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Suborbital Graphs Involving the Action of the Direct Product of Alternating and Cyclic Groups on the Cartesian Product of Two Sets

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Abstract: The combinatorial properties and invariants which includes transitivity, primitivity, ranks and subdegrees of direct product of Alternating group and Cyclic group acting on Cartesian product of two set have been extensively studied. However, the construction of suborbital graphs for this group action remains largely unexplored. As a result, this research paper addresses this gap by constructing suborbital graphs involving direct product of the Alternating group and Cyclic group acting on the Cartesian product of two sets where their respective properties such as connectivity, self-pairedness, girth and vertex degree are analyzed in detail. First, the result shows that all suborbits are $self-paired$ implies that the vertex sets are undirected to each other. Secondly, the constructed suborbital graphs are classified into three parts for any value of $n \geqslant 3$. In $part\ A$, it is proven that the constructed suborbital graphs $\Gamma_1, \Gamma_2, \Gamma_3,..., \Gamma_(n-1)$ are undirected, regular of degree $(n-1)$, disconnected with $n-connected$ components and has a girth of 3. In $part\ B$, the constructed suborbital graph $\Gamma_{(n-1)+1}$ is found to be undirected, regular of degree $(n-1)$, disconnected with $n-connected$ components and has a girth of 3. Lastly in $part\ C$, the graphs $\Gamma_{(n-1)+2}$, $\Gamma_{(n-1)+3}$, $\Gamma_{(n-1)+4}$,..., $\Gamma_{(n-1)+n}$ are found to be undirected, regular of degree $(n-1)^2$, disconnected with $n^2-connected$ components and has a girth of 3.

Keywords: Suborbits, Suborbital Graphs, Direct Product, Cartesian Product, Alternating Group, Cyclic Group

1. Introduction

Suppose that $G = A_n \times C_n$ and $Stab_G(x_1, y_1)$ be the stabilizer of point x in G. Also, let Δ_0 and Δ_j be distinct suborbits in set $X \times Y$ such that $(x_1, y_1) \in \Delta_0$ and $(x_j, y_j) \in \Delta_j\ \forall j = 1, 2, 3, ...$ Then the suborbital denoted as O corresponding to Δ is given as:

$$O = \{g(x_1, y_1), g(x_j, y_j) : g \in Stab_G(x_1, y_1)\ , (x_j, y_j) \in \Delta_j \text{ and } (x_1, y_1) \in \Delta_0\}$$

Suborbital graph Γ_j corresponding to suborbital O is then constructed when an edge is drawn from a point Δ_0 to a point Δ_j such that the following conditions holds $\forall (x_1, y_1) \in \Delta_0, (x_j, y_j) \in \Delta_j$:

1. When $x_1 = x_j$ and $y_1 \neq y_j\ \forall j = 1, 2, 3, ...$
2. When $x_1 \neq x_j$ and $y_1 = y_j\ \forall j = 1, 2, 3, ...$
3. When $x_1 \neq x_j$ and $y_1 \neq y_j\ \forall j = 2, 3, 4, ...$

$Note$: Suborbit Δ_0 has a null graph since it has trivial vertex set. Instead, it forms a loop. Therefore in this case, it is not considered.

2. Notation and Preliminary Results

Definition 2.1. [1] A set G together with a binary operation is said to be a $group$ if it satisfies the axioms of closure, associativity, identity and inverse.

Definition 2.2. [2] Let X be a non-empty set, then a permutation is a bijective mapping from set X to itself. The group is said to be $Symmetric$ if it consist of all permutations of a set X and is denoted as S_n. The subgroup of S_n that consist all even permutations is referred to as the $Alternating$ group and is denoted by A_n whose order is $|A_n| = \frac{n!}{2}$.

Definition 2.3. [3] A $Cyclic$ group is a group that can be generated by a single element $g \in G$ such that $G = \langle g \rangle$. It is denoted as C_n and has an order of n.

Definition 2.4. [4] Consider a suborbit $\Delta_j = (x_j, y_j)$ and $g \in G$. Then, Δ_j is $self-paired$ if $\{g(x_j, y_j) = (x_i, y_i)\}$ and $\{g(x_i, y_i) = (x_j, y_j) \in \Delta_j\}$ such that $\Delta_j = \Delta_j^*$ $\forall i, j = 1, 2, 3, ...$

Definition 2.5. [5] Suppose that G acts on X. Then the $stabilizer$ of point x in G is the set of elements of the group which leaves elements of a set unchanged under group action. It is denoted as $Stab_G(x)$ and given by: $Stab_G(x) = \{g \in G : gx = x\}$.

Definition 2.6. [6] Suppose G acts on X. Then the $orbits of point x$ in G are the partitions under the group action that represents the equivalence classes of elements of a set when mapped by the elements of the group. It is denoted as $Orb_G(x)$ and given by $Orb_G(x) = \{gx : g \in G\}$.

Theorem 2.1. [7] ($Sims\ Theorem$). Let G acts transitively on X. Then the suborbital graph is said to be $undirected$ if the suborbit is self-paired otherwise, it will be regraded as directed. Group action is said to be primitive if all vertices set points are connected together by the edges.

Theorem 2.2. [8] ($Wielandt\ Theorem$). Suppose G acts transitively on X. Then there exists at least one self-paired suborbit of G on X if and only if the order of the group is even.

Theorem 2.3. [9] ($Cameroon\ Theorem$). Suppose G acts on X transitively and for all $g \in G$. Then the number of suborbits that are self-paired is given by: $\frac{1}{|G|} \sum\limits_{g \in G} |Fix(g)^2|$.

Definition 2.7. [10] A $graph$ is a diagram consisting of a pair of ordered sets, namely vertices and edges, denoted as $G(V, E)$. The collection of connected vertices and their adjacent edges is referred to as a $walk$.

Definition 2.8. [11] A $path$ is a walk in which no vertex is repeated, except for the first and last vertices. A path that starts and ends at the same vertex, forming a closed loop, is called a $cycle$.

Definition 2.9. [12] A graph in which every pair of vertices is joined by a path is said to be $connected$. Conversely, a graph that is not connected is termed $disconnected$.

Definition 2.10. [13] A graph is described as $directed$ if its edges have a specified direction, otherwise it is undirected. An edge that connects a vertex to itself is called a $loop$.

Definition 2.11. [11] The shortest length of a cycle in a graph is referred to as $girth$.

Definition 2.12. [12] A $multigraph$ is a graph that contains multiple edges between the same pair of vertices but has no loops. In other words, two or more vertices can be connected by multiple edges. In contrast, a $simple\ graph$ is a graph that has no loops and contains at most one edge between any two vertices.

Definition 2.13. [14] Suppose G acts on X transitively and $Stab_G(x)$ is the stabilizer of points x in G. Then the $suborbits$ are the orbits of the stabilizers which arises when the elements of the stabilizer act on the set. They are denoted as Δ_0 and defined given by: $\Delta_i = \{\Delta_1, \Delta_2, ..., \Delta_{i-1}\}$. The total number of suborbits is the $rank$ and the size of these suborbits is the $subdegrees$.

Theorem 2.4. [15]. Suppose that $G = A_n \times C_n$ acts on $X \times Y$. Then when $n = 3$, the stabilizer of (x_1, y_1) in G is an identity. When $n \geqslant 4$, the size of the stabilizer of (x_1, y_1) in G is given as $|Stab_G(x_1, y_1)| = \frac{(n-1)!}{2}$

Theorem 2.5. [15]. Suppose that $G = A_n \times C_n$ acts on $X \times Y$. Then when $n = 3$, the rank is 9 and the corresponding subdegrees are $1, 1, .., 1$ $(9 - times)$. When $n \geqslant 4$, the rank is $2n$ and subdegrees are $\underbrace{1, 1, ..., 1}_{n}, \underbrace{n-1, n-1, ..., n-1}_{n}$.

3. Main Results

3.1. Suborbital Graphs of $A_3 \times C_3$ Acting on $X \times Y$

From theorem 2.5 when $n = 3$, the rank is 9 and the corresponding subdegrees are $1, 1, 1, 1, 1, 1, 1, 1, 1$. Therefore, the suborbits are arranged as follows:

$\Delta_0 = Orb_H(x_1, y_1) = \{(x_1, y_1)\} = 1$ (Trivial Orbit)
$\Delta_1 = Orb_H(x_1, y_2) = \{(x_1, y_2)\} = 1$
$\Delta_2 = Orb_H(x_1, y_3) = \{(x_1, y_3)\} = 1$
$\Delta_3 = Orb_H(x_2, y_1) = \{(x_2, y_1)\} = 1$
$\Delta_4 = Orb_H(x_3, y_1) = \{(x_3, y_1)\} = 1$
$\Delta_5 = Orb_H(x_2, y_2) = \{(x_2, y_2)\} = 1$
$\Delta_6 = Orb_H(x_2, y_3) = \{(x_2, y_3)\} = 1$
$\Delta_7 = Orb_H(x_3, y_2) = \{(x_3, y_2)\} = 1$
$\Delta_8 = Orb_H(x_3, y_3) = \{(x_3, y_3)\} = 1$

Lemma 3.1. Let $\Delta_j = \{\Delta_1, \Delta_2, \Delta_3, ..., \Delta_8\}$ be the suborbits of $A_3 \times C_3$ acting on $X \times Y$. Then Δ_j is $self-paired$.

$Proof$. From the definition 2.4, consider $\Delta_1 = \{(x_1, y_2)\}$ and $g = (e_x, e_y)$, then $\{(e_x, e_y)(x_1, y_2)\} = \{(x_1, y_2)\}$ and $\{(e_x, e_y)(x_1, y_2)\} = \{(x_1, y_2) \in \Delta_1\}$. $\Rightarrow \Delta_1 = \Delta_1^*$. Therefore, it follows that all suborbits, Δ_j are $self-paired$. By theorem 2.1 ($Sims\ Theorem$), if the suborbits are $self-paired$, then their graphs are said to be $undirected$.

Given that $\Delta_0 = \{(x_1, y_1)\}$, $\Delta_1 = \{(x_1, y_2)\}$ and from theorem 2.4, $Stab_G(x_1, y_1) = \{(e_x, e_y)\}$. Then, the suborbital O_1 corresponding to suborbit Δ_1 is given as:

$O_1 = \{(e_x, e_y)(x_1, y_1), (e_x, e_y)(x_1, y_2)\} = \{(x_1, y_1), (x_1, y_2)\}$. Therefore, a suborbital graph Γ_1 is constructed by drawing edges from vertex set (x_1, y_1) to vertex set (x_1, y_2) such that any first components of ordered pairs are the identical and the second components of ordered pairs are different.

The constructed suborbital graph Γ_1 below is undirected, regular of degree 2 and disconnected with $3-connected$ components of girth 3.

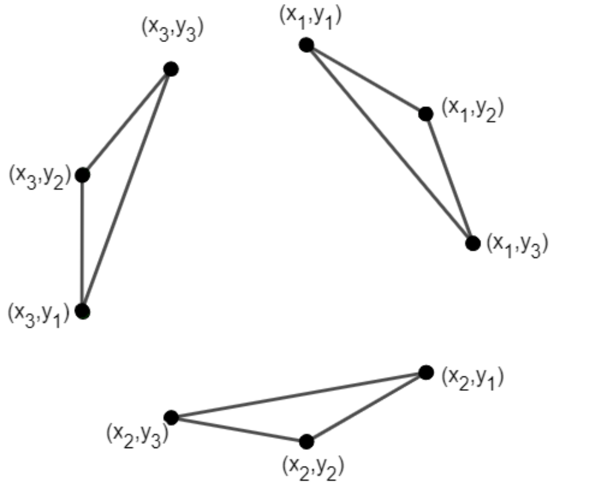

Figure 1. *Suborbital graph Γ_1 corresponding to Δ_1 of $A_3 \times C_3$ acting on $X \times Y$.*

$Note$: The suborbit $\Delta_2 = \{(x_1, y_3)\}$, falls to the same category as that of Δ_1. That is, the first components of the ordered pairs are the identical and the second components of the ordered pairs are different. Therefore, the structure of the graph Γ_2 is identical to that of Γ_1. Hence, it has the same properties as that of Γ_1.

Given that $\Delta_0 = \{(x_1, y_1)\}$, $\Delta_3 = \{(x_2, y_1)\}$ and $Stab_G(x_1, y_1) = \{(e_x, e_y)\}$. Suborbital O_3 corresponding to suborbit Δ_3 is given as:

$O_3 = \{(e_x, e_y)(x_1, y_1), (e_x, e_y)(x_2, y_1)\} = \{(x_1, y_1), (x_2, y_1)\}$ From this equation, suborbital graph Γ_3 is constructed by drawing edges from vertex set (x_1, y_1) to vertex set (x_2, y_1) such that the first components of the ordered pairs are non-identical and the second components of the ordered pairs are identical.

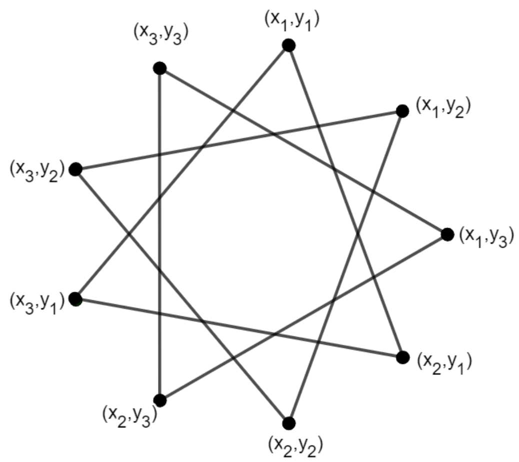

Figure 2. *Suborbital graph Γ_3 corresponding to Δ_3 of $A_3 \times C_3$ acting on $X \times Y$.*

The constructed suborbital graph Γ_3 is undirected, regular of degree 2 and disconnected with $3-connected$ components of girth 3.

$Note$: The suborbit $\Delta_4 = \{(x_3, y_1)\}$, falls to the same category as that of Δ_3. That is, the second components of the ordered pairs are the identical and the first components of the ordered pairs are different. Therefore, the structure of graph Γ_4 is identical as that of Γ_3. Hence, it has the same properties to that of Γ_3.

Given that $\Delta_0 = \{(x_1, y_1)\}$, $\Delta_5 = \{(x_2, y_2)\}$ and $Stab_G(x_1, y_1) = \{(e_x, e_y)\}$. Suborbital O_5 corresponding to suborbit Δ_5 is given as:

$O_5 = \{(e_x, e_y)(x_1, y_1), (e_x, e_y)(x_2, y_2)\} = \{(x_1, y_1), (x_2, y_2)\}$ Suborbital graph Γ_5 is then constructed by drawing edges from vertex set (x_1, y_1) to vertex set (x_2, y_2) such that neither the first components nor the second components of the ordered pairs are identical.

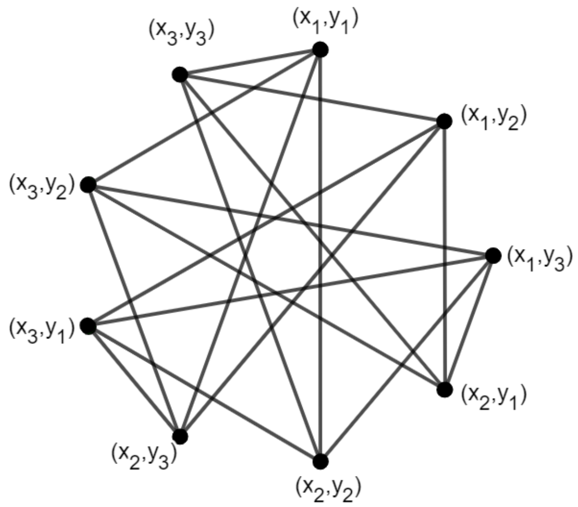

Figure 3. *Suborbital graph Γ_5 corresponding to Δ_5 of $A_3 \times C_3$ acting on $X \times Y$.*

The constructed suborbital graph Γ_5 is undirected, regular of degree 4 and disconnected with $6-connected$ components of girth 3.

$Note$: The suborbits $\Delta_6 = \{(x_2, y_3)\}$, $\Delta_7 = \{(x_3, y_2)\}$ and $\Delta_8 = \{(x_3, y_3)\}$ fall to the same category as that of Δ_5. That is, neither the first nor the second components of the ordered pairs are the identical. Therefore, the structures of the graphs Γ_6, Γ_7 and Γ_8 are identical to that of Γ_5. Hence, they have the same properties as that of Γ_5.

3.2. Suborbital Graphs of $A_4 \times C_4$ Acting on $X \times Y$

From theorem 2.5 when $n = 4$, the rank is 8 and the corresponding subdegrees are $1, 1, 1, 1, 3, 3, 3, 3$. Therefore, the suborbits are arranged as follows:

$\Delta_0 = Orb_H(x_1, y_1) = \{(x_1, y_1)\} = 1$ (Trivial Orbit)
$\Delta_1 = Orb_H(x_1, y_2) = \{(x_1, y_2)\} = 1$
$\Delta_2 = Orb_H(x_1, y_3) = \{(x_1, y_3)\} = 1$
$\Delta_3 = Orb_H(x_1, y_4) = \{(x_1, y_4)\} = 1$
$\Delta_4 = Orb_H(x_2, y_1) = \{(x_2, y_1), (x_3, y_1), (x_4, y_1)\} = 3$
$\Delta_5 = Orb_H(x_2, y_2) = \{(x_2, y_2), (x_3, y_2), (x_4, y_2)\} = 3$

$\Delta_6 = Orb_H(x_2, y_3) = \{(x_2, y_3), (x_3, y_3), (x_4, y_3)\} = 3$
$\Delta_7 = Orb_H(x_2, y_4) = \{(x_2, y_4), (x_3, y_4), (x_4, y_4)\} = 3$

Lemma 3.2. Let $\Delta_j = \{\Delta_1, \Delta_2, \Delta_3, ..., \Delta_7\}$ be the suborbits of $A_4 \times C_4$ acting on $X \times Y$, then, Δ_j is $self-paired$

$Proof$. Consider $\Delta_5 = \{(x_2, y_2), (x_3, y_2), (x_4, y_2)\}$ and $g = ((x_2\ x_3\ x_4), e_y)$, then $\{((x_2\,x_3\,x_4), e_y)(x_2, y_2)\} = \{(x_3, y_2)\}$ and $\{((x_2\,x_3\,x_4), e_y)(x_3, y_2)\} = \{(x_4, y_2) \in \Delta_5\}$. $\Rightarrow \Delta_5 = \Delta_5^*$. Therefore, it follows that all suborbits, Δ_j are $self-paired$. By theorem 2.1 ($Sims\ Theorem$), if the suborbits are $self-paired$, then their graphs are said to be $undirected$.

Given that $\Delta_0 = \{(x_1, y_1)\}$, $\Delta_1 = \{(x_1, y_2)\}$ and from theorem 2.4, there are 3 elements of the stabilizers. They are: $Stab_G(x_1, y_1) = \{(e_x, e_y), ((x_2 x_3 x_4), e_y), ((x_2 x_4 x_3), e_y)\}$.

The suborbital O_1 corresponding to suborbit Δ_1 is given as:

$O_1 = \{(e_x, e_y)(x_1, y_1), (e_x, e_y)(x_1, y_2)\} = \{(x_1, y_1), (x_1, y_2)\}$
$O_1 = \{((x_2\,x_3\,x_4), e_y)(x_1, y_1), ((x_2\,x_3\,x_4), e_y)(x_1, y_2)\} = \{(x_1, y_1), (x_1, y_2)\}$
$O_1 = \{((x_2\,x_4\,x_3), e_y)(x_1, y_1), ((x_2\,x_4\,x_3), e_y)(x_1, y_2)\} = \{(x_1, y_1), (x_1, y_2)\}$

Therefore, a suborbital graph Γ_1 is constructed by drawing an edge from vertex set point (x_1, y_1) to vertex set point (x_1, y_2) such that any first components of ordered pairs are identical and the second components are different.

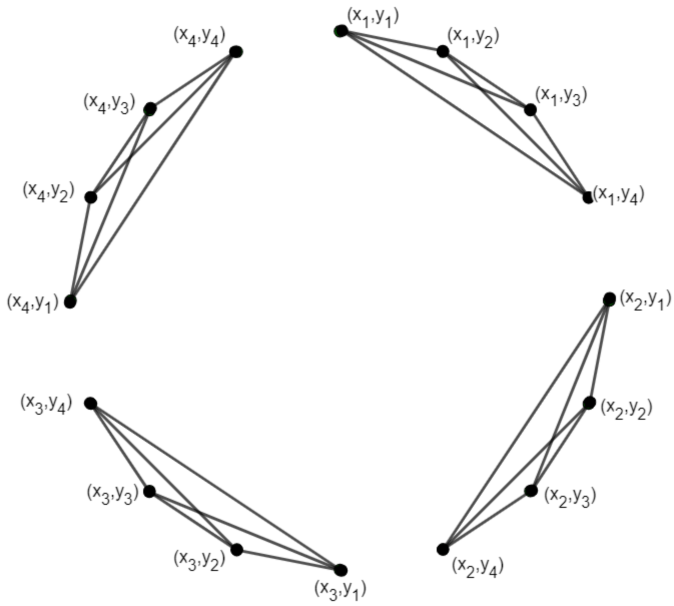

Figure 4. *Suborbital graph Γ_1 corresponding to Δ_1 of $A_4 \times C_4$ acting on $X \times Y$.*

The constructed suborbital graph Γ_1 is undirected, regular of degree 3, disconnected with $4-connected$ components and has a girth of 3.

$Note$: The suborbits $\Delta_2 = \{(x_1, y_3)\}$ and $\Delta_3 = \{(x_1, y_4)\}$ fall to the same category as that of Δ_1. That is, the first components of the ordered pairs are the identical but the second components are not. Therefore, the structures of the graphs Γ_2 and Γ_3 are identical to that of Γ_1. Hence, they have the same properties as that of Γ_1.

Given that

$\Delta_0 = \{(x_1, y_1)\}$, $\Delta_4 = \{(x_2, y_1), (x_3, y_1), (x_4, y_1)\}$ and $Stab_G(x_1, y_1) = \{(e_x, e_y), ((x_2\,x_3\,x_4), e_y), ((x_2\,x_4\,x_3), e_y)\}$.

The suborbital O_4 corresponding to suborbit Δ_4 is given as:

$O_4 = \{(e_x, e_y)(x_1, y_1), (e_x, e_y)(x_2, y_1), (e_x, e_y)(x_3, y_1), (e_x, e_y)(x_4, y_1)\}$
$\quad = \{(x_1, y_1), (x_2, y_1), (x_3, y_1), (x_4, y_1)\}$
$O_4 = \{((x_2\,x_3\,x_4), e_y)\,(x_1, y_1), ((x_2\,x_3\,x_4), e_y)\,(x_2, y_1), ((x_2\,x_3\,x_4), e_y)\,(x_3, y_1), ((x_2\,x_3\,x_4), e_y)\,(x_4, y_1)\}$
$\quad = \{(x_1, y_1), (x_3, y_1), (x_4, y_1), (x_2, y_1)\}$
$O_4 = \{((x_2\,x_4\,x_3), e_y)\,(x_1, y_1), ((x_2\,x_4\,x_3), e_y)\,(x_2, y_1), ((x_2\,x_4\,x_3), e_y)\,(x_3, y_1), ((x_2\,x_4\,x_3), e_y)\,(x_4, y_1)\}$
$\quad = \{(x_1, y_1), (x_4, y_1), (x_2, y_1), (x_3, y_1)\}$

Suborbital graph Γ_4 is then constructed by drawing edges from vertex set point (x_1, y_1) to vertices set points $\{(x_2, y_1), (x_3, y_1), (x_4, y_1)\}$ such that any second components of ordered pairs are the identical but the first components are different.

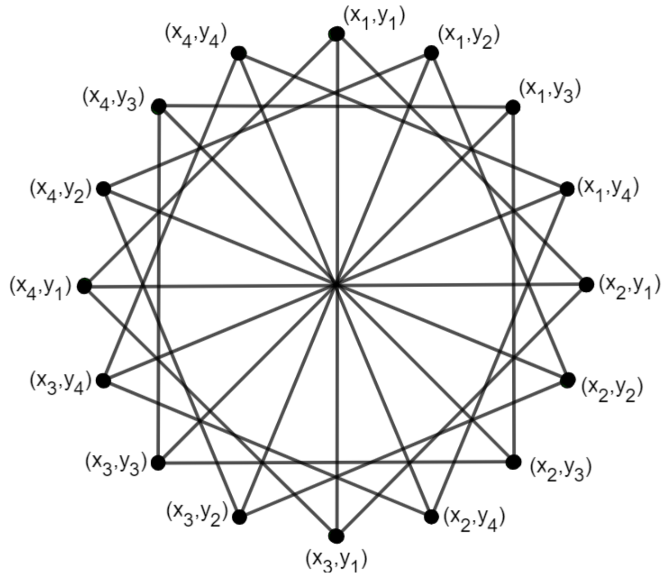

Figure 5. *Suborbital graph Γ_4 corresponding to Δ_4 of $A_4 \times C_4$ acting on $X \times Y$.*

The constructed suborbital graph Γ_4 is undirected, regular of degree 3, disconnected with $4-connected$ components and has a girth of 3.

Given that

$\Delta_0 = \{(x_1, y_1)\}$, $\Delta_5 = \{(x_2, y_2), (x_3, y_2), (x_4, y_2)\}$ and $Stab_G(x_1, y_1) = \{(e_x, e_y), ((x_2\ x_3\ x_4), e_y), ((x_2\ x_4\ x_3), e_y)\}$.

The suborbital O_5 corresponding to suborbit Δ_5 is given as:

$O_5 = \{(e_x, e_y)(x_1, y_1), (e_x, e_y)(x_2, y_2), (e_x, e_y)(x_3, y_2), (e_x, e_y)(x_4, y_2)\}$

$= \{(x_1, y_1), (x_2, y_2), (x_3, y_2), (x_4, y_2)\}$

$O_5 = \{((x_2\ x_3\ x_4), e_y)(x_1, y_1), ((x_2\ x_3\ x_4), e_y)(x_2, y_2), ((x_2\ x_3\ x_4), e_y)(x_3, y_2), ((x_2\ x_3\ x_4), e_y)(x_4, y_2)\}$

$= \{(x_1, y_1), (x_3, y_2), (x_4, y_2), (x_2, y_2)\}$

$O_5 = \{((x_2\ x_4\ x_3), e_y)(x_1, y_1), ((x_2\ x_4\ x_3), e_y)(x_2, y_2), ((x_2\ x_4\ x_3), e_y)(x_3, y_2), ((x_2\ x_4\ x_3), e_y)(x_4, y_2)\}$

$= \{(x_1, y_1), (x_4, y_2), (x_2, y_2), (x_3, y_2)\}$

Suborbital graph Γ_5 is then constructed by drawing edges from vertex set point (x_1, y_1) to vertices set points $\{(x_2, y_2), (x_3, y_2), (x_4, y_2)\}$ such that neither the first nor the second components of ordered pairs are the identical.

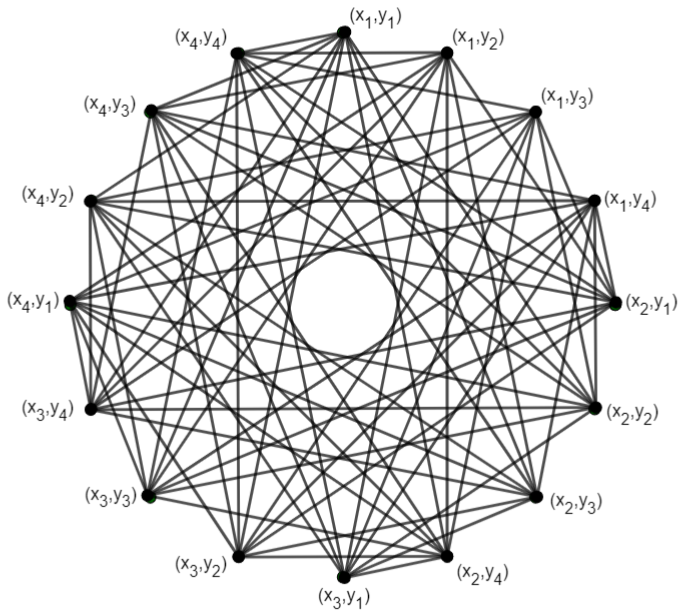

Figure 6. *Suborbital graph Γ_5 corresponding to Δ_5 of $A_4 \times C_4$ acting on $X \times Y$.*

Constructed suborbital graph Γ_5 is undirected, regular of degree 9, disconnected with $16-connected$ components and has a girth of 3.

$Note$: The suborbits $\Delta_6=\{(x_2,y_3),(x_3,y_3),(x_4,y_3)\}$ and $\Delta_7=\{(x_2,y_4),(x_3,y_4),(x_4,y_4)\}$ fall to the same category as that of Δ_5. That is, neither the first nor the second components of the ordered pairs are the identical. Therefore, the structures of the graphs Γ_6 and Γ_7 are identical to that of Γ_5. Hence, they have the same properties as that of Γ_5.

3.3. Suborbital Graphs of $A_5\times C_5$ Acting on $X\times Y$

From theorem 2.5 when $n=5$, the rank is 10 and the corresponding subdegrees are $1,1,1,1,1,4,4,4,4,4$. Therefore, the suborbits are arranged as follows:

$\Delta_0=Orb_H(x_1,y_1)=\{(x_1,y_1)\}=1$ (Trivial Orbit)

$\Delta_1=Orb_H(x_1,y_2)=\{(x_1,y_2)\}=1$

$\Delta_2=Orb_H(x_1,y_3)=\{(x_1,y_3)\}=1$

$\Delta_3=Orb_H(x_1,y_4)=\{(x_1,y_4)\}=1$

$\Delta_4=Orb_H(x_1,y_5)=\{(x_1,y_5)\}=1$

$\Delta_5=Orb_H(x_2,y_1)=\{(x_2,y_1),(x_3,y_1),(x_4,y_1),(x_5,y_1)\}=4$

$\Delta_6=Orb_H(x_2,y_2)=\{(x_2,y_2),(x_3,y_2),(x_4,y_2),(x_5,y_2)\}=4$

$\Delta_7=Orb_H(x_2,y_3)=\{(x_2,y_3),(x_3,y_3),(x_4,y_3),(x_5,y_3)\}=4$

$\Delta_8=Orb_H(x_2,y_4)=\{(x_2,y_4),(x_3,y_4),(x_4,y_4),(x_5,y_4)\}=4$

$\Delta_9=Orb_H(x_2,y_5)=\{(x_2,y_5),(x_3,y_5),(x_4,y_5),(x_5,y_5)\}=4$

Lemma 3.3. Let $\Delta_j=\{\Delta_1,\Delta_2,\Delta_3,...,\Delta_9\}$ be the suborbits of $A_5\times C_5$ acting on $X\times Y$. Then Δ_j is $self-paired$

$Proof$. Consider $\Delta_7=\{(x_2,y_3),(x_3,y_3),(x_4,y_3),(x_5,y_3)\}$ and $g=\{((x_2\,x_5\,x_4),e_y)\}$. Then;

$\{((x_2\,x_5\,x_4),e_y)(x_2,y_3)\}=\{(x_5,y_3)\}and\{((x_2\,x_5\,x_4),e_y)(x_5,y_3)\}=\{(x_4,y_3)\in\Delta_7\}.\Rightarrow\Delta_7=\Delta_7^*$.

Therefore, it follows that all suborbits Δ_j are $self-paired$. By theorem 2.1 ($Sims\ Theorem$), if the suborbits are $self-paired$, then their graphs are said to be $undirected$.

Given that $\Delta_0=\{(x_1,y_1)\}$, $\Delta_1=\{(x_1,y_2)\}$ and from theorem 2.4, there are 12 elements of the stabilizers when $n=5$. They are:

$Stab_G(x_1,y_1)=\{(e_x,e_y),((x_2\,x_4\,x_5),e_y),((x_2\,x_5\,x_4),e_y),((x_2\,x_3\,x_5),e_y),((x_3\,x_5\,x_4),e_y),((x_2\,x_4\,x_3),e_y),$ $((x_3\,x_4\,x_5),e_y),((x_2\,x_5\,x_3),e_y),((x_2\,x_3\,x_4),e_y),((x_2\,x_5)(x_3\,x_4),e_y),((x_2\,x_3)(x_4\,x_5),e_y),((x_2\,x_4)(x_3\,x_5),e_y)\}$.

The suborbital O_1 corresponding to suborbit Δ_1 is given as:

$O_1=\{(e_x,e_y)(x_1,y_1),(e_x,e_y)(x_1,y_2)\}=\{(x_1,y_1),(x_1,y_2)\}$

$O_1=\{((x_2\,x_4\,x_5),e_y)(x_1,y_1),((x_2\,x_4\,x_5),e_y)(x_1,y_2)\}=\{(x_1,y_1),(x_1,y_2)\}$

$O_1=\{((x_2\,x_5\,x_4),e_y)(x_1,y_1),((x_2\,x_5\,x_4),e_y)(x_1,y_2)\}=\{(x_1,y_1),(x_1,y_2)\}$

$\vdots$

$O_1=\{((x_2\,x_4)(x_3\,x_5),e_y)(x_1,y_1),((x_2\,x_4)(x_3\,x_5),e_y)(x_1,y_2)\}=\{(x_1,y_1),(x_1,y_2)\}$

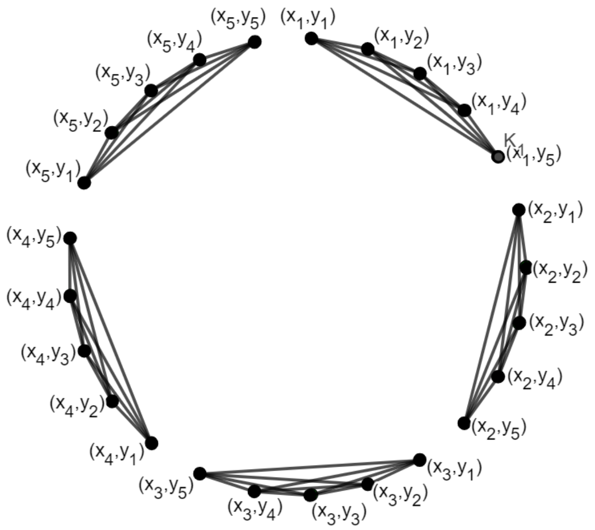

Figure 7. *Suborbital graph Γ_1 corresponding to Δ_1 of $A_5\times C_5$ acting on $X\times Y$.*

Therefore, a suborbital graph Γ_1 is constructed by drawing an edge from vertex set point (x_1,y_1) to vertex set point (x_1,y_2) such that any first components of ordered pairs are identical and the second components are different.

The constructed suborbital graph Γ_1 is undirected, regular of degree 4, disconnected with $5-connected$ components and has a girth of 3.

$Note$: The suborbits $\Delta_2=\{(x_1,y_3)\}$,$\Delta_3=\{(x_1,y_4)\}$ and $\Delta_4=\{(x_1,y_5)\}$ fall to the same category as that of Δ_1. That is, the first components of the ordered pairs are the identical but the second components are not. Therefore, the structures of the graphs Γ_2, Γ_3 and Γ_4 are identical to that of Γ_1. Hence, they have the same properties as that of Γ_1.

Given that $\Delta_0=\{(x_1,y_1)\}$, $\Delta_5=\{(x_2,y_1),(x_3,y_1),$ $(x_4,y_1),(x_5,y_1)\}$ and $Stab_G(x_1,y_1)=\{(e_x,e_y),$ $((x_2\,x_4\,x_5),e_y),((x_2\,x_5\,x_4),e_y),((x_2\,x_3\,x_5),e_y),$ $((x_3\,x_5\,x_4),e_y),((x_2\,x_4\,x_3),e_y),((x_3\,x_4\,x_5),e_y),$

$((x_2\, x_5\, x_3)\,, e_y), ((x_2\, x_3\, x_4)\,, e_y), ((x_2\, x_5)\,(x_3\, x_4)\,, e_y),$ $((x_2\, x_3)\,(x_4\, x_5)\,, e_y), ((x_2\, x_4)\,(x_3\, x_5)\,, e_y)\}$. The suborbital O_5 corresponding to suborbit Δ_5 is given as:

$$
\begin{aligned}
O_5 &= \{(e_x,e_y)(x_1,y_1),(e_x,e_y)(x_2,y_1),(e_x,e_y)(x_3,y_1),(e_x,e_y)(x_4,y_1),(e_x,e_y)(x_5,y_1)\}\\
&= \{(x_1,y_1),(x_2,y_1),(x_3,y_1),(x_4,y_1),(x_5,y_1)\}\\
O_5 &= \{((x_2\,x_4\,x_5),e_y)(x_1,y_1),((x_2\,x_4\,x_5),e_y)(x_2,y_1),((x_2\,x_4\,x_5),e_y)(x_3,y_1),((x_2\,x_4\,x_5),e_y)(x_4,y_1),\\
&\quad ((x_2\,x_4\,x_5),e_y)(x_5,y_1)\}\\
&= \{(x_1,y_1),(x_4,y_1),(x_3,y_1),(x_5,y_1),(x_2,y_1)\}\\
O_5 &= \{((x_2\,x_5\,x_4),e_y)(x_1,y_1),((x_2\,x_5\,x_4),e_y)(x_2,y_1),((x_2\,x_5\,x_4),e_y)(x_3,y_1),((x_2\,x_5\,x_4),e_y)(x_4,y_1),\\
&\quad ((x_2\,x_5\,x_4),e_y)(x_5,y_1)\}\\
&= \{(x_1,y_1),(x_5,y_1),(x_3,y_1),(x_2,y_1),(x_4,y_1)\}\\
&\vdots\\
O_5 &= \{((x_2\,x_4)(x_3\,x_5),e_y)(x_1,y_1),((x_2\,x_4)(x_3\,x_5),e_y)(x_2,y_1),((x_2\,x_4)(x_3\,x_5),e_y)(x_3,y_1),((x_2\,x_4)(x_3\,x_5),e_y)\\
&\quad (x_4,y_1),((x_2\,x_4)(x_3\,x_5),e_y)(x_5,y_1)\}\\
&= \{(x_1,y_1),(x_4,y_1),(x_5,y_1),(x_2,y_1),(x_3,y_1)\}
\end{aligned}
$$

Suborbital graph Γ_5 is then constructed by drawing edges from vertex set point (x_1,y_1) to vertices set $\{(x_2,y_1),(x_3,y_1),(x_4,y_1),(x_5,y_1)\}$ such that any second components of ordered pairs are identical but the first components are different.

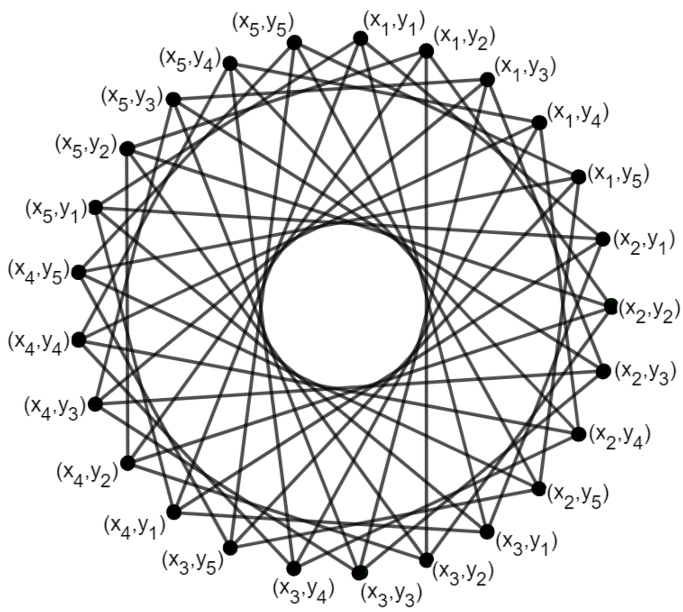

Figure 8. *Suborbital graph Γ_5 corresponding to Δ_5 of $A_5 \times C_5$ acting on $X \times Y$.*

Constructed suborbital graph Γ_5 is undirected, regular of 4 degrees, disconnected with $5-connected$ components and has a girth of 3.

Given that

$\Delta_0 = \{(x_1,y_1)\}$, $\Delta_6 = \{(x_2,y_2),(x_3,y_2),(x_4,y_2),(x_5,y_2)\}$ and $Stab_G(x_1,y_1) = \{(e_x,e_y),((x_2\,x_4\,x_5)\,,e_y),$ $((x_2\,x_5\,x_4)\,,e_y),((x_2\,x_3\,x_5),e_y),((x_3\,x_5\,x_4)\,,e_y),((x_2\,x_4\,x_3)\,,e_y),((x_3\,x_4\,x_5)\,,e_y),((x_2\,x_5\,x_3)\,,e_y),$ $((x_2\,x_3\,x_4)\,,e_y),((x_2\,x_5)\,(x_3\,x_4)\,,e_y),((x_2\,x_3)\,(x_4\,x_5)\,,e_y),((x_2\,x_4)\,(x_3\,x_5)\,,e_y)\}$.

The suborbital O_6 corresponding to suborbit Δ_6 is given as:

$$
\begin{aligned}
O_6 &= \{(e_x,e_y)(x_1,y_1),(e_x,e_y)(x_2,y_2),(e_x,e_y)(x_3,y_2),(e_x,e_y)(x_4,y_2),(e_x,e_y)(x_5,y_2)\}\\
&= \{(x_1,y_1),(x_2,y_2),(x_3,y_2),(x_4,y_2),(x_5,y_2)\}\\
O_6 &= \{((x_2\,x_4\,x_5),e_y)(x_1,y_1),((x_2\,x_4\,x_5),e_y)(x_2,y_2),((x_2\,x_4\,x_5),e_y)(x_3,y_2),((x_2\,x_4\,x_5),e_y)(x_4,y_2),\\
&\quad ((x_2\,x_4\,x_5),e_y)(x_5,y_2)\}\\
&= \{(x_1,y_1),(x_4,y_2),(x_3,y_2),(x_5,y_2),(x_2,y_2)\}
\end{aligned}
$$

$O_6 = \{((x_2\, x_5\, x_4), e_y)(x_1, y_1), ((x_2\, x_5\, x_4), e_y)(x_2, y_2), ((x_2\, x_5\, x_4), e_y)(x_3, y_2), ((x_2\, x_5\, x_4), e_y)(x_4, y_2),$
$((x_2\, x_5\, x_4), e_y)(x_5, y_2)\}$
$= \{(x_1, y_1), (x_5, y_2), (x_3, y_2), (x_2, y_2), (x_4, y_2)\}$

$\vdots$

$O_6 = \{((x_2\, x_4)(x_3\, x_5), e_y)(x_1, y_1), ((x_2\, x_4)(x_3\, x_5), e_y)(x_2, y_2), ((x_2\, x_4)(x_3\, x_5), e_y)(x_3, y_2),$
$((x_2\, x_4)(x_3\, x_5), e_y)(x_4, y_2), ((x_2\, x_4)(x_3\, x_5), e_y)(x_5, y_2)\}$
$= \{(x_1, y_1), (x_4, y_2), (x_5, y_2), (x_2, y_2), (x_3, y_2)\}$

Suborbital graph Γ_6 is then constructed by drawing edges from vertex set point (x_1, y_1) to vertices set points $\{(x_2, y_2), (x_3, y_2), (x_4, y_2), (x_5, y_2)\}$ such that neither the first nor the second components of ordered pairs are identical.

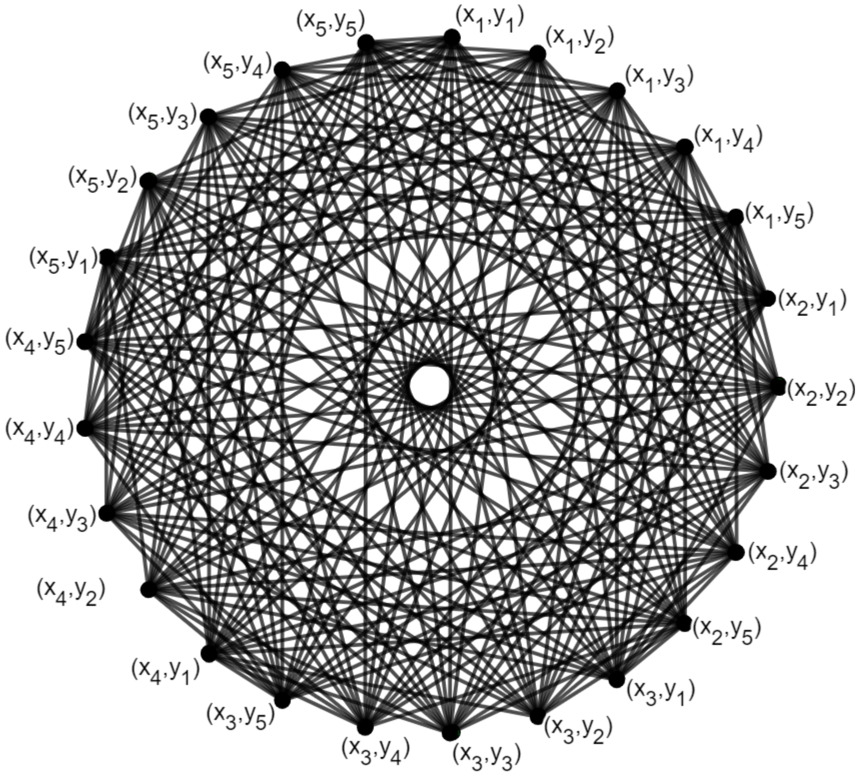

Figure 9. *Suborbital graph Γ_6 corresponding to Δ_6 of $A_5 \times C_5$ acting on $X \times Y$.*

The constructed suborbital graph, Γ_6 is undirected, regular of degree 16, disconnected with $25 - connected$ components and has a girth of 3.

$Note$: The suborbits Δ_7,Δ_8 and Δ_9 fall to the same category as that of Δ_6. That is, neither the first nor the second components are identical. Therefore, the structures of the graphs Γ_7, Γ_8 and Γ_9 are identical to that of Γ_6. Hence, they have same properties as those of Γ_6.

3.4. Suborbital Graphs of $A_n \times C_n$ Acting on $X \times Y$

From the graphs constructed, it is noted that suborbital graphs are classified into three categories:

$Part\ A$. Suborbital graphs are constructed by drawing edges from vertex set (x_1, y_1) to vertices set $\{(x_1, y_2), (x_1, y_3), ..., (x_1, y_n)\}$ such that any first components of ordered pairs are identical but the second components are different.

Therefore, suborbitals O corresponding to the suborbits Δ are given as:

$$O_1 = \{(g)(x_1, y_1), (g)(x_1, y_2)\} = \{(x_1, y_1), (x_1, y_2)\} : g \in G, (x_1, y_2) \in \Delta_1$$
$$O_2 = \{(g)(x_1, y_1), (g)(x_1, y_3)\} = \{(x_1, y_1), (x_1, y_3)\} : g \in G, (x_1, y_3) \in \Delta_2$$
$$O_3 = \{(g)(x_1, y_1), (g)(x_1, y_4)\} = \{(x_1, y_1), (x_1, y_4)\} : g \in G, (x_1, y_4) \in \Delta_3$$
$$\vdots$$
$$O_{(n-1)} = \{(g)(x_1, y_1), (g)(x_1, y_n)\} = \{(x_1, y_1), (x_1, y_n)\} : g \in G, (x_1, y_n) \in \Delta_{(n-1)}$$

Constructed suborbital graphs Γ_1,Γ_2, Γ_3, ..., $\Gamma_(n-1)$ have identical structures showing that they have same properties. That is, the graphs are undirected, regular of degree $(n-1)$, disconnected with $n-connected$ components and has a girth of 3 for $n \geqslant 3$.

$Part\quad B.$ Suborbital graph is constructed by drawing edges from vertex set (x_1, y_1) to vertices set $\{(x_2, y_1), (x_3, y_1), ..., (x_n, y_1)\}$ such that any second components of ordered pairs are identical but the first components are different. The suborbital O corresponding to suborbit Δ is given as:

$$\begin{aligned} O_{(n-1)+1} &= \{(g)(x_1, y_1), (g)(x_2, y_1), (g)(x_3, y_1), (g)(x_4, y_1), ..., (g)(x_n, y_1) \\ &= \{(x_1, y_1), (x_2, y_1), (x_3, y_1), ..., (x_n, y_1)\} : (g) \in G, (x_2, y_1), (x_3, y_1), ..., (x_n, y_1) \in \Delta_{(n-1)+1} \end{aligned}$$

Constructed suborbital graph $\Gamma_{(n-1)+1}$ is undirected, regular of degree $(n-1)$, disconnected with $n-connected$ components and has a girth of 3 for $n \geqslant 3$.

$Part\quad C.$ Suborbital graphs are constructed by drawing edges from vertex set (x_1, y_1) to vertices set $(x_2, y_2), (x_3, y_2), (x_4, y_2), ..., (x_n, y_2)$ such that neither the first nor the second components of the ordered pairs are identical. The suborbitals O corresponding to the suborbits Δ are given as:

$$\begin{aligned} O_{(n-1)+2} &= \{(g)(x_1, y_1), (g)(x_2, y_2), (g)(x_3, y_2), (g)(x_4, y_2), ..., (g)(x_n, y_2)\} \\ &= \{(x_1, y_1), (x_2, y_2), (x_3, y_2), (x_4, y_2), ..., (x_n, y_2)\} \\ &: g \in G, (x_2, y_2), (x_3, y_2), (x_4, y_2), ..., (x_n, y_2) \in \Delta_{(n-1)+2} \\ O_{(n-1)+3} &= \{(g)(x_1, y_1), (g)(x_2, y_3), (g)(x_3, y_3), (g)(x_4, y_3), ..., (g)(x_n, y_3)\} \\ &= \{(x_1, y_1), (x_2, y_3), (x_3, y_3), (x_4, y_3), ..., (x_n, y_3)\} \\ &: g \in G, (x_2, y_3), (x_3, y_3), (x_4, y_3), ..., (x_n, y_3) \in \Delta_{(n-1)+3} \\ O_{(n-1)+4} &= \{(g)(x_1, y_1), (g)(x_2, y_4), (g)(x_3, y_4), (g)(x_4, y_4), ..., (g)(x_n, y_4)\} \\ &= \{(x_1, y_1), (x_2, y_4), (x_3, y_4), (x_4, y_4), ..., (x_n, y_4)\} \\ &: g \in G, (x_2, y_4), (x_3, y_4), (x_4, y_4), ..., (x_n, y_4) \in \Delta_{(n-1)+4} \\ &\vdots \\ O_{(n-1)+n} &= \{(g)(x_1, y_1), (g)(x_2, y_n), (g)(x_3, y_n), (g)(x_4, y_n), ..., (g)(x_n, y_n)\} \\ &= \{(x_1, y_1), (x_2, y_n), (x_3, y_n), (x_4, y_n), ..., (x_n, y_n)\} \\ &: g \in G, (x_2, y_n), (x_3, y_n), (x_4, y_n), ..., (x_n, y_n) \in \Delta_{(n-1)+n} \end{aligned}$$

Constructed suborbital graphs $\Gamma_{(n-1)+2}$, $\Gamma_{(n-1)+3}$, $\Gamma_{(n-1)+4}$,..., $\Gamma_{(n-1)+n}$ have identical structures showing that they have same properties. That is, the graphs are undirected, regular of degree $(n-1)^2$, disconnected with $n^2-connected$ components and has a girth of 3 for $n \geqslant 3$.

$Corollary$ 3.1. The group action of $A_n \times C_n$ acting on $X \times Y$ is imprimitive.

$Proof.$ All suborbital graphs constructed for this group action, are disconnected. Therefore, by theorem 2.1, the group action is imprimitive.

4. Conclusion

In this paper, suborbital graphs of $A_n \times C_n$ on $X \times Y$ for $n \geqslant 3$ were constructed where their respective properties were analyzed. It is proven that all suborbital graphs constructed are undirected, disconnected and each has a girth of 3.

Abbreviations

A_n	Alternating group of degree n whose order Is $\frac{n!}{2}$
C_n	Cyclic group of order n
e_x	An identity element of alternating group
e_y	An identity element of cyclic group
$V = X \times Y$	Cartesian product of SET X and set Y
$\text{Stab}_G(x)$	Stabilizer of a point x in G
$\text{Orb}_G(x)$	The orbit of a point x in G
Δ	Suborbit of G on Set V
$\emptyset$	An empty set
$A_n \times C_n$	Direct product of alternating and cyclic groups
Δ^*	Suborbit which is self-paired with Δ
O	Suborbital of G on the set V
Γ	Suborbital graph which corresponds to suborbital O

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Conflicts of Interest

The authors declare no conflicts of interest.

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