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The Schmidt Decomposition for Entangled System and Nonadiabatic Berry Phases

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Abstract: The time-dependent Hamiltonians are a very important portion in the modeling of real systems. In fact, the dynamic description of an entangled quantum systems is reflected in full coherence with the resolution of a wave function, solution of the Schrödinger equation throughout the entire study path. In this regard, we specify in this paper the system of two-site Bose-Hubbard model that obeys tunnel behavior, as two coupled harmonic oscillators, to examine quantum entanglement. The dynamics of such a system is described by the Schrödinger equation have introduced to the solution, the non-linear Ermakov equations as well as through a passage to the Heisenberg picture approach and the general Lewis and Riesenfeld invariant method compute between coupled harmonic oscillators and the coupled Caldirola Kanai oscillators. We prove that a time exponential increase in the mass of the system brings back to an exponential increase of entanglement and the Heisenberg picture approach is the most stable method to quantum entanglement because, this last has reached very large values. Also, we specify a cyclic time evolution, we find analytically the nonadiabatic Berry phases. In a particular case, such an entangled system acquired a nonadiabatic Berry phases that exhibits the same behavior as the Schmidt parameter.

Keywords: Two-site Bose-Hubbard Model, Schmidt Mode, Entanglement, Nonadiabatic Berry Phases

1. Introduction

The study of the dynamic systems is an issue that raises a large concern in the quantum mechanics and quantum information fields [1-4] particularly, the quantum entanglement [5-7]. From numerous applications, The explanation of an entangled system is obtained in particular thanks to the quantum communication tools, we speak here for example of data processing on the basis of quantum physics; the quantum computing and algorithms [8, 9]. The security of the Internet communications opens thus the way on a necessary parade: quantum cryptography, teleportation etc [10-13]. In this context, an entangled system is a system composed of two subsystems described in Hilbert space by the Hamiltonian $H = H_1 \bigoplus H_2$. The dynamics analysis of entangled systems using the Schmidt decomposition is considerably more complicated than that of stationary systems. In fact, we have to decompose a state vector on a stationary basis to derive the Schmidt mode. However, the calculation of the Schmidt mode for a dynamic state vector is performed only in special cases. Furthermore, the Schmidt mode is not available by a direct Schmidt decomposition of the time-dependent Schrödinger equation $(TDSE)$ solution through the non-linear Ermakov equation. Another important aspect that interests us; the analogy between the behavior of a state vector and the Schmidt decomposition for a periodic evolution leads to the definition of the Berry phases. These geometrical phases have associated with various interesting quantum phenomena: the theory of macroscopic polarization with the King-Smith-Vanderbilt formula[14,15], the whole quantum Hall effect (QHE) incarnated by the Thouless Kohmoto-Nightingale-Nijs formula [16, 17] etc. The consequences of the existence of the non-trivial Berry phases are the cyclical evolution of the system and the existence of the localized state [18-20]. The Berry phases associated with the entanglement of a Gaussian wave functions for a bipartite system is a question to be discussed in this paper. We try to find some properties of these two phenomena, entanglement and nonadiabatic Berry phases for the system of two bosons undergoing the tunneling effect under a full dynamicity.

2. Generalization of the Problem: Two-site Bose-Hubbard Model Hamiltonian

The central feature of the discussion is focused herein on the two-site Bose-Hubbard model Hamiltonian. On the way back from the ref [21], it is expressed as follows:

$$H = \omega_{BH1}(t)\left(a_1^+ a_1 + \frac{1}{2}\right) + \omega_{BH2}(t)\left(a_2^+ a_2 + \frac{1}{2}\right) - J(t)\left(a_1^+ a_2 + a_2^+ a_1\right). \tag{1}$$

To describe the time evolution of (1), we need to solve the $(TDSE)$ following such a procedure $i\frac{\partial\psi}{\partial t} = H\psi$. a_k, a_k^+ in (1) are respectively the annihilation and creation operators $(k = 1.2)$. A direct analysis of the basic characteristics of the system is not efficient. To start, we need to diagonalize the Hamiltonian in consonance with the new operators positions $x_k = \frac{1}{\sqrt{2}}\left(a_k^+ + a_k\right)$ and moments $p_k = \frac{i}{\sqrt{2}}\left(a_k^+ - a_k\right)$ to get the Hamiltonian with a form:

$$H = \frac{\omega_{BH1}(t)}{2}\left(x_1^2 + p_1^2\right) + \frac{\omega_{BH2}(t)}{2}\left(x_2^2 + p_2^2\right) - J(t)\left(x_1 x_2 + p_1 p_2\right). \tag{2}$$

We specify the coordinate system $x_1, x_2; p_1, p_2$, the rotation is performed by the algebraic matrix

$$I = L_{ij} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix},$$

and we choose

$$\alpha = \frac{1}{2}\tan^{-1}\left(\frac{2J(t)}{\omega_{BH1}(t) - \omega_{BH2}(t)}\right). \tag{3}$$

It is thus possible to transmit our physical system from one branch to another through these mathematical tools by reformulating a new spatial phase denominated with position coordinates x_i, momenta p_i, commutable between them and satisfying the new transformation $x_i = L_{ij}x_j$, $p_i = L_{ij}p_j$, $(i = +.-, j = 1.2)$, so we can be with the form

$$H = \frac{A_{BH+}(t)}{2}\left(x_+^2 + p_+^2\right) + \frac{A_{BH-}(t)}{2}\left(x_-^2 + p_-^2\right). \tag{4}$$

Noticing that

$$A_{BH\pm} = \omega_{BH1} \pm J(t)\tan\alpha, \tag{5}$$

To generate a new Hamiltonian from (4) and better described by two decoupled harmonic oscillators with new frequency formalism, therefore, we define a scale transformation reaching respectively, the position $\tilde{x_1} = \sqrt[4]{\mu^{-1}}x_+$, $\tilde{x_2} = \sqrt[4]{\mu}x_-$ and momentum variables $\tilde{p_1} = \sqrt[4]{\mu}p_+$, $\tilde{p_2} = \sqrt[4]{\mu^{-1}}p_-$ in order to obtain

$$H = \frac{d}{2}\left(\tilde{p_1^2} + \tilde{p_2^2}\right) + \frac{1}{2}d_1\tilde{x_1^2} + \frac{1}{2}d_2\tilde{x_2^2}, \tag{6}$$

where

$$d = \sqrt{A_{BH+}A_{BH-}}, \quad d_1 = A_{BH+}\sqrt{\mu}, \quad d_2 = A_{BH-}\sqrt{\mu^{-1}}, \tag{7}$$

and

$$\mu = \frac{A_{BH+}}{A_{BH-}}. \tag{8}$$

The formalism (6) is very useful for the analysis of entangled states. Our concern is to examine the time-evolution entanglement and the cyclic nonadiabatic Berry phases of two-site Bose-Hubbard model in the tunnelling regime when the Hamiltonian parameters have evolved with time. A direct consequence is the solution of the $(TDSE)$, it will be generalized by the non-linear Ermakov equations [22, 23] which asserts the existence of the Schmidt decomposition. In that meaning, the solution of the non-linear Ermakov equations is discussed by ref [24, 25] to examine entanglement. The procedures followed by the references [22, 23] and the Hamiltonian (6) give us the wave function

$$\begin{aligned}\psi_{n,m}(x_1,x_2;t) =&\frac{1}{\sqrt{2^{n+m}n!m!}}\left(\frac{\varepsilon_1\varepsilon_2}{\pi^2}\right)^{\frac{1}{4}}\exp\Big[-i[\alpha_n(t)+\alpha_m(t)]\Big]\\ &\times\exp\left[-\frac{\varepsilon_1}{2}\left(\sqrt[4]{\mu^{-1}}x_1\cos\alpha-\sqrt[4]{\mu}x_2\sin\alpha\right)^2-\frac{\varepsilon_2}{2}\left(\sqrt[4]{\mu^{-1}}x_1\sin\alpha+\sqrt[4]{\mu}x_2\cos\alpha\right)^2\right]\\ &\times\exp\left[i\frac{b_1^{'}(t)}{2b_1(t)}\big(\sqrt[4]{\mu^{-1}}x_1\cos\alpha-\sqrt[4]{\mu}x_2x_2\sin\alpha\big)^2+i\frac{b_2^{'}(t)}{2b_2(t)}\big(\sqrt[4]{\mu^{-1}}x_1\sin\alpha+\sqrt[4]{\mu}x_2\cos\alpha\big)^2\right]\\ &\times H_n\left[\sqrt{\varepsilon_1}\left(\sqrt[4]{\mu^{-1}}x_1\cos\alpha-\sqrt[4]{\mu}x_2\sin\alpha\right)\right]H_m\left[\sqrt{\varepsilon_2}\left(\sqrt[4]{\mu^{-1}}x_1\sin\alpha+x_2\cos\alpha\right)\right],\end{aligned} \tag{9}$$

where α_n,α_m are defined by the relation:

$$\alpha_n(t)=E_n\tau_1\text{and}\quad\alpha_m(t)=E_m\tau_2. \tag{10}$$

E_n, E_m are respectively the corresponding energies. They are given as

$$E_n=-\left(n+\frac{1}{2}\right)d_1\text{and}\quad E_m=-\left(m+\frac{1}{2}\right)d_2, \tag{11}$$

with

$$\tau_1=\int_0^t\frac{d\iota}{b_1^2(\iota)},\quad\tau_2=\int_0^t\frac{d\iota}{b_2^2(\iota)}, \tag{12}$$

and

$$\varepsilon_1=\frac{d_1}{b_1^2},\quad\varepsilon_2=\frac{d_2}{b_2^2}. \tag{13}$$

The parameters b_1,b_2 satisfy the non-linear Ermakov equations $\ddot{b_1}+d_1^2(t)b_1=\frac{d_1^2(0)}{b_1^3}$ and $\ddot{b_2}+d_2^2(t)b_2=\frac{d_2^2(0)}{b_2^3}$. They are given in the ref [26], where

$$b_1(t)=\left(\frac{\Delta d_1}{2d_1(f)}\cos(2\sqrt{d_1(f)}t)+\frac{\Delta d_1}{d_1(f)}\right)^{\frac{1}{2}},$$

and

$$b_2(t)=\left(\frac{\Delta d_2}{2d_2(f)}\cos(2\sqrt{d_2(f)}t)+\frac{\Delta d_2}{2d_2(f)}\right)^{\frac{1}{2}}. \tag{14}$$

$\Delta d_k=d_k(f)-d_k(i)$, with $d_k(i)$ and $d_k(f)$ are respectively the frequencies at the initial and the final time. The initial condition is defined by the time $t=0$, the parameter $b_k(0)=1$ and the wave function

$$\psi_{n,m}\left(\tilde{x}_1,\tilde{x}_2;0\right)=\frac{1}{\sqrt{2^{n+m}n!m!}}\left(\frac{d_1(0)d_2(0)}{\pi^2}\right)^{\frac{1}{4}}\exp\left[-\frac{d_1(0)}{2}\tilde{x}_1^2-\frac{d_2(0)}{2}\tilde{x}_2^2\right]H_n\left(\sqrt{d_1(0)}\tilde{x}_1\right)H_m\left(\sqrt{d_2(0)}\tilde{x}_2\right). \tag{15}$$

3. Non-cyclic State and Entanglement

We will try to decompose in the orthonormalization time-dependent Schmidt basics, the vector $\psi_{n,m}(x_1,x_2;t)$ presented in the formula (9). Start by considering at time $t=0$ a global system consisting with two separeted subsystems (harmonic oscillator 1 and 2), leads to $J=0$, and are described by the $\eta_i\left(x_1\right)$ and $\sigma_j\left(x_2\right)$ wave functions. From the Schmidt theorem [28, 29], now it is easy to decompose the considered wave function into (15) as:

$$\psi_{n,m}\left(\tilde{x}_1,\tilde{x}_2;0\right)=\sum_{i,j}d_{n,m}^{i,j}\eta_i\left(x_1\right)\sigma_j\left(x_2\right), \tag{16}$$

where $d_{n,m}^{i,j}$ is the coefficient of expansion. It is defined throughout the orthogonality relation and is given as $d_{n,m}^{i,j} = \langle\psi_{n,m}\left(\tilde{x}_1, \tilde{x}_2; 0\right)|\eta_i\left(x_1\right)\sigma_j\left(x_2\right)\rangle$. To calculate $d_{n,m}^{i,j}$, we then consider a less restrictive approximation such as $J \simeq 0$, leads to $d_1 \simeq d_2$, $d_1 - d_2 \leq J$ and $b_1(t) \simeq b_2(t)$. After these simplifications, the parameter μ and the wave function in (15) becomes respectively: $\mu = 1$ and

$$\begin{aligned}\psi_{n,m}(\tilde{x}_1, \tilde{x}_2; 0) =& \frac{1}{\sqrt{2^{n+m}n!m!}}\Big(\frac{\omega_{BH1}(0)}{\pi}\Big)^{\frac{1}{2}} \exp\Big[-\frac{\omega_{BH1}(0)}{2}\big(x_1\cos\alpha - x_2\sin\alpha\big)^2 \\ &-\frac{\omega_{BH1}(0)}{2}\big(x_1\sin\alpha + x_2\cos\alpha\big)^2\Big] \\ &\times H_n\left(\sqrt{\omega_{BH1}(0)}\left(x_1\cos\alpha - x_2\sin\alpha\right)\right) H_m\left(\sqrt{\omega_{BH1}(0)}\left(x_1\sin\alpha + x_2\cos\alpha\right)\right).\end{aligned} \tag{17}$$

Following similar problem that is developed in refs [29- 31], $d_{n,m}^{i,j}$ becomes

$$d_{n,m}^{i,j} = (-1)^n \frac{\tan^{i+n}(\alpha)}{\left(1+\tan^2(\alpha)\right)^{\frac{m+n}{2}}}\sqrt{\frac{m!n!}{i!j!}} P_n^{(-(1+m+n), m-i)}\left(1 + \frac{2}{\tan^2(\alpha)}\right). \tag{18}$$

$P_n^{(-(1+m+n), m-i)}\left(1 + \frac{2}{\tan^2(\alpha)}\right)$ is the Jacobi polynomial.

At this stage, we realize a given basic of the dynamic pure state $\psi_{n,m}(x_1, x_2; t)$ in (9) by defining a set of unitary operators able to describe this basic. When the system evolves from $t = 0$ to any t, based on the refs[27, 32], we define a unit evolution depicted in the orthogonal base of the unbound system by:

$$u_i^{'}\left(x_1; t\right) = \left(\frac{1}{b_1^2}\right)^{\frac{1}{4}} e^{\frac{i}{2}\left(\frac{b_1^{'}}{b_1}\right)x_1^2} = \left(\frac{1}{b_1^2}\right)^{\frac{1}{4}} u_i\left(x_1; t\right)$$

and

$$v_j^{'}\left(x_2; t\right) = \left(\frac{1}{b_2^2}\right)^{\frac{1}{4}} e^{\frac{i}{2}\left(\frac{b_2^{'}}{b_2}\right)x_2^2} = \left(\frac{1}{b_2^2}\right)^{\frac{1}{4}} v_j\left(x_2; t\right). \tag{19}$$

So, the pure state defined in (9) can be reexpressed using the Schmidt decomposition in the original base of the initial state (17) to give

$$\psi\left(x_1, x_2; t\right) = \sum_{i,j} A_{i,j}(t)\eta_i^{'}\left(\frac{x_1}{b_1}; t\right)\sigma_j^{'}\left(\frac{x_2}{b_2}; t\right), \tag{20}$$

where

$$\eta_i^{'}\left(x_1; t\right) = u_i^{'}\left(x_1; t\right)\eta_i\left(\frac{x_1}{b_1}; t\right) \text{ and } \quad \sigma_j^{'}\left(\frac{x_2}{b_2}; t\right) = v_j^{'}\left(x_2; t\right)\sigma_j\left(\frac{x_2}{b_2}; t\right). \tag{21}$$

Due to the form of the wave function in (20), it is not possible to derive the coefficient of expansion $A_{i,j}(t)$ directly. Suppose thus a new eigenstate alike to the time-independent eigenstate of two decoupled harmonic oscillators and is given as:

$$\begin{aligned}\psi_{n,m}(\tilde{\xi}_1, \tilde{\xi}_2; t) =& \frac{1}{\sqrt{2^{n+m}n!m!}}\left(\frac{d_1 d_2}{\pi^2}\right)^{\frac{1}{4}} \exp\Big[-i[\alpha_n(t) + \alpha_m(t)]\Big] \\ &\times \exp\left[-\frac{d_1}{2}\tilde{\xi}_1^2 - \frac{c_2}{2}\tilde{\xi}_2^2\right] H_n\left[\sqrt{d_1}\tilde{\xi}_1\right] H_m\left[\sqrt{d_2}\tilde{\xi}_2\right],\end{aligned} \tag{22}$$

where $\tilde{\xi}_k = \frac{\tilde{x_k}}{b_k}$. We put an analogy with the wave function of the initial state assumed solution of the stationary Schrödinger equation. In accordance with this, we rewrite our wave function (20) from a new variables $(\frac{x_1}{b_1}, \frac{x_2}{b_2})$ as:

$$\psi\left(\xi_1, \xi_2; t\right) = \sum_{n,m} b_{n,m} e^{-i(\alpha_n(t)+\alpha_m(t))} \psi_{n,m}\left(\tilde{\xi}_1, \tilde{\xi}_2; 0\right). \tag{23}$$

$\psi_{n,m}\left(\tilde{\xi}_1, \tilde{\xi}_2; 0\right)$ is determined by expression (15) to replace the variables $(\tilde{x}_1, \tilde{x}_2)$ with $(\tilde{\xi}_1, \tilde{\xi}_2)$.

We follow the same procedures as with the wave function of (16), we obtain an integral similar to that given by $d^{i,j}_{n,m}$ with a variables $(\tilde{x}_1, \tilde{x}_2)$ where $b_{n,m} = d^{i,j}_{n,m}$. They are given by expression (18). We can rewrite the wave function $\psi\left(\xi_1, \xi_2; t\right)$ in (23), according to eigenfrequencies $\eta_i\left(\xi_1\right)$ and $\sigma_j\left(\xi_2\right)$ of the separated system as:

$$\psi\left(\xi_1, \xi_2; t\right) = \sum_{i,j} A_{i,j}(t) e^{-i\alpha_k(t)} \eta_i\left(\xi_1\right) e^{-i\alpha_p(t)} \sigma_j\left(\xi_2\right). \tag{24}$$

The coefficient of expansion $A_{i,j}(t)$ is achieved by making an analogy between the two expressions (23), (24) and using the orthogonality condition. The nondegenerate eigenvalues are expressed by setting the condition $k + p = n + m$, we obtain

$$A_{i,j}(t) = \sum_{n=0}^{k+p} d^{k,p}_{n,k+p-n} d^{*i,j}_{n,k+p-n} e^{-i\left[\alpha_n + \alpha_{k+p-n} - \alpha_k - \alpha_p\right]}. \tag{25}$$

With particular report to the initial state basis vectors presented in (16) at an approximate factor, the time-dependent wave function in (20) becomes:

$$\psi\left(x_1, x_2; t\right) = \sum_{i,j} \frac{A_{i,j}(t)}{(b_1^2)^{\frac{1}{4}} (b_2^2)^{\frac{1}{4}}} u\left(x_1, t\right) \eta_i\left(\frac{x_1}{b_1}; t\right) v\left(x_2, t\right) \sigma_j\left(\frac{x_2}{b_2}; t\right). \tag{26}$$

When the condition $k + p = i + j$ is fulfilled, the reduced density matrices corresponding to the wave function in (24) are given as

$$\rho_A\left(\xi_1, \xi_1^{'}\right) = \sum_{i=0}^{k+p} |A_{i,k+p-i}(t)|^2 \eta_i\left(\xi_1\right) \eta_i^*\left(\xi_1^{'}\right), \tag{27}$$

and

$$\rho_B\left(\xi_2, \xi_2^{'}\right) = \sum_{i=0}^{k+p} |A_{i,k+p-i}(t)|^2 \sigma_{k+p-i}\left(\xi_2\right) \sigma_{k+p-i}^*\left(\xi_2^{'}\right). \tag{28}$$

Going back to the original variables (x_1, x_2), (20) becomes:

$$\psi\left(x_1, x_2; t\right) = \sum_{i,j} \frac{A_{i,j}(t)}{(b_1^2)^{\frac{1}{4}} (b_2^2)^{\frac{1}{4}}} u\left(x_1, t\right) \eta_i\left(\frac{x_1}{b_1}; t\right) v\left(x_2, t\right) \sigma_j\left(\frac{x_2}{b_2}; t\right). \tag{29}$$

With the above, the corresponding reduced density matrices are denoted as:

$$\rho_A^{'}\left(x_1, x_1^{'}\right) = u_i\left(x_1; t\right) \frac{\rho_A\left(\frac{x_1}{b_1}, \frac{x_1^{'}}{b_1}\right)}{(b_1^2)^{\frac{1}{4}} (b_2^2)^{\frac{1}{4}}} u_i^*\left(x_1^{'}; t\right), \tag{30}$$

and

$$\rho_B^{'}\left(x_2, x_2^{'}\right) = v_j\left(x_2; t\right) \frac{\rho_B\left(\frac{x_2}{b_2}, \frac{x_2^{'}}{b_2}\right)}{(b_1^2)^{\frac{1}{4}} (b_2^2)^{\frac{1}{4}}} v_j^*\left(x_2^{'}; t\right). \tag{31}$$

From expression (21) and its conjugate $\psi^*\left(x_1^{'}, x_2^{'}; t\right)$ [33, 34], we define the Schmidt mode $\gamma_k(t)$ and the entanglement von Neumann entropy respectively as

$$\gamma_k(t) = \left|\frac{A_{i,k+p-i}(t)}{(b_l)^{\frac{1}{2}}(b_2)^{\frac{1}{2}}}\right|^2, \tag{32}$$

and

$$S = -\sum_k \gamma_k(t) ln\gamma_k(t). \tag{33}$$

The Schmidt parameter is described following refs [35, 36] as

$$\Lambda(t) = \frac{1}{\sum_k \gamma_k^2(t)}. \tag{34}$$

If we consider a cyclic evolution around a closed circuit during a period T, then in this context we can talk to a second concept of quantum mechanics: the Berry phases. We focus in the following section on the Berry phases in the nonadiabatic context.

4. Cyclic State and Nonadiabatic Berry Phases

This section is based on the discussion of the entangled cyclic state to calculate the nonadiabatic Berry phases. The adiabatic approximation of the topological Berry phase extends to the nonadiabatic evolution case by the appropriate choice of the initial state $\psi_{n,m}\left(x_1, x_2; 0\right)$ [37].

Referring to (9), note that at $t = 0$, the system is in the state:

$$\begin{aligned}\psi_{n,m}(x_1, x_2; 0) =& \frac{1}{\sqrt{2^{n+m}n!m!}}\left(\frac{d_1 d_2}{\pi^2}\right)^{\frac{1}{4}} \exp\Big[-i[\alpha_n(t)+\alpha_m(t)]\Big] \\ &\times \exp\left[-\frac{d_1}{2}\left(\sqrt[4]{\mu^{-1}}x_1\cos\alpha - \sqrt[4]{\mu}x_2\sin\alpha\right)^2 - \frac{d_2}{2}\left(\sqrt[4]{\mu^{-1}}x_1\sin\alpha + \sqrt[4]{\mu}x_2\cos\alpha\right)^2\right] \\ &\times H_n\left[\sqrt{d_1}\left(\sqrt[4]{\mu^{-1}}x_1\cos\alpha - \sqrt[4]{\mu}x_2\sin\alpha\right)\right] H_m\left[\sqrt{d_2}\left(\sqrt[4]{\mu^{-1}}x_1\sin\alpha + x_2\cos\alpha\right)\right].\end{aligned} \tag{35}$$

When the parameters $\omega_{BHi}(t)$ and $b_i(t)$ with $(i = 1.2)$, evolves periodically throughout the period T, the system returns after this period to the initial state to acquire the state

$$\psi_{n,m}(x_1, x_2; T) = \exp\Big[-i[\alpha_n(T)+\alpha_m(T)]\Big]\psi_{n,m}\left(x_1, x_2; 0\right), \tag{36}$$

$\psi_{n,m}\left(x_1, x_2; 0\right)$ is defined by expression (35). According to the stationary eigenstate in (16) and the Schmidt theorem, $\psi_{n,m}(x_1, x_2; T)$ can be written as:

$$\psi\left(x_1, x_2; T\right) = \sum_{i,j} g_{n,m}^{i,j} e^{-i\left[\alpha_n(T) - i\alpha_m(T)\right]}\psi_{n,m}\left(\tilde{x}_1, \tilde{x}_2; 0\right). \tag{37}$$

Expression (37) can be developed in the form of two unrelated subsystems as:

$$\psi\left(x_1, x_2; T\right) = \sum_{i,j} B_{i,j}(T)\eta_i\left(x_1; T\right)\upsilon_j\left(x_2; T\right), \tag{38}$$

where

$$B_{i,j}(T) = g_{n,m}^{i,j} g_{n,m}^{*k,p} e^{-i\left[\alpha_n(T)-\alpha_{k+p-n}(T)-\alpha_k(T)-\alpha_p(T)\right]}. \tag{39}$$

Following the same procedures used to derive the integral (18), we have $g_{n,m}^{i,j} = d_{n,m}^{i,j}$. From $t = 0$ to $t = T$, the eigenstate in (36) evolves with the dynamical phase $\phi_t = -i\int_0^T \langle\psi_{n,m}(T)|\dot{\psi}_{n,m}(T)\rangle dt$, herein it is given as

$$\phi_t = -\left(n+\frac{1}{2}\right)d_1\int_0^T \frac{dt}{b_1^2(t)} - \left(m+\frac{1}{2}\right)d_2\int_0^T \frac{dt}{b_2^2(t)}. \tag{40}$$

After studying the dynamic phase, the entangled state accumulate a Berry phases expressed on the basis of the Schmidt decomposition [38, 39] for a nonadiabatic cyclic evolution by

$$\beta(T) = \sum_k \phi_t\Big|B_{i,k+p-i}(T)\Big|^2. \tag{41}$$

5. Applications

We can discuss on planar graphs our analytical results previously found for entanglement (33), (34) and the nonadiabatic Berry phases (41), taking into consideration the influence of the number states $|k, p\rangle$ of harmonic oscillators 1, 2 and the tunnel parameter J. To facilitate the numerical computing, we adjust expression $\Delta\alpha = \alpha_n + \alpha_{k+p-n} - \alpha_k - \alpha_p$ given in (25) to $n\Omega(t) = n\frac{J(t)\left(\frac{\omega_{BH2}(t)-\omega_{BH1}(t)}{J(t)}\right)+\tan(\alpha)}{\sqrt{d_1(t)}+\sqrt{d_2(t)}}$ that can be considered energy of the system and $\Omega(t)$ the corresponding time-dependent frequency. α is determined by Eq(3). Our objective is to show on particular cases how a generalization of their formulations corresponds to the current development of the entangled states and the nonadiabatic Berry phases within the potential wells and then we discuss how it develops for a solve from the general Lewis and Riesenfeld and the Heisenberg picture approach methods.

5.1. Non-linear Ermakov Equations

Consider at this section two particular models using the solution of the Ermakov equation given in (14). We begin with a simple coupled harmonic oscillators with constant, masses $\frac{1}{d}$ and angular frequencies $\sqrt{dd_j}$. Only J is time-dependent, we have

$$b_1(t) = \sqrt{\left(\frac{d_{1f} - d_{1i}}{2d_{1f}}\right)\cos\left(2\sqrt{dd_{1f}}t\right) + \left(\frac{d_{1f} - d_{1i}}{2d_{1f}}\right)}, \tag{42}$$

and

$$b_2(t) = \sqrt{\left(\frac{d_{2f} - d_{2i}}{2d_{2f}}\right)\cos\left(2\sqrt{dd_{2f}}t\right) + \left(\frac{d_{2f} - d_{2i}}{2d_{2f}}\right)}. \tag{43}$$

We have processed for the second model, a coupled Caldirola-Kanai oscillators with an increasing exponential masses $\frac{e^{2\nu_j t}}{d}$, and angular frequencies $\sqrt{dd_j}$. It is a good example of an exact solution, (14) becomes

$$b_1(t) = \sqrt{\left(\frac{d_{1f} - d_{1i}}{2d_{1f}}\right)\cos\left(2\sqrt{de^{-2\nu_1 t}d_{1f}}t\right) + \left(\frac{d_{1f} - d_{1i}}{2d_{1f}}\right)}, \tag{44}$$

and

$$b_2(t) = \sqrt{\left(\frac{d_{2f} - d_{2i}}{2d_{2f}}\right)\cos\left(2\sqrt{de^{-2\nu_2 t}d_{2f}}t\right) + \left(\frac{d_{2f} - d_{2i}}{2d_{2f}}\right)}. \tag{45}$$

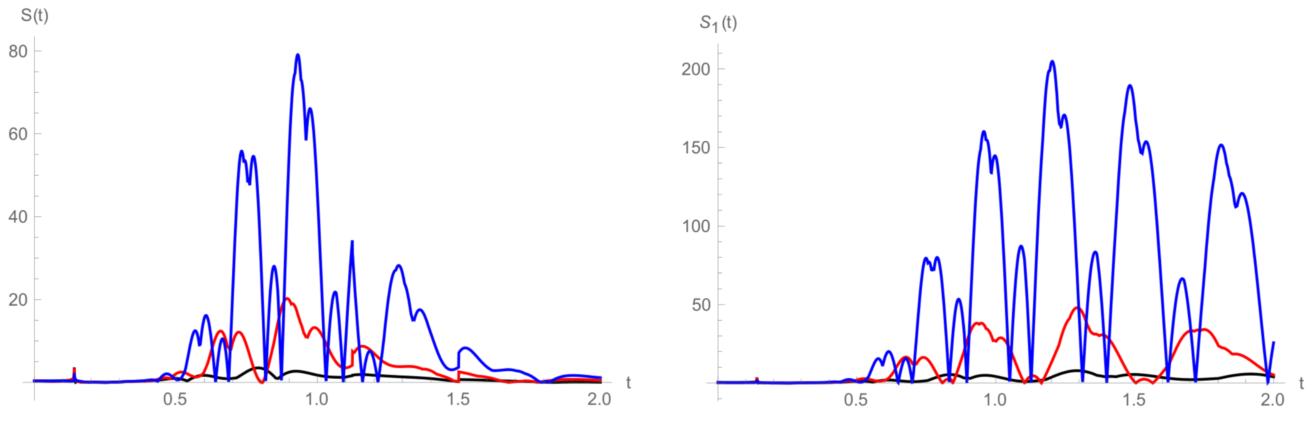

Figure 1. *An example of entanglement entropies in expression* (33)*;* $S(t)$ *from simple coupled harmonic oscillators and* $S_1(t)$ *from the coupled Caldirola-Kanai oscillators by setting* $\omega_{BH1} = 0.3$*,* $\omega_{BH2} = 0.7$*,* $J = 1.8$*,* $\nu_1 = 0.6$*,* $\nu_2 = 1.1$ *and different values of the quantum number* (k, p) *respectively* (k, p)*={(5,5) (black solid line), (5,10) (red solid line), (10,10) (blue solid line)}..*

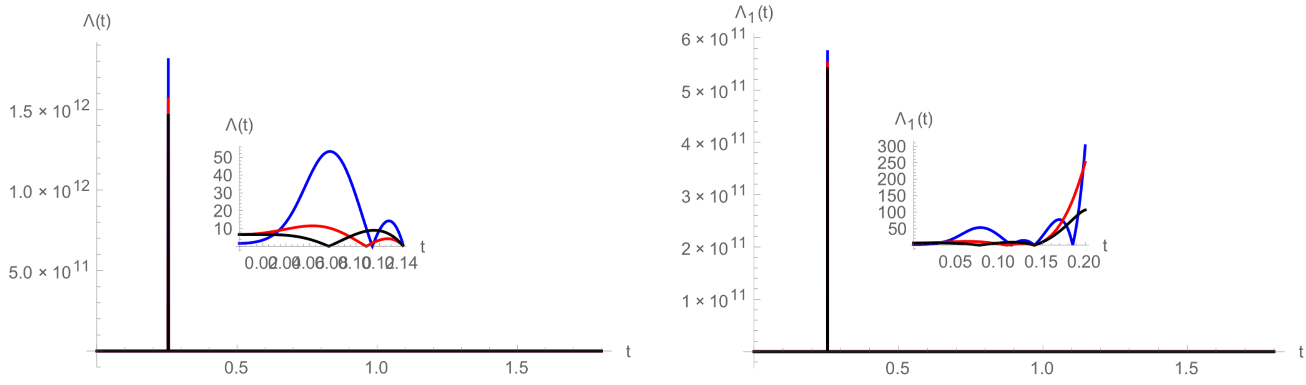

Figure 2. *Same as Figure* 1 *but from results of expression* (34)*;* $\Lambda(t)$ *from simple coupled harmonic oscillators and* $\Lambda_1(t)$ *from the coupled Caldirola-Kanai oscillators.*

5.2. General Lewis and Riesenfeld Method

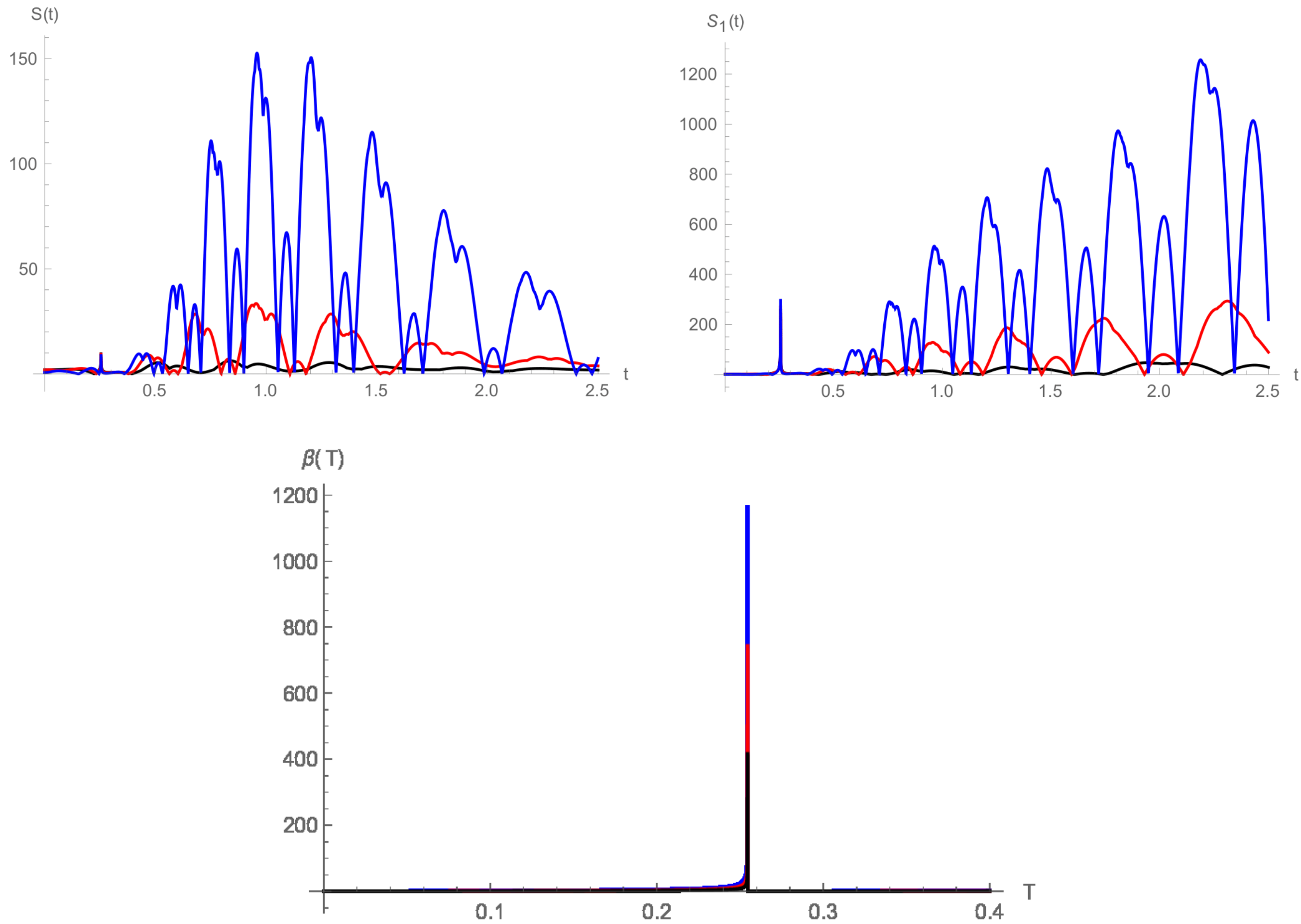

Figure 3. *Same as Figure* 1 *for* $S(t)$ *and* $S_1(t)$ *but from the representation associated with the results of the general Lewis and Riesenfeld method solution and* $\beta(T)$ *in* (41) *to the case of the simple coupled harmonic oscillators for different quantum number* (k,p)*, respectively* (k,p)*=*{*(0,2) (black solid line), (2,2) (red solid line), (2,4) (blue solid line)*}*.*

Let us now see how this ignorance will lead to a general treatment of Lewis and Riesenfeld method. We assimilate the wave function in expression (27) of the ref [40] to that given in (9). From the case of the first system; coupled harmonic oscillators, we have $\rho_j(t)$ in (27) of the ref [40] are the solutions of the second order differential equations

$$\frac{d^2}{dt^2}\left(\rho_j(t)\right)+\gamma(t)\frac{d}{dt}\left(\rho_j(t)\right)+dd_j\rho_j(t)=\frac{d^2}{\rho_j^3(t)}, \tag{46}$$

and they are denoted by $\rho_j(t) = \sqrt{\frac{d}{\sqrt{dd_j}}}$. Going back to (14), $b_j(t)$ becomes

$$b_j(t) = (dd_j)^{\frac{1}{4}}\,. \tag{47}$$

Next, consider the coupled Caldirola-Kanai oscillators. In basic terms, the $\rho_j(t)$ are prepared as:

$$\rho_1(t) = \sqrt{\frac{de^{-2\nu_1 t}}{\Omega_+}}, \tag{48}$$

and

$$\rho_2(t) = \sqrt{\frac{de^{-2\nu_2 t}}{\Omega_-}}, \tag{49}$$

where

$$\Omega_\pm^2 = de^{-2\nu_j t}d_j - \frac{\nu_j^2}{4}. \tag{50}$$

$\pm$ correspond respectively to $(j = 1.2)$.
Thus, $b_j(t)$ in (14) should be equal to

$$b_1(t) = \sqrt{d_1\frac{de^{-2\nu_1 t}}{\Omega_+}}, \tag{51}$$

and

$$b_2(t) = \sqrt{d_2\frac{de^{-2\nu_2 t}}{\Omega_-}}. \tag{52}$$

5.3. Heisenberg Picture Approach

To see this mathematical dependence, we formulate the expression of the wave function (3.6) solution of the ref [41] to that in expression (9). From this, $b_j(t)$ becomes

$$b_j(t) = \sqrt{\frac{d_j g_{-j}}{\omega_{Ij}}}, \tag{53}$$

where (j=1.2) and

$$g_{-j} = c_{1j}f_{1j}^2 + c_{2j}f_{1j}(t)f_{2j}(t) + c_{3j}f_{2j}^2(t), \tag{54}$$

are the solutions of the differentials equations given from same Lie algebra. c_{lj} are constant numbers and $f_{lj}(t)$ are the solutions of the second order linear differential equations. They are a differential equations of the form:

$$\frac{d^2}{dt^2}\left(f_{lj}(t)\right) + \frac{d}{dt}\left(\frac{d}{dt}\left(f_{lj}(t)\right)\right) + dd_j f_{lj}(t) = 0, \tag{55}$$

$(l = 1.2)$ and

$$\omega_{Ij}^2 = g_{+j}g_{-j} - g_{0j}^2. \tag{56}$$

g_{+j} and g_{0j} have checked the system in the form

$$g_{0j}(t) = -\frac{1}{2d}\left(\frac{dg_{-j}(t)}{dt}\right), \tag{57}$$

$$g_{+j}(t) = \frac{d_j}{d}g_{-j}(t) - \frac{1}{d}\left(\frac{dg_{0j}(t)}{dt}\right). \tag{58}$$

To start, we use for the simple coupled harmonic oscillators model the ref [42]. We have $c_{1j} = d$, $c_{2j} = 0$ and $c_{3j} = dS_j^4$. S_j are the Schrödinger operators and they are given from ref[43] by $S_j = S_0 e^{i\sqrt{dd_j}t}$. The $f_{lj}(t)$ in (5.14) becomes

$$f_{1j}(t) = \cos\sqrt{dd_{1j}}t, \tag{59}$$

$$f_{2j}(t) = \sin\sqrt{dd_{2j}}t. \tag{60}$$

Using (5.18) and (5.19), expression (5.13) give

$$g_{-j}(t) = d_j\left(\cos^2\sqrt{dd_j}t + S_j^4\sin^2\sqrt{dd_j}t\right). \tag{61}$$

Consequently (5.16), (5.17) yield

$$g_{0j}(t) = \left(S_j^4 - 1\right)\sqrt{dd_j}\cos\sqrt{dd_{1j}}t\sin\sqrt{dd_{2j}}t, \tag{62}$$

$$g_{+j}(t) = d_j\left(\sin^2\sqrt{dd_j}t + S_j^4\cos^2\sqrt{dd_j}t\right), \tag{63}$$

and

$$\omega_{Ij} = \begin{cases} \omega_{I1} = \sqrt{dd_1}S_1^2 \\ \omega_{I2} = \sqrt{dd_2}S_2^2. \end{cases} \tag{64}$$

From coupled Caldirola-Kanai oscillators model [44], we have

$$b_j(t) = \sqrt{d_j\frac{de^{-2\nu_j t}}{\Omega_{Ij}}}, \tag{65}$$

where

$$\Omega_{Ij}^2 = g_{1j}(t)g_{3j}(t) - g_{2j}^2(t), \tag{66}$$

$$g_{1j}(t) = de^{-2\nu_j t}, \tag{67}$$

$$g_{2j}(t) = \nu_j, \tag{68}$$

$$g_{3j}(t) = \frac{d_j^2}{de^{-2\nu_j t}}, \tag{69}$$

and

$$\Omega_{Ij} = \begin{cases} \Omega_{I1} = \sqrt{de^{-2\nu_1 t}d_1 - \nu_1^2} \\ \Omega_{I2} = \sqrt{de^{-2\nu_2 t}d_2 - \nu_2^2}. \end{cases} \tag{70}$$

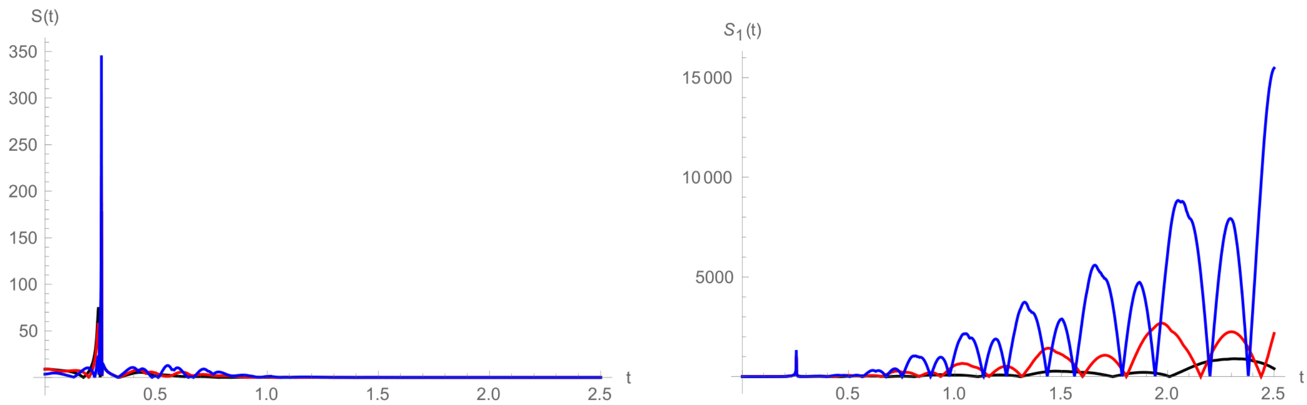

Figure 4. *The same as Figure 1 and Figure 3 for $S(t)$ and $S_1(t)$ but from the Heisenberg picture approach configuration.*

This method is develloped by the works [45, 46] to examine entanglement and some quantum physics notions.

5.4. Discussion

We represent on three particular cases Figures $1, 3$ and 4; the entanglement entropies linked to the positioned variation of the quantum states $|k, p\rangle$ to show the effect of the classical functions solution of the differential equations used in the treatement on the variation of entanglement. We prove that the Heisenberg picture approach is the most efficient method to examine entanglement, here the quantum entanglement reaches a very large values. The unexpected feature is that an exponential increase in the time of the mass of the harmonic oscillator results in an exponential increase of the quantum entanglement such that we have reached extremely important values to the coupled Caldirola Kanai harmonic oscillators with respect the simple coupled harmonic oscillators. A very rapid increase in the Schmidt parameter dropped the oscillatory behavior that is occurs in expression (3.19) and we can discern only the linear behavior. The particles in the Schmidt mode are distributed according to a conditional probability so the increase of the Schmidt parameter with the quantum numbers (k, p) means that when a pair of harmonic oscillators is in a pure bipartite state $|k, p\rangle$, its probability of being in another state $|k^{'}, p^{'}\rangle$ is non-zero but in the particular case where the system is at a time t_1 in the state $|k, p\rangle$, its probability of being in another state $|k^{'}, p^{'}\rangle$ is thus zero. The same behavior is repeated with the nonadiabatic Berry phases because of expression $\left|B_{i,k+p-i}(T)\right|^2$ so nonadiabatic Berry phases represents the probability of capturing the system in the states $|k, p\rangle$ when the particle remains confined in a closed region.

6. Conclusion

The system of two-site Bose-Hubbard model in the tunneling regime is interpreted in such a way as of two coupled harmonic oscillators. The dynamics of entanglement and the nonadiabatic Berry phases of the considered system is studied based on the Schmidt decomposition employs the wave function solution of $(TDSE)$ to arbitrary quantum numbers. As an important result, we make known the dynamics of entanglement because it can be immense for an exponential time growth of the mass. The Heisenberg picture approach acts as a good investigator of quantum entanglement with respect the solution via the non-linear Ermakov equations and the general Lewis and Riesenfeld method. On a particular case, it is possible now to recognize the properties of such an entangled system by examining its nonadiabatic Berry phases because it presents a similar behavior to that of the Schmidt parameter.

Abbreviations

TDSE	Time-dependent Schrödinger equation
QHE	Quantum Hall effect

Author Contributions

Ahlem Abidi et Adel Trabelsi are the authors. The authors reads and approved the final manuscript.

Conflicts of Interest

The authors declare no conflicts of interest.

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