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Weak Solutions to the Three-Dimensional Chemotaxis-Stokes System with Active Transport

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Abstract: In this paper, we study the following chemotaxis-Stokes system with active transport

$$\begin{cases} c_t + u\cdot\nabla c = \nabla\cdot(\alpha c(\nabla c - \chi\nabla n)) - nc, \\ n_t + u\cdot\nabla n = \nabla\cdot(\beta n(\nabla n - \chi\nabla c)) + \lambda nc, \\ u_t = \Delta u + (n+c)\nabla\phi - \nabla P, \quad \nabla\cdot u = 0 \end{cases}$$

in a bounded domain with positive parameters α, β, χ and λ. Here, c and n denote the density of the nutrient acting as a chemoattractant and the density of the cell, respectively and u denotes the velocity of the fluid. The parameters α, β, χ and λ are positive constants and λ represents the cell growth rate occurred by the nutrient supply. The novelty of this system is that there is not only a chemotaxis term, which reflects the movement of the cells toward nutrient sources, but also an active transport one, which means the nutrient is moving towards the cells. In order to prove the existence of weak solutions to the above problem, we introduce the regularized problem using the the Yosida approximation of $B := -\Delta + 1$ under homogeneous Neumann boundary conditions in $L^2(\Omega)$. Based on a priori estimates for the solutions of the regularized problem, it is shown that under no-flux boundary condition and for any suitably regular initial data, an associated initial value problem possesses a global weak solution provided $0 < \chi < 1$.

Keywords: Chemotaxis-Stokes, Active Transport, Weak Solution

1. Introduction

This paper deals with the following chemotaxis-Stokes system with active transport

$$\begin{cases} c_t + u\cdot\nabla c = \nabla\cdot(\alpha c(\nabla c - \chi\nabla n)) - nc, \\ \quad x\in\Omega, \quad t>0, \\ n_t + u\cdot\nabla n = \nabla\cdot(\beta n(\nabla n - \chi\nabla c)) + \lambda nc, \\ \quad x\in\Omega, \quad t>0, \\ u_t = \Delta u + (n+c)\nabla\phi - \nabla P, \quad x\in\Omega, \quad t>0, \\ \nabla\cdot u = 0, \quad x\in\Omega, \quad t>0, \end{cases} \tag{1}$$

for the unknown functions c, n and u in a bounded domain $\Omega \subset \mathbb{R}^3$ with sufficiently smooth boundary. Here, c and n denote the density of the nutrient acting as a chemoattractant and the density of the cell, repectively and u denotes the velocity of the fluid. The parameters α, β, χ and λ are positive constants and λ represents the cell growth rate occurred by the nutrient supply.

Chemotaxis means an oriented movement of cells towards higher concentration chemical materials and this mechanism happens in a lot of biological process ([1, 2]).

Baterica living in water, often follow the oxigen which they consume as well as diffuse through the water. Moreover, they are transported by the fluid. This bialogical process could be described by the following PDE(Partial Differential Equation):

$$\begin{cases} c_t + u \cdot \nabla c = \Delta c - f(x, n, c), \\ n_t + u \cdot \nabla n = \nabla \cdot (D(n)\nabla n - nS(n, c)\nabla c) + g(n), \\ u_t + \kappa(u \cdot \nabla)u = \Delta u - \nabla P + n\nabla\phi, \\ \nabla \cdot u = 0, \end{cases} \tag{2}$$

where $\kappa \in \mathbb{R}$. Here, D repesents the diffusivity and S denotes the chemotactic sensitivity. The given functions g and f reflect the growth of bacterial and the oxygen consumption, respectively. The problem (2) and its modeified ones have been studied in many literatures (see for instance [3–11]). Especially, chemotaxis mechnism appears in the Cahn-Hilliard systems for tumor growth, where chemical species acts as the nutrient for the tumor cells([12], [13–15]). An absorbing interest in these models is that there is not only a chemotaxis term, which reflects the movement of the tumor cells toward nutrient sources, but also an active transport one, which means the nutrient is moving towards the tumor cells([12]). Furthermore, they say that the mechanisms like this have already been observed in the malign tumor colony.

To the best of our knowledge, excepting the Cahn-Hilliard systems with chemotaxis, we are not to able to find any literatures studying chemotaxis models with active transport. Motivated by these work, we are studying chemotaxis systems with active transport, starting from the simple problems, and we consider the problem (1). The purpose of this work is to show the global existence of weak solutions to (1).

In this pater, the norms in $L^q(\Omega)^3$ and $L^q(\Omega)$ are denoted by $\|\cdot\|_q (1 \le q \le \infty)$ and let $\|\cdot\| = \|\cdot\|_2$. Also, $(\cdot,\cdot)$ means the inner product in $L^2(\Omega)^3$ and $L^2(\Omega)$. Define the following function spaces,

$$C^\infty_{0,\sigma}(\Omega) := \{u \in C^\infty_0(\Omega)^3, \nabla \cdot u = 0\},$$

$$L^2_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{L^2(\Omega)^3}},$$

$$W^{1,2}_{0,\sigma} := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,2}(\Omega)^3}}$$

and let A be the Stokes operator with domain $D(A) = W^{1,2}_{0,\sigma} \cap W^{2,2}(\Omega)^3$.

Suppose the following assumptions for the parameters and the initial data,

$$\alpha, \beta, \lambda > 0, \quad 0 < \chi < 1, \tag{3}$$

$$\begin{aligned} &c_0, n_0 \in W^{1,\infty}(\Omega) \\ \text{with} \quad &c_0 \ge 0 \quad \text{and} \quad n_0 \ge 0, \quad u_0 \in D(A^{1/2}). \end{aligned} \tag{4}$$

Definition 1.1. Let $0 < T < \infty$ and $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Assume that (3) and (4) are fulfilled. Then, a pair of functions (c, n, u) is called a weak solution of (1) on $[0, T]$, if it satisfies

$$c, n \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega)), u \in L^\infty(0, T; L^2_\sigma(\Omega)^3) \cap L^2(0, T; W^{1,2}_{0,\sigma}(\Omega)^3),$$

$$-\int_0^T\int_\Omega c\varphi_t - \int_\Omega c_0\varphi(\cdot, 0) - \int_0^T\int_\Omega cu\nabla\varphi = -\alpha\int_0^T\int_\Omega c(\nabla c - \chi\nabla n)\cdot\nabla\varphi - \int_0^T\int_\Omega nc\varphi, \tag{5}$$

$$-\int_0^T\int_\Omega n\varphi_t - \int_\Omega n_0\varphi(\cdot, 0) - \int_0^T\int_\Omega nu\nabla\varphi = -\beta\int_0^T\int_\Omega n(\nabla n - \chi\nabla c)\cdot\nabla\varphi + \lambda\int_0^T\int_\Omega nc\varphi, \tag{6}$$

$$-\int_0^T\int_\Omega u\zeta_t - \int_\Omega u_0\zeta(\cdot, 0) = -\int_0^T\int_\Omega \nabla u \cdot \nabla\zeta + \int_0^T\int_\Omega (n + c)\nabla\phi \cdot \zeta, \tag{7}$$

for any $\varphi \in C^\infty_0(\Omega \times [0, T))$ and $\zeta \in C^\infty_{0,\sigma}(\Omega \times [0, T))$.

Theorem 1.1. Let $0 < T < \infty$ and $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Assume that (3) and (4) are fulfilled. Then, there exists a weak solution of (1) on $[0, T]$.

In this paper, the letter C is a positive constant only related to the parameters $\alpha, \beta, \lambda, \chi$ and the initial data, whose value may be changed from line to line, even in the same line.

2. Proof of the Main Result

To prove Theorem 1.1, we introduce the following regularized problems for $\varepsilon \in (0, 1)$:

$$\begin{cases} c_{\varepsilon t}+u_\varepsilon\cdot\nabla c_\varepsilon=\alpha\nabla\cdot((c_\varepsilon+\varepsilon)\nabla c_\varepsilon-\chi c_\varepsilon\nabla J_\varepsilon n_\varepsilon)-n_\varepsilon c_\varepsilon, \quad x\in\Omega, \quad t>0,\\ n_{\varepsilon t}+u_\varepsilon\cdot\nabla n_\varepsilon=\beta\nabla\cdot((n_\varepsilon+\varepsilon)\nabla n_\varepsilon-\chi n_\varepsilon\nabla J_\varepsilon c_\varepsilon)+\lambda n_\varepsilon c_\varepsilon, \quad x\in\Omega, \quad t>0,\\ u_{\varepsilon t}=\Delta u_\varepsilon-\nabla P_\varepsilon+(n_\varepsilon+c_\varepsilon)\nabla\phi, \quad x\in\Omega, t>0,\\ \nabla\cdot u_\varepsilon=0, \quad x\in\Omega, t>0,\\ \nabla c_\varepsilon\cdot\nu=\nabla n_\varepsilon\cdot\nu=0, \quad u_\varepsilon=0, \quad x\in\partial\Omega, t>0,\\ c_\varepsilon(x,0)=c_0(x), \quad n_\varepsilon(x,0)=n_0(x), \quad x\in\Omega\\ u_\varepsilon(x,0)=u_0, \quad x\in\Omega, \end{cases} \tag{8}$$

where $J_\varepsilon=(I+\varepsilon B)^{-1}$ is the Yosida approximation of $B:=-\Delta+1$ under homogeneous Neumann boundary conditions in $L^2(\Omega)$.

The following lemma shows the existence of classical solution to (8) and it is proved by the similar way from [16], [17], so we omit the proof here.

Lemma 2.1. Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with smooth boundary. Assume that (3) and (4) are fulfilled. Then for each $\varepsilon\in(0,1)$, (8) has a classical solution $(c_\varepsilon, n_\varepsilon, u_\varepsilon)$ such that

$$c_\varepsilon, n_\varepsilon\in C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)),$$

$$u_\varepsilon\in C^0(\bar{\Omega}\times[0,\infty))\cap C^{2,1}(\bar{\Omega}\times(0,\infty))$$

with $c_\varepsilon\geq 0$ and $n_\varepsilon\geq 0$.

Lemma 2.2. Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with smooth boundary. Assume that (3) and (4) are fulfilled. Then,

$$\|c_\varepsilon(t)\|_1+\|n_\varepsilon(t)\|_1+\int_0^t\|n_\varepsilon c_\varepsilon\|_1\leq C \tag{9}$$

holds for any $t>0$.

Proof This lemma is directly proved by integrating the first and second equation of (8).

Lemma 2.3. Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with smooth boundary. Assume that (3) and (4) are fulfilled. Then,

$$\int_0^t\|\nabla c_\varepsilon\|^2+\|\nabla n_\varepsilon\|^2ds\leq C(t+1) \tag{10}$$

holds for any $t>0$.

Proof Test the first equation and the second one of (8) by $\beta\ln c_\varepsilon$ and $\alpha\ln n_\varepsilon$, respecitvely, to obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega\beta(c_\varepsilon\ln c_\varepsilon-c_\varepsilon)dx\\ &\quad+\alpha\beta\|\nabla c_\varepsilon\|^2+2\alpha\beta\varepsilon\|\nabla\sqrt{c_\varepsilon}\|^2\\ &=\alpha\beta\chi(\nabla c_\varepsilon,\nabla J_\varepsilon n_\varepsilon)-\beta\int_\Omega n_\varepsilon c_\varepsilon\ln c_\varepsilon dx, \end{aligned} \tag{11}$$

$$\begin{aligned} &\frac{d}{dt}\int_\Omega\alpha(n_\varepsilon\ln n_\varepsilon-n_\varepsilon)dx\\ &\quad+\alpha\beta\|\nabla n_\varepsilon\|^2+2\alpha\beta\varepsilon\|\sqrt{n_\varepsilon}\|^2\\ &=\alpha\beta\chi(\nabla n_\varepsilon,\nabla J_\varepsilon c_\varepsilon)+\alpha\lambda\int_\Omega c_\varepsilon n_\varepsilon\ln n_\varepsilon dx, \end{aligned} \tag{12}$$

where we use $(u_\varepsilon\cdot\nabla c_\varepsilon,\ln c_\varepsilon)=(u_\varepsilon\cdot\nabla n_\varepsilon,\ln n_\varepsilon)=0$ due to $\nabla\cdot u_\varepsilon=0$. Adding (11) and (12), leads to

$$\begin{aligned} &\frac{d}{dt}\int_\Omega(\beta c_\varepsilon\ln c_\varepsilon+\alpha n_\varepsilon\ln n_\varepsilon-\beta c_\varepsilon-\alpha n_\varepsilon)dx\\ &\quad+\alpha\beta(\|\nabla c_\varepsilon\|^2+\|\nabla n_\varepsilon\|^2)\\ &\leq\alpha\beta\chi((\nabla c_\varepsilon,\nabla J_\varepsilon n_\varepsilon)+(\nabla n_\varepsilon,\nabla J_\varepsilon c_\varepsilon))\\ &\quad-\int_\Omega n_\varepsilon c_\varepsilon(\beta\ln c_\varepsilon-\alpha\lambda\ln n_\varepsilon)dx. \end{aligned} \tag{13}$$

From the Hölder inequality and Young's inequality, we have

$$\begin{aligned} &(\nabla c_\varepsilon,\nabla J_\varepsilon n_\varepsilon)+(\nabla n_\varepsilon,\nabla J_\varepsilon c_\varepsilon)\\ &\qquad\leq\|\nabla c_\varepsilon\|\|\nabla J_\varepsilon n_\varepsilon\|+\|\nabla n_\varepsilon\|\|\nabla J_\varepsilon c_\varepsilon\|\\ &\qquad\leq\frac{1}{2}(\|\nabla c_\varepsilon\|^2+\|\nabla n_\varepsilon\|^2)\\ &\qquad\quad+\frac{1}{2}(\|\nabla J_\varepsilon c_\varepsilon\|^2+\|J_\varepsilon\nabla n_\varepsilon\|^2)\\ &\qquad\leq\frac{1}{2}(1+\frac{1}{\chi})(\|\nabla c_\varepsilon\|^2+\|\nabla n_\varepsilon\|^2)\\ &\qquad\quad+C(\|c_\varepsilon\|_1^2+\|n_\varepsilon\|_1^2) \end{aligned}$$

where we use

$$\begin{aligned} \|\nabla J_\varepsilon v\|^2&=(-\Delta J_\varepsilon v, J_\varepsilon u)=(BJ_\varepsilon v-J_\varepsilon v, J_\varepsilon v)\\ &=\|B^{1/2}J_\varepsilon v\|^2-\|J_\varepsilon v\|^2\leq\|B^{1/2}v\|^2\\ &=(u,Bv)=(u,-\Delta v+v)\\ &=\|\nabla v\|^2+\|v\|^2\\ &\leq\|\nabla v\|^2+C(\|\nabla v\|^{6/5}\|v\|_1^{4/5}+\|v\|_1^2)\\ &\leq\|\nabla v\|^2+(\frac{1}{\chi}-1)\|\nabla v\|^2+C\|v\|_1^2. \end{aligned} \tag{14}$$

Also, we get

$$\begin{aligned} &n_\varepsilon c_\varepsilon(\beta\ln c_\varepsilon-\alpha\lambda\ln n_\varepsilon)\\ &\quad\leq C\int_\Omega n_\varepsilon(1+c_\varepsilon^{3/2})+c_\varepsilon(1+n_\varepsilon^{3/2})dx\\ &\quad\leq C(\|c_\varepsilon\|_1+\|n_\varepsilon\|_1+\|c_\varepsilon\|_{5/2}^{5/2}+\|n_\varepsilon\|_{5/2}^{5/2})\\ &\quad\leq C(1+\|c_\varepsilon\|_1^{5/2}+\|n_\varepsilon\|_1^{5/2}+\|\nabla c_\varepsilon\|^{9/5}\|c_\varepsilon\|_1^{7/10}\\ &\qquad+\|\nabla n_\varepsilon\|^{9/5}\|n_\varepsilon\|_1^{7/10})\\ &\quad\leq\frac{\alpha\beta(1-\chi)}{4}(\|\nabla c_\varepsilon\|^2+\|\nabla n_\varepsilon\|^2)\\ &\qquad+C(1+\|c_\varepsilon\|_1^7+\|n_\varepsilon\|_1^7), \end{aligned}$$

where we use $|y\ln y| < C(1+y^{3/2}), y>0$ and the Gagliardo-Nirenberg inequality

$$\|v\|_{5/2}^{5/2} \le C(\|\nabla v\|^{9/5}\|v\|_1^{7/10} + \|v\|_1^{5/2}).$$

By the above estimations, (13) becomes

$$\begin{aligned}&\frac{d}{dt}\int_\Omega(\beta c_\varepsilon \ln c_\varepsilon + \alpha c_\varepsilon \ln c_\varepsilon - \beta c_\varepsilon - \alpha n_\varepsilon)dx\\ &\qquad + \frac{\alpha\beta(1-\chi)}{4}(\|\nabla c_\varepsilon\|^2 + \|\nabla n_\varepsilon\|^2)\\ &\quad \le C(1+\|c_\varepsilon\|_1^{7/2} + \|n_\varepsilon\|_1^{7/2}).\end{aligned} \tag{15}$$

Integrate the above inequality on $(0,t)$, to get

$$\begin{aligned}&\int_\Omega \beta c_\varepsilon \ln c_\varepsilon + \alpha n_\varepsilon \ln n_\varepsilon dx\\ &\qquad + \frac{\alpha\beta(1-\chi)}{4}\int_0^t \|\nabla n_\varepsilon\|^2 + \|\nabla c_\varepsilon\|^2 ds\\ &\quad \le Ct + C.\end{aligned} \tag{16}$$

Considering $\int_\Omega c_\varepsilon \ln c_\varepsilon dx \ge -C|\Omega|$ and $\int_\Omega n_\varepsilon \ln n_\varepsilon dx \ge -C|\Omega|$, due to $y\ln y \to 0, y\to 0$, (16) completes the proof.

Lemma 2.4. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Assume that (3) and (4) are fulfilled. Then,

$$\begin{aligned}&\|A^{1/2}u_\varepsilon(t)\|^2 + \|c_\varepsilon(t)\|^2 + \|n_\varepsilon(t)\|^2\\ &\qquad + \int_0^t \|Au_\varepsilon\|^2 ds \le Ce^{Ct} + C,\end{aligned} \tag{17}$$

holds for any $t > 0$.

Proof Test the first equation of (8) by $c_\varepsilon - \chi J_\varepsilon n_\varepsilon$ to obtain

$$\begin{aligned}&(c_{\varepsilon t}, c_\varepsilon - \chi J_\varepsilon n_\varepsilon) - \chi(u_\varepsilon\cdot\nabla c_\varepsilon, J_\varepsilon n_\varepsilon)\\ &\qquad + \alpha\|(c_\varepsilon+\varepsilon)^{1/2}\nabla(c_\varepsilon - \chi J_\varepsilon n_\varepsilon)\|^2\\ &\quad = -(n_\varepsilon c_\varepsilon, c_\varepsilon - \chi J_\varepsilon n_\varepsilon)\\ &\qquad + \alpha\varepsilon(\nabla J_\varepsilon n_\varepsilon, \nabla(c_\varepsilon - \chi J_\varepsilon n_\varepsilon)).\end{aligned} \tag{18}$$

Similarly, test the second equation of (8) by $n_\varepsilon - \chi J_\varepsilon c_\varepsilon$ to obtain

$$\begin{aligned}&(n_{\varepsilon t}, n_\varepsilon - \chi J_\varepsilon c_\varepsilon) - \chi(u_\varepsilon\cdot\nabla n_\varepsilon, J_\varepsilon c_\varepsilon)\\ &\qquad + \beta\|(n_\varepsilon+\varepsilon)^{1/2}\nabla(n_\varepsilon - \chi J_\varepsilon c_\varepsilon)\|^2\\ &\quad = \lambda(n_\varepsilon c_\varepsilon, n_\varepsilon - \chi J_\varepsilon c_\varepsilon)\\ &\qquad + \beta\varepsilon(\nabla J_\varepsilon c_\varepsilon, \nabla(n_\varepsilon - \chi J_\varepsilon c_\varepsilon)).\end{aligned} \tag{19}$$

Add the above equations, then we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|c_\varepsilon(t)\|^2 + \|n_\varepsilon(t)\|^2 - 2\chi(J_\varepsilon^{1/2}n_\varepsilon(t), J_\varepsilon^{1/2}c_\varepsilon(t)))\\ &\qquad + \alpha\|c_\varepsilon^{1/2}\nabla(c_\varepsilon - \chi J_\varepsilon n_\varepsilon)\|^2 + \beta\|n_\varepsilon^{1/2}\nabla(n_\varepsilon - \chi J_\varepsilon c_\varepsilon)\|^2 + \alpha\varepsilon\|\nabla(c_\varepsilon - \chi J_\varepsilon n_\varepsilon)\|^2 + \beta\varepsilon\|\nabla(n_\varepsilon - \chi J_\varepsilon c_\varepsilon)\|^2\\ &\quad = -(n_\varepsilon c_\varepsilon, (1+\lambda\chi)c_\varepsilon - (\lambda+\chi)n_\varepsilon) + \chi((u_\varepsilon\cdot\nabla c_\varepsilon, J_\varepsilon n_\varepsilon) + (u_\varepsilon\cdot\nabla n_\varepsilon, J_\varepsilon c_\varepsilon))\\ &\qquad + \alpha\varepsilon(\nabla J_\varepsilon n_\varepsilon, \nabla(c_\varepsilon - \chi J_\varepsilon n_\varepsilon)) + \beta\varepsilon(\nabla J_\varepsilon c_\varepsilon, \nabla(n_\varepsilon - \chi J_\varepsilon c_\varepsilon)).\end{aligned} \tag{20}$$

Let

$$z(t) := \|c_\varepsilon(t)\|^2 + \|n_\varepsilon(t)\|^2 - 2\chi(J_\varepsilon^{1/2}n_\varepsilon(t), J_\varepsilon^{1/2}c_\varepsilon(t)),$$

which fulfills

$$\begin{aligned}z(t) &= \|c_\varepsilon(t)\|^2 + \|n_\varepsilon(t)\|^2 - 2\chi(J_\varepsilon^{1/2}n_\varepsilon(t), J_\varepsilon^{1/2}c_\varepsilon(t))\\ &\ge \|c_\varepsilon(t)\|^2 + \|n_\varepsilon(t)\|^2 - 2\chi\|J_\varepsilon^{1/2}n_\varepsilon(t)\|\|J_\varepsilon^{1/2}c_\varepsilon(t)\|\\ &\ge \|c_\varepsilon(t)\|^2 + \|n_\varepsilon(t)\|^2 - 2\chi\|n_\varepsilon(t)\|\|c_\varepsilon(t)\|\\ &\ge (1-\chi)(\|c_\varepsilon(t)\|^2 + \|n_\varepsilon(t)\|^2)\end{aligned} \tag{21}$$

by the Hölder inequality and Young's inequality.

Now, using the Hölder inequality and Young's inequality, we have

$$\begin{aligned}&-(n_\varepsilon c_\varepsilon, (1+\lambda\chi)c_\varepsilon - (\lambda+\chi)n_\varepsilon)\\ &\quad \le C(\|c_\varepsilon\|_3^3 + \|n_\varepsilon\|_3^3)\\ &\quad \le C(\|\nabla c_\varepsilon\|^{3/2}\|c_\varepsilon\|^{3/2} + \|\nabla n_\varepsilon\|^{3/2}\|n_\varepsilon\|^{3/2}) + C\\ &\quad \le C(\|\nabla c_\varepsilon\|^2\|c_\varepsilon\|^2 + \|\nabla n_\varepsilon\|^2\|n_\varepsilon\|^2\\ &\qquad + \|\nabla c_\varepsilon\|^2 + \|\nabla n_\varepsilon\|^2 + \|c_\varepsilon\|^2 + \|n_\varepsilon\|^2) + C\\ &\quad \le Cz(\|\nabla c_\varepsilon\|^2 + \|\nabla n_\varepsilon\|^2 + 1)\\ &\qquad + C(\|\nabla c_\varepsilon\|^2 + \|\nabla n_\varepsilon\|^2) + C,\end{aligned} \tag{22}$$

where we use (21) and the Gagliardo-Nirenberg inequality

$$\|v\|_3 \le C(\|\nabla v\|^{1/2}\|v\|^{1/2} + \|v\|_1).$$

And using Hölder inequality and Young's inequality yield

$$\begin{aligned}&(u_\varepsilon\cdot\nabla c_\varepsilon, J_\varepsilon n_\varepsilon) + (u_\varepsilon\cdot\nabla n_\varepsilon, J_\varepsilon c_\varepsilon)\\ &\quad \le \|\nabla c_\varepsilon\|^2 + \|\nabla n_\varepsilon\|^2\\ &\qquad + C\|u_\varepsilon\|_6^2(\|J_\varepsilon n_\varepsilon\|_3^2 + \|J_\varepsilon c_\varepsilon\|_3^2)\\ &\quad \le \|\nabla c_\varepsilon\|^2 + \|\nabla n_\varepsilon\|^2\\ &\qquad + C\|A^{1/2}u_\varepsilon\|^2(\|\nabla n_\varepsilon\|^2 + \|\nabla c_\varepsilon\|^2 + 1).\end{aligned} \tag{23}$$

Also, we have

$$\begin{aligned}&\alpha\varepsilon(\nabla J_\varepsilon n_\varepsilon, \nabla(c_\varepsilon - \chi J_\varepsilon n_\varepsilon))\\ &\quad\le \alpha\varepsilon\|\nabla J_\varepsilon n_\varepsilon\|\|\nabla(c_\varepsilon - \chi J_\varepsilon n_\varepsilon)\|\\ &\quad\le \alpha\varepsilon\|\nabla(c_\varepsilon - \chi J_\varepsilon n_\varepsilon)\|^2 + C\|\nabla J_\varepsilon n_\varepsilon\|^2\\ &\quad\le \alpha\varepsilon\|\nabla(c_\varepsilon - \chi J_\varepsilon n_\varepsilon)\|^2 + C(\|\nabla n_\varepsilon\|^2 + \|n_\varepsilon\|_1^2)\end{aligned} \tag{24}$$

and

$$\begin{aligned}&\beta\varepsilon(\nabla J_\varepsilon c_\varepsilon, \nabla(n_\varepsilon - \chi J_\varepsilon c_\varepsilon))\\ &\quad\le \beta\varepsilon\|\nabla J_\varepsilon c_\varepsilon\|\|\nabla(n_\varepsilon - \chi J_\varepsilon c_\varepsilon)\|\\ &\quad\le \beta\varepsilon\|\nabla(n_\varepsilon - \chi J_\varepsilon c_\varepsilon)\|^2 + C\|\nabla J_\varepsilon c_\varepsilon\|^2\\ &\quad\le \beta\varepsilon\|\nabla(n_\varepsilon - \chi J_\varepsilon c_\varepsilon)\|^2 + C(\|\nabla c_\varepsilon\|^2 + \|c_\varepsilon\|_1^2),\end{aligned} \tag{25}$$

where we use (14).

On the other hand, test the third equation of (8) by Au_ε to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|A^{1/2}u_\varepsilon\|^2 + \|Au_\varepsilon\|^2\\ &\quad= ((n_\varepsilon + c_\varepsilon)\nabla\phi, Au_\varepsilon))\\ &\quad\le \frac{1}{2}\|Au_\varepsilon\|^2 + C(\|n_\varepsilon\|^2 + \|c_\varepsilon\|^2)\\ &\quad\le \frac{1}{2}\|Au_\varepsilon\|^2 + C(\|\nabla n_\varepsilon\|^2 + \|\nabla c_\varepsilon\|^2).\end{aligned} \tag{26}$$

Add (20) and (26) using the estimations (22)-(25) to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(z(t) + \|A^{1/2}u_\varepsilon\|^2) + \alpha\|c_\varepsilon^{1/2}\nabla(c_\varepsilon - \chi J_\varepsilon n_\varepsilon)\|^2\\ &\quad + \beta\|n_\varepsilon^{1/2}\nabla(n_\varepsilon - \chi J_\varepsilon c_\varepsilon)\|^2 + \frac{1}{2}\|Au_\varepsilon\|^2\\ &\le Cz(\|\nabla c_\varepsilon\|^2 + \|\nabla n_\varepsilon\|^2 + 1)\\ &\quad + \|A^{1/2}u_\varepsilon\|^2(\|\nabla c_\varepsilon\|^2 + \|\nabla n_\varepsilon\|^2 + 1)\\ &\quad + C(\|\nabla c_\varepsilon\|^2 + \|\nabla n_\varepsilon\|^2 + 1).\end{aligned} \tag{27}$$

Applying the Gronwall's inequality to (27) considering (10), proves this lemma.

Proof of Theorem 1.1 Let $0 < T < \infty$. Applying Aubin-Lions lemma considering Lemma 2.3 and Lemma 2.4, we can choose $\{\varepsilon\}, \varepsilon \to 0$ (still denote by itself) and take the functions c, n and u, such that

$$c, n \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)),$$

$$u \in L^\infty(0,T;D(A^{1/2})) \cap L^2(0,T;D(A)),$$

$$c_\varepsilon \to c \quad in \quad L^2(0,T;L^2(\Omega)),$$

$$c_\varepsilon \rightharpoonup c \quad in \quad L^2(0,T;W^{1,2}(\Omega)),$$

$$n_\varepsilon \to n \quad in \quad L^2(0,T;L^2(\Omega)),$$

$$n_\varepsilon \rightharpoonup n \quad in \quad L^2(0,T;W^{1,2}(\Omega)),$$

$$u_\varepsilon \to n \quad in \quad L^2(0,T;D(A^{1/2})),$$

$$u_\varepsilon \rightharpoonup u \quad in \quad L^2(0,T;D(A)).$$

Now, prove that (c, n, u) is the weak solution of (1) on $[0, T]$. First, (8) leads to

$$\begin{aligned}&-\int_0^T\int_\Omega c_\varepsilon\varphi_t - \int_\Omega c_0\varphi(\cdot,0) - \int_0^T\int_\Omega c_\varepsilon u_\varepsilon\nabla\varphi\\ &\quad= -\alpha\int_0^T\int_\Omega((c_\varepsilon + \varepsilon)\nabla c_\varepsilon - c_\varepsilon\chi\nabla J_\varepsilon n_\varepsilon)\cdot\nabla\varphi\\ &\qquad - \int_0^T\int_\Omega n_\varepsilon c_\varepsilon\varphi,\end{aligned} \tag{28}$$

$$\begin{aligned}&-\int_0^T\int_\Omega n_\varepsilon\varphi_t - \int_\Omega n_0\varphi(\cdot,0) - \int_0^T\int_\Omega n_\varepsilon u_\varepsilon\nabla\varphi\\ &\quad= -\beta\int_0^T\int_\Omega((n_\varepsilon + \varepsilon)\nabla n_\varepsilon - n_\varepsilon\chi\nabla J_\varepsilon c_\varepsilon)\cdot\nabla\varphi\\ &\qquad + \lambda\int_0^T\int_\Omega n_\varepsilon c_\varepsilon\varphi,\end{aligned} \tag{29}$$

$$\begin{aligned}&-\int_0^T\int_\Omega u_\varepsilon\zeta_t - \int_\Omega u_0\zeta(\cdot,0)\\ &\quad= -\int_0^T\int_\Omega\nabla u_\varepsilon\cdot\nabla\zeta + \int_0^T\int_\Omega(n_\varepsilon + c_\varepsilon)\nabla\phi\cdot\zeta,\end{aligned} \tag{30}$$

for any $\varphi \in C_0^\infty(\Omega\times[0,T))$ and $\zeta \in C_{0,\sigma}^\infty(\Omega\times[0,T))$.

From the above convergence and the boundedness of $c_\varepsilon, n_\varepsilon$ and u_ε, we can easily see that

$$\int_0^T\int_\Omega n_\varepsilon\varphi_t \to \int_0^T\int_\Omega n\varphi_t, \quad \int_0^T\int_\Omega c_\varepsilon\varphi_t \to \int_0^T\int_\Omega n\varphi_t,$$

$$\int_0^T\int_\Omega u_\varepsilon\zeta_t \to \int_0^T\int_\Omega u\zeta_t, \quad \int_0^T\int_\Omega n_\varepsilon c_\varepsilon\varphi \to \int_0^T\int_\Omega nc\varphi,$$

$$\int_0^T\int_\Omega n_\varepsilon u_\varepsilon\nabla\varphi \to \int_0^T\int_\Omega nu\nabla\varphi,$$

$$\int_0^T\int_\Omega c_\varepsilon u_\varepsilon\nabla\varphi \to \int_0^T\int_\Omega cu\nabla\varphi$$

$$\int_0^T\int_\Omega(c_\varepsilon + \varepsilon)\nabla c_\varepsilon\cdot\nabla\varphi \to \int_0^T\int_\Omega c\nabla c\cdot\nabla\varphi,$$

$$\int_0^T\int_\Omega(n_\varepsilon + \varepsilon)\nabla n_\varepsilon\cdot\nabla\varphi \to \int_0^T\int_\Omega n\nabla n\cdot\nabla\varphi,$$

$$\int_0^T\int_\Omega\nabla u_\varepsilon\nabla\zeta \to \int_0^T\int_\Omega\nabla u\nabla\eta,$$

$$\int_0^T\int_\Omega(n_\varepsilon + c_\varepsilon)\nabla\phi\cdot\zeta \to \int_0^T\int_\Omega(n+c)\phi\cdot\zeta$$

when $\varepsilon \to 0$.

To prove the convergence of $\int_0^T\int_\Omega c_\varepsilon\nabla J_\varepsilon n_\varepsilon\cdot\nabla\varphi$, we use following estimations:

$$\begin{aligned}
&\int_0^T\int_\Omega c_\varepsilon \nabla J_\varepsilon n_\varepsilon \cdot \nabla\varphi - \int_0^T\int_\Omega c\nabla n\cdot\nabla\varphi \\
&\quad = \int_0^T\int_\Omega c_\varepsilon \nabla J_\varepsilon (n_\varepsilon - n)\cdot\nabla\varphi + \int_0^T\int_\Omega c_\varepsilon \nabla (J_\varepsilon n - n)\cdot\nabla\varphi \\
&\qquad + \int_0^T\int_\Omega (c_\varepsilon - c)\nabla n\cdot\nabla\varphi,
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\int_0^T\int_\Omega c_\varepsilon \nabla J_\varepsilon (n_\varepsilon - n)\cdot\nabla\varphi + \int_0^T\int_\Omega (c_\varepsilon - c)\nabla n\cdot\nabla\varphi \\
&= -\int_0^T\int_\Omega J_\varepsilon (n_\varepsilon - n)\nabla c_\varepsilon\cdot\nabla\varphi - \int_0^T\int_\Omega c_\varepsilon J_\varepsilon (n_\varepsilon - n)\Delta\varphi + \int_0^T\int_\Omega (c_\varepsilon - c)\nabla n\cdot\nabla\varphi \\
&\le C(\|n_\varepsilon - n\|_{L^2(0,T;L^2(\Omega))}\|\nabla c_\varepsilon\|_{L^2(0,T;L^2(\Omega))} + \|n_\varepsilon - n\|_{L^2(0,T;L^2(\Omega))}\|c_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \\
&\quad + \|c_\varepsilon - c\|_{L^2(0,T;L^2(\Omega))}\|\nabla n\|_{L^2(0,T;L^2(\Omega))}) \\
&\to 0, \quad \varepsilon\to 0.
\end{aligned} \tag{32}$$

$$\begin{aligned}
&\int_0^T\int_\Omega c_\varepsilon \nabla (J_\varepsilon n - n)\cdot\nabla\varphi \\
&\quad = -\int_0^T\int_\Omega (J_\varepsilon n - n)\nabla c_\varepsilon\cdot\nabla\varphi - \int_0^T\int_\Omega (J_\varepsilon n - n)c_\varepsilon\Delta\varphi \\
&\quad \le C\|J_\varepsilon n - n\|_{L^2(0,T;L^2(\Omega))}\|\nabla c_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \\
&\qquad + C\|J_\varepsilon n - n\|_{L^2(0,T;L^2(\Omega))}\|c_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \\
&\quad \le C\|J_\varepsilon n - n\|_{L^2(0,T;L^2(\Omega))} \to 0, \quad \varepsilon\to 0,
\end{aligned} \tag{33}$$

where we use Lebesgue's dominated convergence lemma considering

$$\|J_\varepsilon n - n\|^2 \le (\|J_\varepsilon n\| + \|n\|)^2 \le 4\|n\|^2, \quad \int_0^T \|n\|^2 < \infty,$$

and

$$\lim_{\varepsilon\to 0}\|J_\varepsilon n - n\|^2_{L^2(0,T;L^2(\Omega))} = \lim_{\varepsilon\to 0}\int_0^T \|J_\varepsilon n - n\|^2 = \int_0^T \lim_{\varepsilon\to 0}\|J_\varepsilon n - n\|^2 = 0. \tag{34}$$

The convergence of $\int_0^T\int_\Omega n_\varepsilon \nabla J_\varepsilon c_\varepsilon\cdot\nabla\varphi$ is obtained by the silmilar way.

Bringing together above calculations, yields that (c, n, u) satisfies (5)-(7) for any $\varphi \in C_0^\infty(\Omega\times[0,T))$ and $\zeta \in C_{0,\sigma}^\infty(\Omega\times[0,T))$.

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Abbreviations

PDE Partial Differential Equation

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Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] E. Keller, L. Segel, Model for chemotaxis, J. Theor. Biol. 30 (1979) 225–234.

[2] T. Hillen, K. Painter, A user's guide to PDE models for chemotaxis, J. Math. Biol. 58 (2009) 183–217.

[3] T. Nagai, Blow-up of radially symmetric solutions to a chemotaxis system, Adv. Math. Sci. Appl. 5 (1995) 581–601.

[4] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, J. Differential Equations 248 (2010) 2889–2905.

[5] N. Mizoguchi, P. Souplet, Nondegeneracy of blow-up points for the parabolic Keller-Segel system, Ann. Inst. H. Poincaé Anal. Non Linéaire 31 (2014) 851–875.

[6] N. Bellomo, A. Bellouquid, Y. Tao, M. Winker, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci. 25 (2015) 1663–1763.

[7] Y. Wang, M. Winkler, Z. Xiang, Global classical solutions in a two-dimensional chemotaxis-Navier-Stokes system with subcritical sensitivity, Annali della Scuola Normale Superiore di Pisa-Classe di Scienze XVIII (2018) 2036–2145.

[8] A. Khelghati, K. Baghaei, Boundedness of classical solutions for a chemotaxis model with rotational flux terms, Z. Agew. Math. Mech 98 (2018) 1864–1877.

[9] Q. Zhang, W. Tao, Bounedness and stabilization in a two-species chemotaxis system with signal absorption, Computers and Mathematics with Applications 78 (2019) 2672–2681.

[10] K. Nam, K. Li, Y. Kim, Boundedness of solutions to a 2D chemotaxis-Navier-Stokes system with general sensitivity and nonlinear diffusion, Nonlinear Anal. RWA 73 (2023) 103906.

[11] Y. Kim, K. Li, K. Nam, Global solvability and stabilization in a three-dimensional coral fertilization model involving the Navier-Stokes equations, J. Math. Anal. Appl. 534 (2024) 128048.

[12] H. Garcke, K. Lam, E. Sitka, V. Styles, A Cahn-Hilliard-Darcy model for tumour growth with chemotaxis and active transport, Math. Models Methods Appl. Sci. 26 (2016) 1095–1148.

[13] H. Garcke, K. Lam, Analysis of a Cahn-Hilliard system with non-zero Dirichlet conditions modeling tumor growth with chemotaxis, Discrete Contin. Dyn. Syst. 37 (2017) 4277–4308.

[14] H. Garcke, K. Lam, Well-posedness of a Cahn-Hilliard system modelling tumour growth with chemotaxis and active transport, European J. Appl. Math. 28(2) (2017) 284–316.

[15] M. Ebenbeck, H. Garcke, Analysis of a Cahn-Hilliard-Brinkman model for tumour growth with chemotaxis, J. Differential Equations 266(9) (2019) 5998–6036.

[16] Y. Tao, M. Winkler, A chemotaxis-haptotaxis model: The roles of nonlinear diffusion and logistic source, SIAM J. Math. Anal. 43 (2011) 685–704.

[17] M. Winkler, A three-dimensional Keller-Segel-Navier-Stokes system with logistic source: Global weak solutions and asymptotic stabilization, J. Functional Analysis 276 (2019) 1339–1401.