Neutrinos

We consider (37) with only field torsion (without deformation) as follows:

$$\gamma^i\gamma^3\gamma^0 U^T \partial_i\, exp\left((I + \gamma)\frac{p \bullet x}{2}\right) = 0, \tag{38}$$

where

$U^T = (u_0, u_1, u_2, u_3)^T$ is a constant column vector;

γ^i are the Dirac matrices in the Weyl representation;

$p \bullet x = Ip_i x^i$, p_i denotes the four-vector energy momentum.

In two-component (block) spinors, the solutions of (38) are given by:

$$\psi_{31,2} = \left(\begin{vmatrix} -p_3 \\ p_1 + ip_2 \end{vmatrix} + \begin{vmatrix} p_1 - ip_2 \\ p_3 \end{vmatrix}\right)\frac{e^{(I+\gamma)\frac{p \bullet x}{2}}}{p_0} \tag{39}$$

$$\psi_{33,4} = -\left(\begin{vmatrix} -p_3 \\ p_1 + ip_2 \end{vmatrix} + \begin{vmatrix} p_1 - ip_2 \\ p_3 \end{vmatrix}\right)\frac{e^{(I+\gamma)\frac{p \bullet x}{2}}}{p_0} \tag{40}$$

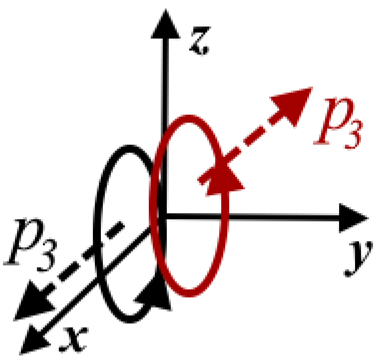

Figure 2. *Neutrino.*

It is evident that the solutions $\psi_{31,2}$ and $\psi_{33,4}$ differ only in sign. Simply put, $\psi_{31,2}$ and $\psi_{33,4}$ describe the same particle. A change in helicity is observed when a particle transitions from state ψ_{31} to ψ_{32} as well as from ψ_{33} to ψ_{34}. This phenomenon is illustrated in Figure 2. If the helicity was left-handed, then it would be right-handed in the opposite direction. Helicity is not a *P*-parity-invariant quantity.

The particle is its own antiparticle, that is, solution (38) does not change if we change the

$$exp((I+\gamma)p \bullet x/2) \text{ to the } exp(-(I+\gamma)p \bullet x/2).$$

However, each state, $\psi_{\alpha 1}$ and $\psi_{\alpha 2}$ (also $\psi_{\alpha 3}$ and $\psi_{\alpha 4}$), behaves as independent, separate particles.

Solutions (39) and (40) can be written in compact form as follows:

$$\psi_3 = (\sigma_1 p_1 + \sigma_2(\pm p_2) + \sigma_3(\mp p_3))\frac{exp\left((I+\gamma)\frac{p \bullet x}{2}\right)}{p_0} \quad (41)$$

The upper signs before the impulses refer to the solution $\psi_{31,2}$, and the lower signs refer to $\psi_{33,4}$.

ψ_α and $\tilde{\psi}_\alpha$ are obtained from the torsion of the field (without deformation); thus, they must be antisymmetric functions. To verify this, we considered the wave function of a system comprising two identical non-interacting particles and performed a swap operation on them:

$$\psi_{\alpha,\beta} = \frac{p\, exp\left((I+\gamma)\frac{p \bullet x}{2}\right)}{p_0} \wedge \frac{q\, exp\left((I+\gamma)\frac{q \bullet x}{2}\right)}{q} =$$

$$= -\frac{q\, exp\left((I+\gamma)\frac{q \bullet x}{2}\right)}{q} \wedge \frac{p\, exp\left((I+\gamma)\frac{p \bullet x}{2}\right)}{p_0} = -\psi_{\beta,\alpha}$$

The wave function of the superposition of two identical, non-interacting particles changes sign when the particles are permuted. The particles described by the function ψ_α are fermions. Equations (36) and (37) are equivalent to Dirac equations for fermions.

Because ψ_3 is a spinor and the mass of this fermion is zero, the particle itself is one of the three generations of neutrinos. The equations for the remaining generations can be solved similarly.

3.2. Solutions for Photon

Now, we consider (37) without field torsion, that is, only with deformation:

$$\gamma^i U^T \partial_i\, exp\left((I+\gamma)\frac{p \bullet x}{2}\right) = 0 \quad (42)$$

In two-component spinors, the solutions of (42) are as follows:

$$\chi_{31,2} = \left(\begin{vmatrix} p_3 \\ p_1 + ip_2 \end{vmatrix} + \begin{vmatrix} p_1 - ip_2 \\ -p_3 \end{vmatrix}\right)\frac{exp\left((I+\gamma)\frac{p \bullet x}{2}\right)}{p_0} \quad (43)$$

$$\chi_{33,4} = -\left(\begin{vmatrix} p_3 \\ p_1 + ip_2 \end{vmatrix} + \begin{vmatrix} p_1 - ip_2 \\ -p_3 \end{vmatrix}\right)\frac{exp\left((I+\gamma)\frac{p \bullet x}{2}\right)}{p_0} \quad (44)$$

Solutions (43) and (44), that is, $\chi_{31,2}$ and $\chi_{33,4}$, differ only in terms of sign.

Functions $\chi_{31,2}$ and $\chi_{33,4}$ describe the same particle. Moreover, the particle is its own antiparticle with a mass of zero.

Upon transitioning from state χ_{31} to χ_{32} (and similarly from χ_{33} to χ_{34}), the helicity of the particle remained unaltered, maintaining a right-handed polarization (Figure 3). There is no difference between particles and antiparticles.

Solutions (43) and (44) can be written in compact form as follows:

$$\chi_3 = (\sigma_1 p_1 + \sigma_2(\pm p_2) + \sigma_3(\pm p_3))\frac{exp\left((I+\gamma)\frac{p \bullet x}{2}\right)}{p_0} \quad (45)$$

The upper signs before the impulses refer to the solution $\chi_{31,2}$, and the lower signs refer to $\chi_{33,4}$.

It is clear that χ_α and $\tilde{\chi}_\alpha$ are derived from field deformation (without torsion), and they must be symmetric functions. To verify this, we consider the wave function of a system of two identical noninteracting particles and swap them as follows:

$$\chi_{\alpha,\beta} = \frac{p\, exp\left((I+\gamma)\frac{p \bullet x}{2}\right)}{p_0} \bullet \frac{q\, exp\left((I+\gamma)\frac{q \bullet x}{2}\right)}{q} =$$

$$= \frac{q\, exp\left((I+\gamma)\frac{q \bullet x}{2}\right)}{q} \bullet \frac{p\, exp\left((I+\gamma)\frac{p \bullet x}{2}\right)}{p_0} = \chi_{\beta,\alpha}$$

The wave function of the superposition of two identical noninteracting particles remained unchanged when the particles were rearranged. The particles described by the function χ_α are bosons. Equations (43) and (44) are equivalent to Dirac equations for bosons.

Given that function (45) is an function χ_3 for a boson, it can be inferred that the particle corresponding to it is a boson. The mass of this boson is zero, and there exist only two possible states. Based on these observations, it was concluded that the particle in question is a photon.

We have considered a photon in a flat isotropic space. In such a space, all three "generations" of photons are identical. In a Riemannian, and even more so in the anisotropic space $|\tau_{\alpha 0}|$, the photon (boson) will be different for each "direction" of α0.

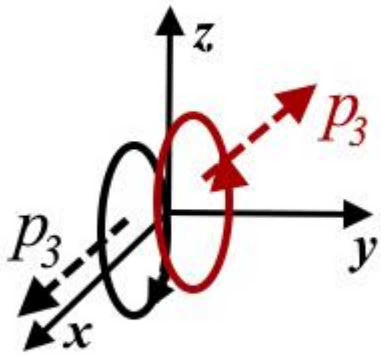

Figure 3. *Photon.*

4. Conclusions

1) Clifford algebra in the pseudo-Riemannian space $R^{1,3}$ with torsion and spinor bundles can be employed to unify gravitational, electromagnetic, and spinor fields into a single structure. Field deformation is a prerequisite for gravity, and torsion is a prerequisite for electromagnetism. The existence of a spinor field is contingent on the presence of deformation, torsion, and the spinor bundle.
2) The tensor basis method and Clifford product of vectors permit the consideration of both deformation and torsion within the field, thereby facilitating the unification of the Einstein equation and Maxwell system of equations. Einstein and Maxwell's equations are equivalent. Einstein's equations describe spatial quantities including the Ricci tensor and scalar curvature. Maxwell's equations describe field quantities such as the electromagnetic field tensor, 4-current, etc.
3) The inhomogeneity of the *DA* field can be represented by the sum of the three biquaternions, each of which splits into a pair of bispinors. Differentiation of the bispinors yields three Dirac-type equations for the bosons and fermions. So the solution of Dirac-type equations with deformation describes a boson, whereas the solution with torsion describes a fermion.

   In a four-dimensional space, there are only three biquaternions: R_{10}, R_{20}, and R_{30}. Consequently, there could only be three particle generations.

   Massless particles, photon, and three generations of neutrinos (γ, ν_e, ν_μ, ν_τ) were combined into a single structure: singlet (photon) + triplet (three neutrinos).
4) The divergence gradient ($D(D\bullet A)$) is a 4-dimensional electromagnetic current. This formulation of the four-current gives the nonlinearity of the inhomogeneous Maxwell system and the laws of conservation of the four-current and eddy currents. It also clearly shows the geometric meaning of singularities – field "holes," i.e., current sources and the need for calibrations such as Lorentz, Coulomb, and others.

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Author Contributions

Alimzhan Kholmuratovich Babaev is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The author declares no conflicts of interest.

Appendix

$$D\bullet(\Box A - Ric\bullet A) = D\bullet(\mu T\bullet A)$$

In coordinate form:

$$e^n\bullet e^k D_n (\Box A_k - A_p R^p{}_k) = e^n\bullet e^k D_n (\mu A_p T^p{}_k)$$

By simplifying we get:

$$g^{nk}D_n (g^{ij}D_i D_j A_k) - g^{nk}D_n (A_p R^p{}_k) = \mu g^{nk}D_n (A_p T^p{}_k)$$

$$D_k (g^{ij}D_i D_j A^k) - D_k (A^i R^k{}_i) = \mu D_k (A^i T^k{}_i)$$

We open brackets under differentiation:

$$D_k D^i D_j A^k - D_k A^i R^k{}_i - A^i D_k R^k{}_i = \mu D_k A^i T^k{}_i + \mu A^i D_k T^k{}_i \quad \text{(A-1)}$$

It is known that [2] $D_k R^k{}_i = 0.5 D_i R$
and according to formula (16) $\mu D_k A^i T^k{}_i = D_k j^k$.
Then, (A-1) takes the form:

$$D_k D^i D_j A^k - D_k A^i R^k{}_i - 0.5A^i D_i R = D_k j^k + \mu A^i D_k T^k{}_i \quad \text{(A-2)}$$

In the first term we change the differentiation order (k with i).

$$D_n D_i D_j A^k = D_i D_n D_j A^k,$$

because

$$g^{ij} (D_k D_i D_j A^k - D_i D_k D_j A^k) =$$
$$= g^{ij} (D_p A^k R^p{}_{jik} - D_j A^p R^k{}_{pik}) = - D_p A^k R^p{}_k + D_j A^p R^j{}_p = 0.$$

Now in $D_i D_k D_j A^k$ we change the order of differentiation (k with j):

$$g^{ij} D_i (D_k D_j A^k - D_j D_k A^k) = g^{ij} D_i (- A^p R^k{}_{pjk}) =$$

$$= D_i (A^p R^i{}_p) = D_i A^p R^i{}_p + A^p D_i R^i{}_p$$

or $g^{ij}\, D_i D_k D_j A^k = g^{ij}\, D_i D_j D_k A^k + D_i A^p R^j_p + A^p D_i R^j_p$

Considering that according to formula (10) $e^j D_j D_k A^k = j$ is a four-dimensional electromagnetic current, from the last equality, we obtain:

$$g^{ij} D_k D_i D_j A^k = D_i j^i + D_i A^p R^j_p + A^p D_i R^j_p \qquad \text{(A-3)}$$

(A-3) substitute into (A-2):

$$D_k j^k + D_k A^p R^k_p + A^p D_k R^k_p - D_k A^p R^k_p - 0.5\, A^i D_i R =$$

$$= D_k j^k + \mu A^p D_k T^k_p$$

Now, we reduce the four currents and simplify:

$$D_k R^k_p - 0.5\, \delta^k_i D_k R = \mu D_k T^k_p$$

"Getting rid" of differentiation, we obtain Einstein's equation:

$$R^k_p - 0.5\ \delta^k_i\ R + \delta^k_i\ \lambda_i = \mu\ T^k_p \qquad \text{(A-4)}$$

λ_i are the cosmological constants or eigenvalues of the Bochner operator.

References

[1] Chris J. L. Doran. Geometric Algebra and its Application to Mathematical Physics. Sidney Sussex College. A dissertation submitted for the degree of Doctor of Philosophy in the University of Cambridge. February 1994, https://doi.org/10.17863/CAM.16148

[2] L. D. Landau and E. V. Lipschitz, The Classical Theory of Fields, vol. 2. pp. 70 – 80, 295 – 301. https://haidinh89.wordpress.com/wp-content/uploads/2015/08/landau-l-d-lifshitz-e-m-course-of-theoretical-physics-vol-02-the-classical-theory-of-fields-3305.pdf

[3] Wikipedia. Dirac equation, Available from: https://en.wikipedia.org/wiki/Dirac_equation

[4] Wikipedia. Minkowski spacetime, Available from: https://en.wikipedia.org/wiki/Minkowski_space

[5] Wikipedia. Einstein – Cartan theory. Available from: https://en.wikipedia.org/wiki/Einstein%E2%80%93Cartan_theory

[6] A. Kh. Babaev. Biquaternions, rotations and spinors in the generalized Clifford algebra. sci-article, №45 (May) 2017, https://sci-article.ru/number/05_2017.pdf pp. 296 – 304. (In Russian).

[7] Wikipedia. Tensor product, Available from: https://en.wikipedia.org/wiki/Tensor_product

[8] Wikipedia. Torsion tensor, Available from: https://en.wikipedia.org/wiki/Torsion_tensor

[9] Wikipedia. Dirac operator, Available from: https://en.wikipedia.org/wiki/Dirac_operator

[10] Wikipedia. Covariant derivative, Available from: https://en.wikipedia. org/wiki/Covariant derivative

[11] Wikipedia. Main invariants of the tensor. Available from: https://en.wikipedia.org/wiki/Invariants_of_tensors

[12] A. Kh. Babaev. An alternative formalism based on Clifford algebra. sci-article, №40 (December) 2016, (In Russian). https://doi.org/10.24108/preprints-3112490

[13] Wikipedia. Levi - Civita epsilon. Available from: https://en.wikipedia.org/wiki/Levi-Civita_symbol

[14] Wikipedia. Gauge fixing. Available from: https://en.wikipedia.org/wiki/Gauge_fixing

[15] A. Kh. Babaev. Four-dimensional current conservation law in a Clifford algebra-based formalism. sci-article, №42 (February) 2017. pp. 27 – 33. Preprint https://doi.org/10.24108/preprints-3112478 (In Russian).

[16] Faraday's law. https://en.wikipedia.org/wiki/Faraday%27s_law_of_induction

[17] Peter Petersen. Demystifying the curvature term in Lichnerowicz Laplacians. https://doi.org/10.3842/SIGMA.2020.064

[18] Bochner Laplacian. https://en.wikipedia.org/wiki/Laplace_operators_in_differential_geometry

[19] A. Kh. Babaev. Equivalence of the inhomogeneous system of Maxwell's equations and Einstein's equations. sci-article, №43 (March) 2017. https://doi.org/10.13140/RG.2.2.33648.47362 (In English)

[20] Wikipedia. Electromagnetic tensor. Available from: https://en.wikipedia.org/wiki/Electromagnetic_tensor

[21] Wikipedia. Matrix exponential. Available from: https://en.wikipedia.org/wiki/Matrix_exponential

[22] A. Kh. Babaev. Description of Lorentz transformations, the Doppler effect, Hubble's law, and related phenomena in curvilinear coordinates by generalized biquaternions. December 2024, https://www.researchgate.net/publication/386334104_Description_of_Lorentz_transformations_the_Doppler_effect_Hubble's_law_and_related_phenomena_in_curvilinear_coordinates_by_generalized_biquaternions.

[23] Wikipedia. Prime ideal. Available from: https://en.wikipedia.org/wiki/Prime_ideal

[24] Wikipedia. Spinors. Available from: https://en.wikipedia.org/wiki/Spinor

[25] A. Kh. Babaev. Derivation of the Dirac equation from the inhomogeneity of space and solution for neutrino generations. sci-article, №52 (December) 2017. https://sci-article.ru/number/12_2017.pdf pp. 237- 244 (In Russian)

Biography

Alimzhan Kh. Babaev defended his thesis for a PhD in physical and math sciences on the topic "Multiple collisions of particles and fragmentations of 22Ne nuclei in a photoemulsion at P/A=4.1 GeV/c" in 1989 at the Institute of Nuclear Physics of the Academy of Sciences of Uzbekistan (Tashkent). He worked at the Department of Nuclear Physics and Cosmic Rays at the National University (Tashkent, Uzbekistan) as an associate professor. Since 2000, he has worked as a lecturer at the Novosibirsk State Technical University (Novosibirsk, Russian Federation) in the Department of Higher Mathematics. At present, he is working as an independent researcher in the field of applications of methods of abstract algebra to classical and quantum field physics. He is the author (co-author) of more than 30 scientific papers published in peer-reviewed journals.

Research Field

Alimzhan Kholmuratovich Babaev: Clifford algebra, Gravity, Electromagnetism, Biquaternions and Spinor fields, Partial differential equations, Unified field theory.