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Research Article

Analysis of Short-Term Interest Rate Models with Stochastic Volatility

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Abstract

The examination of short-term interest-rate behaviour is of critical importance in financial analysis, risk management, and in formulating monetary policy. Fluctuations in financial markets in recent times have emphasised the need for strong and reliable models that can effectively model behaviors and dynamics involved in short-term interest-rate fluctuations. Conventional approaches, including Vasicek's, have been universally embraced; yet such techniques often face difficulty in explaining clustering and autocorrelated volatility in real-world data. This study explores short-term interest rate models with stochastic volatility and evaluates their effectiveness in comparison to Autoregressive Conditional Heteroskedasticity (ARCH) and Generalised ARCH (GARCH) models. Using historical data from Nigerian financial instruments, we carried out Ljung-Box Q-statistic and ARCH tests to examine autocorrelation and volatility clustering. Results indicate that the data exhibits strong autocorrelation and significant volatility clustering. The predictive performance of our stochastic volatility model was measured by 10-day ahead volatility forecasts, which reached the sum of squared deviations of 1.3095, while ARCH had 2.0001 and GARCH had 2.1433. Our findings suggest that the stochastic volatility model outperforms the traditional ones, such as ARCH and GARCH, for interest rate change forecasting. Based on the performance realised, observed stochastic volatility models are recommended to better forecast interest rates, particularly for the emerging markets, where financial data could be volatile.

Keywords

Interest Rate Models, Stochastic Volatility, ARCH, GARCH

1. Introduction

In financial terms, interest is basically the cost of borrowing money [2]. Just as in any free-market economy, price ultimately comes down to supply and demand. Such that when demand is weak, lenders charge less for them to release their cash. However, when demand is great, they increase the price by charging a higher interest rate. Interest rate futures trading has grown dramatically during the previous two decades, with the introduction of various new products. Interest rate derivatives are priced based on interest rate levels and models; hence, effective market practice and financial theory are dependent

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on sound statistical analysis and interest rate modelling [1]. Similarly, interest rate characteristics are as common when economic conditions change. This is where the government plays a major role.

Central banks like the Central Bank of Nigeria (CBN) tend to purchase government debt when an economy is experiencing a contraction, pumping much-needed cash into the stagnant economy for new loans [43]. This increased supply, coupled with weak demand, drives the interest rates down. During an upturn, this scenario is exactly opposite. Still, at face value, a hike or drop in interest rates is quite unsure. Therefore, a number of interest rate models have been developed over the years in the attempt to predict or assess the characteristics of various interest rates. An interest rate model is generally regarded as a probabilistic description of the future development of interest rates [42]. In the context of rapid development, innovation, and expansion of the interest rate markets, it is increasingly important to keep pace with practical and theoretical advances. Future interest rates cannot be predicted, in fact, contemporary information-interest rate models characterise that uncertainty [26]. Models such as these have to be applied in quantitative analysis of securities with rate-dependent cash flows to find the present value of the uncertainty. Such models often possess stochastic volatility.

The word stochastic could be seen as just another word for random or uncertain [6]. The concept of interest rate itself encompasses the time value of money; how money in the present value is worth more than money in the future [26]. With such a principle, lenders have to add on an interest component to balance the two values—the interest rate. How interest rates change as time goes by has an air of unpredictability to it. And with uncertainties comes actuaries, which is used to develop models.

In Nigeria for example, the CBN is responsible for setting the interest rates. Each country's government just sets interest rates to what they feel is best for their country's economic situation. Normally a government would set a higher interest rate to combat inflation or low-interest rates to try and increase growth [2]. In other words, a low-interest rate increases growth and a high-interest rate combat inflation. However, what would warrant these changes in interest rates is unpredictable or uncertain. This is where and why actuaries come in handy and interest rate models are developed [6]. Nevertheless, there is a myriad of interest rate models with stochastic volatility which are been used to predict interest rates. This research mainly analysed popular short rate models with stochastic volatility.

Whilst there are several interest rate models for use, this research compares the performance of a wide variety of well-known short term interest models in capturing their stochastic behaviour. This can serve as a common framework in which different models could be nested and their performance benchmarked. The research could equally serve as a resource material for scholars in the field of financial engineering and economics.

2. Literature Review and Statement of Research Problem

2.1. Conceptual Review

2.1.1. Concept of Interest Rate and Modelling

Interest is the percentage of a totality of money charged for its use; it is a price paid for borrowing money or payment received for lending money. Interest rate more particularly is "either the cost of borrowing money or the reward for saving it and it is calculated as a percentage of the amount borrowed" [7]. From the definition of the Bank of England, one can infer that an interest rate is the amount of interest due per period, as a proportion of the amount lent, deposited or borrowed [7]. This lent amount is often referred to as the principal sum. According to [2], "interest rates rates represent the compensation provided by a borrower (debtor) to a lender (creditor) for the utilisation of funds over a specified duration, expressed as an annual percentage."

As a standalone, "interest" is of two types—simple and compound interest [1]. Simple interest is computed based on the initial or principal sum of a loan. In contrast to simple interest, the amount of compound interest will vary each year since it accounts for the interest accrued in prior periods. Characteristic to compound interest, interest rates can be either in fixed or variable (floating) terms. Fixed interest rates are rates that do not change over the lifespan of the loan or investment—irrespective of any changing economic conditions [2]. A variable or floating interest rate, on the other hand, is subject to change and is often secured to an underlying index such as the one-year Treasury Bill rate, or even the popular London Interbank Offering Rate (LIBOR) rate. The LIBOR rate is extensively published on short-term European money market loans and has affected the foreign lending rates of big banks in the United States of America, as well as regional and smaller institutions, globally [1].

An interest rate model is a hypothetical description of a complex process regarding the rate of interest. They are frequently employed to calculate probable interest rate change. Interest rate models are pertinent in futures, bonds, money markets and/or other processes where the interest rate is extended or compounded over a period which may require predictive analysis. A model would try to identify the elements or conditions that are thought to explain the movements of interest rates [21]. These factors are unpredictable or stochastic, hence it is impossible to forecast with certainty the future level of any factor. [48] opined that interest rate modelling involves formulas that can get complex. An unambiguous scheme of notation must be used and carried across a range of different models; which can equally be useful for calculations. Furthermore, to produce a somewhat realistic depiction of interest rate behaviour, an interest-rate model must define a statistical procedure that characterises the stochastic nature of the elements thought to explain interest rate dynamics [21].

Over the years, there have been several developments in interest rate models. This research mainly covers short interest rate models, nevertheless, there are other interest rate models such as instantaneous forward rate models and market-rate models. Even though some of these models has been used today, many of them can be calibrated and modified to suit unique conditions in contemporary times [16].

2.1.2. Short Rate Models

When used in interest rate derivatives, the short rate model is a mathematical model that describes how the short rate will change over time to determine how interest rates will change in the future. It is represented by r_t [5, 41].

The formula for the short rate model is given by:

$$dr = (\alpha + \beta r)\, dt + \sigma\, (r^{\gamma})\, dZ \tag{1}$$

Where:

dr = change in interest rates

dt = time interval

σ = variance of the rate changes

dZ = random variable

r = risk-less interest rate

According to [5], the spot rate at a certain moment in time—also known as the instantaneous interest rate—is the stochastic state variable in short rate modelling. As a result, the annualised, constantly compounded interest rate at which an organisation can borrow money for an incredibly short time t on the yield curve is known as the short rate, or r_t. According to [32], defining the present short rate does not define the yield curve as a whole. However, if one models the evolution of r_t as a stochastic process under a risk-neutral measure Q, then under some fairly relaxed technical conditions, no-arbitrage arguments based on [27] no-arbitrage theory demonstrate that the price at time t of a zero-coupon bond maturing at time T with a payoff of 1 is given by:

$$P(t,T) = E^{Q}\left[\exp\left(-\int_t^T r_s\, ds\right) | \mathcal{F}_t\right]$$

Where $\mathcal{F}$ is the natural filtration for the process. Natural filtration is the process that records the "past behaviour" of a stochastic process at each time [22]. The interest rates indicated by zero-coupon bonds constitute a yield curve, or, more specifically, a zero curve. As a result, specifying a short-term rate model determines future bond prices. This indicates that instantaneous forward rates are also specified by the typical formula:

$$f(t,T) = -\frac{\partial}{\partial T} \ln(P(t,T))$$

Interest rate term structure model may be grouped into endogenous and exogenous models [36, 51]. In an endogenous term structure model, the current term structure of rates is an output rather than an input of the model. Examples of the endogenous short rate models covered in this research include the Vasicek model of 1977 [49] and the Cox, Ingersoll and Ross model of 1985 [24]. Exogenous models are, in turn, so the currently adopted term structure of rates becomes input and not an outcome from a model. Representatives of exogenous short rate models include those: Hull-White model of 1990 [30]; Black-Derman-Toy model of 1990 [12], as well as the Black-Karasinski model [10], this work would look into a part of all three these three kinds. These models were described not only for their historical importance but also for letting one treat them more clearly.

The interest rate model described by [49] stipulates the interest rate movements that are driven by only one source of market risk. It provides the expected destination of interest rates at the close of a defined period, assuming prevailing volatility of the market, the long-term mean value of interest rate, and a market risk factor. The model assumed that, under the real-world measure, the instantaneous spot rate follows an Ornstein-Uhlenbeck process with constant coefficients [13]. Indeed [49], was the first model to incorporate mean reversion, which is a typical property of interest rates and sets them apart from other financial prices.

Cox et al. [24] created a model that calculates interest rate movements as a function of current volatility, the mean rate, and spreads as a spinoff of the Vasicek model. The market risk component is also introduced by [24]. The model assumes mean reversion towards a long-term normal interest rate level and also introduces a square root factor that excludes negative rates. This is known as the CIR model, or Cox-Ingersoll-Ross model. As such, the CIR model is a one-factor equilibrium model that guarantees that the computed interest rates are always positive (non-negative) by using a "square-root" diffusion process. The Vasicek model is a one-factor modelling technique, just like the CIR model.

However, interest rates can become negative in the Vasicek model since it lacks a square root component.

This model, as developed by [29], assumes that short rates are normally distributed. Therefore, the Hull-White model assumes that the short rates are normally distributed and subject to mean reversion. It extends the Vasicek Model and part of the CIR model. [32] reported that volatility is likely to be low when the rates are around zero reflected in a bigger mean reversion in the Hull-White model.

The study [12] introduced the Black-Derman-Toy (BDT) model, a yield-based model applied for pricing bonds and interest-rate options. The BDT is one of the most popular and celebrated models within the fixed-income interest rate theory [35]. The underlying concept of a BDT model is to calculate a binomial tree of the short-term rates of interest which would be adequately flexible in nature to fit into the data sets. The aim of the BDT model, thus, shall be to obtain correct pricing bond options, swaptions, and other related interest rate options. It works together to identify the mean-reverting behaviour of short-term rate with a lognormal distribution [35]. Black-Karasinski model developed by [10] defines interest rate movements as conducted by one source of randomness.

Black-Karasinski model, interest rate fluctuations are caused by a single source of randomness [10]. The model is an extension of the BDT model, which was created in 1990, the year prior. In its most basic version, the model may fit the current values of a series of floors, caps, or European swaptions, as well as the prices of zero-coupon bonds. Approximately thirty years have passed since [10] introduced their own mean-reverting lognormal short rate model for interest rates as a substitute for Hull-White's normal model [30], which, while reasonably tractable, does not ensure positive rates [46]. The model has gained popularity among practitioners and financial engineers because to its relatively strong fitting quality to market data, particularly the swaption volatility surface [15].

The aforementioned short-rate models are known as one-factor short rate models. However, there are multiple-factor short rate models that fall outside the scope of this study. Multi-short rate models include two-factor models like the Longstaff-Schwartz model and the Chen model, which has three factors. Multiple-factor short rate models assume more than one stochastic element, whereas short rate models assume that the future development of interest rates is determined by only one stochastic factor [31, 13]. Although unrealistic, the short rate models offer reasonable estimates of the term structure of interest rates, because the numerous factors that impact interest rates are intimately interconnected.

2.2. Theoretical Review

There are several theories of interest rate that related to this research, viz. the real/classical theory of interest rate; Black-Scholes option-pricing theory [11]; no-arbitrage theory [27] and Black-76 theory [9]. However, the no-arbitrage theory and the Black-76 theory proved most useful to this research.

No-arbitrage theory is useful in this research since interest rate derivatives are priced using the no-arbitrage or arbitrage-free principle. To ensure that no trader may earn risk-free by purchasing one and selling the other, its price is fixed at the same level as the replicating portfolio's value. [4] claim that any arbitrage opportunities that arise will vanish because traders who take advantage of them would drive the derivative's price up until it matches the value of replicating portfolios. Finally, the Black-76 theory has much relevance to this research in a sense that it rests on several assumptions, among which the log-normal distribution of future prices and the expected change in the futures' price being zero. The fact that Black's updated model models the value of a futures option at maturity using forward prices rather than the spot prices Black-Scholes utilised is one significant distinction between their 1976 and 1973 models [40]. Additionally, it makes the assumption that volatility is time-dependent rather than constant. Additionally, recent forward market rate models are built to make sense with the interest rate caplet formula with its fixed maturity developed by [9].

2.3. Empirical Review

A lot of work has gone into creating models for interest rate claims. However, the theoretical developments in this field have not kept pace with the empirical assessment of these models, particularly in the market. Furthermore, whilst many extant studies have focused on one to three interest rate models, very few have compared several interest rate models with stochastic volatility for interest rate claims. Building models to establish interest rates as a crucial financial key and estimate has been the main economic emphasis in recent years. More specifically, a lot of work has gone into creating interest rate claim pricing models [6]. However, these models' empirical assessment has lagged, particularly in the LIBOR and swaption markets. Much of the literature on interest rate with stochastic volatility reviewed, focused on one to three interest rate models. Very few compared several interest rate models with stochastic volatility for interest rate claims. Therefore, despite the importance of analysing interest rates with stochastic volatility, there is still a wide divergence of opinion on how best to value these claims.

Of the previous studies reviewed, [43, 33, 17], did the most extensive comparison of different interest rate models with stochastic volatility. They carried out an empirical comparison over eight different interest rate models that were predominantly short term. [44, 45] equally carried out empirical comparisons of interest rate models but focused on three interest rate models each. [45], assessed the risk measures and behaviours for bonds under stochastic short interest rate models like Merton [39], Vasicek and CIR models. Whereas [44] assessed two short rate models and one market model viz. the CIR model, the BDT model and the forward-LIBOR market model. [28] also reviewed works of literature on four interest rate models viz. Vasicek, CIR, Hull-White and HJM models but, carried out an empirical analysis on the Nelson-Siegel model.

Some studies [3, 47] carried out analyses on the HJM model. They believed that the model is the best model that incorporated the stochastic volatility of bond prices not spanned by movements in the yield curve. However, using empirical evidence [23] was able to identify short rate models like the Vasicek and CIR models with mean-reversion that equally exhibited unspanned stochastic volatility. [20] also studied the HJM framework, but for pricing of long-dated commodity derivatives. [34, 17], assessed optimal portfolio selection under stochastic volatility and stochastic interest rates. Whilst [34] used the Vasicek model a general model for the interest rate, [18] used the CIR model. However, both of them got similar results.

In the bid to extend a short rate model with multifactor tendencies, [19, 37], extended the Hull-White [29] model. From their analyses, they developed a dual-curve short rate model with multi-factor stochastic volatility. Whilst [19] focus their research on methods for approximating the pricing formula, [37] measured the volatility with the SABR stochastic model. [38] averred that the SABR is a good feat for measuring the volatility of interest rate. [50] equally extends a model with multifactor tendencies, rather than use any short rate model, he used a market model, since market models are easier to calibrate. [50] analysed the Lévy LIBOR model, as it appears to be a coherent framework that best incorporates negative interest rates.

Even though several available research centred on building a model to establish interest rate claims, [26] carried out the study to conclude which of the different interest rate modelling approaches has the best estimate accuracy. Their finding gave insights on how best to estimate interest rates.

Much of the available literature on interest rates with stochastic volatility, focused on one to three interest rate models. Very few compared several interest rate models with stochastic volatility for interest rate claims. Even those who have empirically compared interest models like [43, 33, 17], only focused on short rate models empirically tested in the United States. More understanding on interest rate models would come to be should the short rate models be tested in Africa, typically with the Central Bank of Nigeria. This presented a geographical gap that this research seeks to cover.

2.4. Statement of Research Problem

Across the various categories of interest rates, over 20 different interest models have been developed. Even, several of the classical short-rate models have been extended to produce some sort of hybrid models to fit particular scenarios. Also, most of these interest rate models are still been used in contemporary times. However, relatively little is known regarding how well these models represent the fundamental behaviour of the short-term riskless rate. This is most likely due to the absence of a standard framework that would allow many models to be layered and their performance to be compared. Furthermore, it is challenging to evaluate comparable performance accurately without a common framework. This represents the thrust of this research.

This study examines how effectively a wide range of well-known models capture the stochastic behaviour of the short-term rate using a straightforward econometric framework. Our method takes use of the fact that several single-factor and multifactor term structure models suggest dynamics for the short-term riskless rate r that may be nested inside *equation* (1). The idea is to find a way some of the popular interest rate models with stochastic volatility can compare with each other. Even though each model differs fundamentally.

3. Methodology

The conditional variance in this section was modelled using the Autoregressive Conditional Heteroscedasticity (ARCH) and Generalised Autoregressive Conditional Heteroscedasticity (GARCH) models. [25] was the first to offer ARCH models. The ARCH processes contain constant unconditional variances but non-constant variances that depend on the past. The variance in these procedures is expressed as a linear function of the squared errors' recent historical values.

3.1. General Specifications of Short Rate Models

The class of continuous-time Markov processes, whose dynamics are controlled by an Ito process, is commonly used to describe the behaviour of the instantaneous interest rate. This model's features include the inclusion of both conditional variance and conditional mean variables. The following procedure is considered to be the generic stochastic differential equation of the interest rate:

$$dr_t = (\alpha + \beta r_t)dt + \sigma r_t^{\gamma} dW_t \tag{2}$$

Where r_t is the interest rate, dW_t is a standard Brownian motion, σ is a constant, and α, β, and γ are parameters. This model implies that the dynamics of interest rate changes are determined by its conditional mean and variance.

The short rate's dynamics under the risk-neutral measure have been modelled in a number of ways. These models outline a certain type of $\mu(t, r(t)$ and $\sigma(t, r(t)$. Most short rate models are either normally or lognormally distributed [13].

Remark 3.1.1. Short rate models that are normally distributed include Vasicek model and Hull-White model.

Remark 3.1.2. Short rate models that are lognormally distributed include: Black-Karasinki and Black-Derman-Toy.

Since the Cox-Ingersolt-Ross (CIR) model is a non-central Chi-squared distribution model with a regularly distributed (Gaussian) short rate as the most analytically tractable, it does not fall into any of these two categories [15].

3.2. Selected Short Rate Models

The table below represents the short rate models empirically analysed in this research.

Table 1. *Selected Short Rate Models.*

Model	**Drift $\mu(t, r(t)$**	**Diffusion $\sigma(t, r(t)$**
Vasicek	$\theta - \alpha r(t)$	σ
Hull-White	$\theta(t) - \alpha r(t)$	$\sigma(t)$
Black-Karasinki	$r(t)(\theta(t) - \alpha \ln r(t))$	$\sigma r(t)$
Black-Derman-Toy	$\theta(t) r(t)$	$\sigma(t) r(t)$
Cox-Ingersolt-Ross	$\alpha(\beta - r(t))$	$\sigma\sqrt{r(t)}$

3.3. Vasicek Model

Definition 3.3.1. (Short rate dynamics in the Vasicek Model) In the Vasicek model, the short rate according to [49] is assumed to satisfy the following stochastic differential equation (SDE):

$$dr_t = (\theta - \alpha r_t)dt + \sigma dW_t \quad (3)$$

Where $\theta, \alpha, \sigma > 0$, and W falls within the risk measure and is a typical Brownian motion. The Q-measure, often referred to as a risk-neutral measure, is a method of calculating probability in which the total of the anticipated future payouts, discounted at the risk-free rate, represents the current worth of a financial asset.

Vasicek used a mean-reverting drift to simulate the short rate, which is regarded as the first realistic model of the short rate. The drift is positive when r_t is below $\frac{\theta}{\alpha}$

The SDE for r(t) can be solved explicitly observe that:

$$d(e^{at}r(t)) = \theta e^{\alpha t}dt + \sigma e^{\alpha t}dW_t \quad (4)$$

Integrating from t to time $s \geq t$, and multiplying both sides of the equality by $e^{\{-\alpha s\}}$, we get:

$$r(s) = r(t)^{-\alpha(s-t)} + \theta\int_t^s e^{-\alpha(s-u)}du + \sigma\int_t^s e^{-\alpha(s-u)}dW(t) \quad (5)$$

The short rate is normally distributed under the risk-neutral measure Q, as shown by *equation* (5), with mean and variance dependent on $F(s)$

$$E(r(t)\,|\,F(s) = r(0)e^{-\alpha s} + \theta\int_0^s e^{-\alpha(s-u)}du$$

$$= r(0)e^{-\alpha s} + \frac{\theta(1-e^{-\alpha s})}{\alpha}$$

and variance given by the Isometry as

$$\mathrm{E}I^2(t) = \mathrm{E}\int_0^t \Delta^2(u)du \quad (6)$$

$$Var(r(s)) = \sigma^2\int_0^s e^{-2\alpha(s-u)}du$$

$$= \frac{\sigma^2(1-e^{-2\alpha s})}{2\alpha}$$

Remark 3.4.1. *Convergence:* For any time *t*, the short rate r_t has a positive probability of being negative. This is one of the Vasicek model's main shortcomings. Nonetheless, the Vasicek model's short rate is mean reverting, meaning that rates return to their historical mean level because

$$\mathrm{E}(r(s)) \to \frac{\theta}{\alpha} \text{ as } s \to \infty,$$

$$\mathrm{E}(r(s)) = r(0)e^{-\alpha s} + \theta\left(\frac{1-e^{-\alpha s}}{\alpha}\right) \to r(0)(0) + \frac{\theta}{\alpha}(1-0) = \frac{\theta}{\alpha} \text{ as } s \to \infty$$

3.4. Vasicek Bond Pricing Formula

Computing the integral of r(s) from t to T in the *equation* (6), we have

$$\int_t^T r(s)ds = r(t)\int_t^T e^{-\alpha(s-t)}ds + \theta\int_t^T\left(\int_t^s e^{-\alpha(T-u)}du\right)ds + \sigma\int_t^T\left(\int_t^s e^{-\alpha(T-u)}dW(t)\right)ds \quad (7)$$

Let us denote the integral in the first term on the right-hand side of *equation* (7) by

$$D(t,T) = \int_t^T e^{-\alpha(s-t)}ds = \frac{1-e^{-\alpha(T-t)}}{\alpha}$$

To compute the second and third terms observe that

$$d\left(\int_t^s e^{-\alpha(s-u)}du\right) = ds - \alpha\left(\int_t^s e^{-\alpha(s-u)}du\right)ds \quad (8)$$

$$d\left(\int_t^s e^{-\alpha(s-u)}dW(u)\right) = d\omega(s) - \alpha\left(\int_t^s e^{-\alpha(s-u)}dW(u\right)ds \quad (9)$$

Hence

$$\left(\int_t^s e^{-\alpha(s-u)}du\right)ds = d\left(\int_t^s \frac{1-e^{-\alpha(s-u)}}{\alpha}\right)du$$

$$= d\left(\int_t^s D(u,s)du\right)$$

Also,

$$\left(\int_t^s e^{-\alpha(s-u)}du\right)ds = d\left(\int_t^s \frac{1-e^{-\alpha(s-u)}}{\alpha}\right)dW(u)\,.$$

$$= d\left(\int_t^s D(u,s)dW(u)\right)$$

Integrating from t to T, we have

$$\int_t^T r(s)ds = r(t)D(t,T) + \theta\int_t^T D(u,T)du + \sigma\int_t^T D(u,T)dW(u) \quad (10)$$

It follows that

$$X = \theta\int_t^T D(u,T)du + \sigma\int_t^T D(u,T)dW(u) \quad (11)$$

is a random variable independent of F_t, normally distributed with mean

$$m = \theta\int_t^T D(u,T)du \quad (12)$$

and variance is given by the Ito Isometry as

$$\mathrm{E}I^2(t) = \mathrm{E}\int_0^t \Delta^2(u)du\,. \quad (13)$$

$$s^2 = \theta^2\int_t^T D(u,T)^2 du \quad (14)$$

Remark 3.5.1. Under the risk-neutral measure Q. The expectation of e^{-x} is $e^{-m+\frac{1}{2}s^2}$. Hence, using the Merton bond pricing formula by [39], we arrive at the result above.

3.5. Black-Karasinki Model (Exponential Vasicek)

Definition 3.5.1. In the Black-Karasinki model, the short rate is given by

$$r(t) = e^y(t) \Rightarrow y(t) = \ln r(t) \quad (15)$$

Therefore, the Black-Karasinki model is

$$d\ln r(t) = k(\theta - \ln r(t))dt + \sigma dW(t)\,,$$

where $k, \theta, \sigma > 0$ and W is a Brownian motion under the risk-neutral measure.

Theorem 3.6.1. According to [10], the short rate in the Black-Karasinki model satisfies the stochastic differential equation

$$dr(t) = (k\theta + \frac{\sigma^2}{2} - k\ln(r(t))dt) + \sigma r(t)dW(t) \quad (16)$$

let $0 \le s \le t \le T$ Then $r(t)$ is given by

and is conditionally on $F(s)$ lognormally distributed with

$$r(t) = \exp\left\{\ln(r(s))e^{-k(t-s)} + \theta(1-e^{-k(t-s)}) + \sigma\int_s^t e^{-k(t-u)dW(t)}\right\} \quad (17)$$

$$\mathrm{E}(r(t)\,|\,F_{(s)}) = \exp\{\ln\left(r(s)e^{-k(t-s)} + \theta(1-e-k(t-s))\right) + \frac{\sigma^2}{4k}(1-e^{-2k(t-s)}) \quad (18)$$

$$Var(r(t)\,|\,F_{(s)}) = \exp\left\{2\ln(r(s))e^{-k(t-s)} + 2\theta(1-e^{-k(t-s)})\right\}\times \exp\left\{\frac{\sigma^2}{2k}(1-e^{-2k(t-s)})\right\}\left[\exp\left\{\frac{\sigma^2}{2k}\left(1-e^{-2k(t-s)}\right)\right\}\right] \quad (19)$$

and

Remark 3.5.1. The Black-Karasinki model's short rate r_t is always positive since it is lognormally distributed. The inability of $P(t, T)$ to be clearly estimated is a drawback.

Remark 3.5.2. Convergence Criterion: An advantage of the Black- Karasinki model is that r is always mean-reverting provided

$$\psi^* = \lim_{t\to\infty}\left\{k\int_0^t \theta(u)e^{-k(t-u)}du\right\} \quad (20)$$

exist, and then

$$\mathrm{E}(r(t)\,|\,F_{(s)}) \to \exp(\psi^* + \frac{\sigma^2}{4k}) \text{ as } t \to \infty$$

and

$$Var(r(t)\,|\,F(s)) \to \exp\left(2\psi^* + \frac{\sigma^2}{2k}\right)\left[\exp(\frac{\sigma^2}{4k}) - 1\right] \text{ as }$$
$$t \to \infty$$

3.6. Hull-White Model (Extended Vasicek)

A time-varying parameter is required in order to replicate the original zero-coupon curve precisely [32]. The purpose of this parameter is to precisely mirror the original word structure. The Hull-White model [29] is arguably the most widely used model with time-dependent parameters, with the values corresponding to θ and σ that shows in the Vasicek model are selected to be functions of time that are deterministic;

$$dr(t) = (\theta(t) - \alpha r(t))dt + \sigma(t)dW(t) \tag{21}$$

where α is constant and W_t is a Brownian motion under the risk-neutral measure Q.

Integrating *equation*(3.19) from t to $s \geq t$, we have

$$r(s) = r(t)e^{-\alpha(s-t)} + \int_t^s \theta(u)e^{-\alpha(s-u)}du + \int_t^s \sigma(u)e^{-\alpha(s-u)}dW(u) \tag{22}$$

$r(t)$ is conditionally on $F(s)$ normally distributed with

$$\mathrm{E}(r(t)\,|\,F(s)) = r(t)e^{-\alpha(s-t)} + \alpha\int_t^s \theta(u)e^{-\alpha(s-u)}du \tag{23}$$

and

$$Var(r(t)\,|\,F(s)) = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(s-t)}\right) \tag{24}$$

Remark 3.6.1. As in the Vasicek model, the short rate $r(t)$ in the extended Vasicek model, for each time t, can be negative with positive probability, namely with probability

$$\phi\left(-\frac{r(o)e^t + \alpha\int_0^t \theta(u)e^{-}\alpha(s-t)du}{\sqrt{\frac{\sigma 2}{2\sigma}(1-e^{-2\alpha t})}}\right) \tag{25}$$

which is often negligible in practice.

Remark 3.6.2. Convergence Criterion: The short rate in the Hull-White model is mean-reverting provided.

$$\psi^* = \lim_{t\to\infty}\left\{\alpha\int_0^s \theta(u)e^{-\alpha(s-u)}du\right\} \tag{26}$$

exist, and then

$$\mathrm{E}(r(t)) \to \psi^* \text{ as } t \to \infty$$

3.7. Hull-White Zero Coupon Bond Price

Adopting an approval analogous to that in the Vasicek model, we can derive an analytic expression for the Zero-Coupon bond price using the risk-neutral pricing formula from the Merton model [14]. This yields

$$B(t,T) = \exp\left(-r(t)D(t,T) - \int_t^T \sigma(u)^2 D(u,T)^2 du\right) \tag{27}$$

Where the time-dependent parameter $\theta(t)$ can be chosen to match the current term structure. From *equation* (26) it can be seen that what is really needed is an expression for the integral $\int_t^T \sigma(u)D(u,T)du - \int_0^T \sigma(u)D(u,T)du$ rather than $\theta(t)$ itself, Hence, we take

$$\begin{aligned}\frac{\ln B(0,T)}{B(0.,T)} &= -r(0)(D(0,r) - D(0,t)) - \int_t^T \theta(u)D(u,T)du - \\ &\int_0^t \theta(u)(D(u,T) - D(u,T))du + \frac{1}{2}\int_0^t \sigma(u)^2 D(u,t)^2 du + \\ &\frac{1}{2}\int_0^t \sigma(u)^2 (D(u,T)^2 - D(u,t)^2)du\end{aligned} \tag{28}$$

The integrals from 0 to t can be rewritten by using the relation

$$D(u,T) = D(u,t) + D_t(u,t)D(t,T) \tag{29}$$

where $D_t(u,t)$ is the partial derivative of $D(u,t)$ with respect to t, to yield

$$\frac{\ln B(0,T)}{B(0.,T)} = -r(0)D_t(0,r)D(0,t) - \int_t^T \theta(u)D(u,T)du - D(t,T)\int_0^t \theta(u)D_t(u,T)du + \frac{1}{2}\int_0^t \sigma(u)^2 D(u,t)^2 du + D(t,T)\int_0^t \sigma(u)^2 D(u,T)D_t(u,t)du + \frac{1}{2}\theta(t,T)^2\int_0^t \sigma^2(u)(D(u,T)^2 - D(u,t)^2)du \quad (30)$$

From *equation* (27), the formula for the instantaneous forward rate. We then obtain

$$f(0,t) = r(0)D_t(0,t) + \int_0^t \theta(u)D_t(u,t)du - \int_0^t \sigma^2(u)D(u,t)D_t(u,t) \quad (31)$$

It follows that

$$\ln\frac{B(0,T)}{B(0.,T)} = -f(0,t)D(t,T) - \int_t^T \theta(u)D_t(u,T)du + \frac{1}{2}\int_t^T \sigma(u)^2 D(u,t)^2 du + \frac{1}{2}D(t,T)^2\int_0^t \sigma(u)^2 D_t(u,t)^2 du \quad (32)$$

Equation (30) gives the desired expression for $\int_t^T \theta(u)D(u,T)du$ in terms of $B(0,t), B(0,T)$ and $f(o,t)$ that is in terms of the current term structure. Substituting this expression into *equation* (28), we get the following result.

Proposition 3.8.1. In the Hull white model, the zero bond price at time $t \geq 0$ that gives an exact fit to the term structure of interest rates at time 0 is

$$B(t,T) = \frac{B(0,T)}{B(0,t)}\exp\left(-(r(t) - f(0,t))D(t,T) - \frac{1}{2}D(t,T)^2\int_0^t \sigma(u)^2 D_t(u,t)^2 du\right) \quad (33)$$

where $D(t,T)$ is given by *equation* (8).

3.8. Black-Derman-Toy (BDT) Model

The Black-Derman-Toy Model (BDT) is a well-known short rate model in mathematical finance that is used to price bond options, swaptions, and other interest rate derivatives [35, 40]. The short-rate's mean-reverting characteristic was initially combined with a log-normal distribution in the model. The short rate is assumed in the BDT model satisfies the stochastic differential equation;

$$d\ln(r) = \left[\theta_t + \frac{\sigma_t^1}{\sigma_t}\ln(r)\right]dt + \sigma_t dW_t \quad (34)$$

where,

r = the instantaneous short rate at time t

θ_t = value of the underlying asset at option expiry

σ_t = instant short rate volatility

W_t = a standard Brownian motion under a risk-neutral probability measure.

For constant (time-independent) short rate volatility σ, the model is

$$d\ln(r) = \theta_t dt + \sigma dW_t \quad (35)$$

3.9. Data Description

A one-month interest rate is used as a stand-in for the short interest rate. A range of one-month rates were employed:

Federal Government of Nigeria bonds (FGN): Weekly observations of a one-month data stream from January 2011 to December 2020 for Federal Government of Nigeria bonds (FGN), yielding 522 observations for each series.

Open Market Operation (OMO): Weekly observations totalling 1079 observations from January 2005 to September 2020.

Nigeria Treasury Bill yields (NTB): 307 observations are included in the sample period, which runs from January 2020 to December 2020.

3.10. ARCH, GARCH and Discretisation Methods

As earlier mentioned, we used the ARCH and Generalised ARCH (GARCH) models to model the conditional variance. It was [25] who first introduced the ARCH models used in research today. There are constant unconditional variances in the ARCH processes, but non-constant variations that depend on the past. A linear function of the recent historical values of the squared errors is used to characterise the variance in these processes.

3.10.1. Discretisation

To draw inferences about the time series data, we have to rely on 3 discrete realisations. Thus, the sample period [0, T] are divided into 3 intervals corresponding to the discrete-time data of Central Bank of Nigeria (CBN) securities [FBN Bond, OMO, NTB]. Then, the continuous-time process is replaced with a piecewise constant process and in each interval $[t_i, t_{i+1})$ $i = 1, 2, \cdots, N-1$. It is assumed that the process is constant but from one interval to the next it is changing.

Remark 3.10.1. We adopted the Euler Scheme for the discretisation of our data.

3.10.2. ARCH Model

Consider the process

$$r_t = \phi_0 + \phi_1 r_{t-1} + \varepsilon_t, |\phi_1| < 1 \tag{36}$$

with the error term

$$\varepsilon_t = u_t \sigma_t \tag{37}$$

where u_r has a standard normal distribution, the conditional variance of ε_t given ε_{t-1}

$$Var(\varepsilon_t \mid \varepsilon_{t-1}) = Var(u_t \sigma_t) = \sigma_t^2 Var(u_t) = \sigma_t^2 \tag{38}$$

the autocorrelation in volatility is modelled by allowing the conditional variance of the error term σ_t^2 to depend on the immediately previous value of the squared error.

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 \tag{39}$$

Remark 3.10.2. As α_0 and α_1 are constant, the above model is known as an ARCH(1), since the conditional variance depends only on one lagged squared error. The generalisation of this model can be gotten by including more lags of ε_t.

Thus ARCH(q) model is

$$\begin{aligned} \sigma_t^2 &= \alpha_0 + \alpha_1 \varepsilon_{t-r}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2 \\ &= \alpha_0 + \sum_{i=1}^{2} \alpha_i \varepsilon_{t-i}^2 \end{aligned} \tag{40}$$

The unconditional variance is denoted σ^2 and defined as

$$E(\sigma_t^2) = E(\alpha_0 + \alpha_0 E(\varepsilon_{t-1}^2) + \cdots + \alpha_q E(\varepsilon_{t-q}^2) \tag{41}$$

using $E(\varepsilon_{t-1}^2) = \sigma_{t-i}^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 + \cdots + \alpha_q \sigma_{t-q}^2$ $\Rightarrow \sigma^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 + \cdots + \alpha_q \sigma^2$

to arrive at

$$\sigma^2 = \frac{\alpha_0}{1 - \sum_{i=1}^{q} \alpha_i} \tag{42}$$

Stationarity of the ARCH(q) models imposes condition on the α_i coefficients. ARCH(1) model has stationary moments of order 2 and φ, if $3\alpha^2 < 1$, and these moments are

$$E(\varepsilon_t^2) = \frac{\alpha_0}{1 - \alpha_1} \tag{43}$$

$$E(\varepsilon_t^{\varphi}) = \frac{3\alpha_0^2}{(1-\alpha_1)^2} \frac{1-\alpha_1^2}{1-3\alpha_1^2} \tag{44}$$

$$k = \frac{E(\varepsilon_t^{\varphi})}{(E(\varepsilon_t^{\varphi}))^2} = \frac{3(1-\alpha_1^2)}{1-3\alpha_1^2} > 3 \tag{45}$$

where $k = 3$ for the normal distribution; $k = 3$ indicates left kurtosis.

3.10.3. Generalised ARCH (GARCH) Model

GARCH models were created separately as an extension of ARCH models in order to produce more adaptable outcomes. The conditional variance equation is expressed as follows because the GARCH models let the conditional variance to depend on valuable own lags, such that:

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{46}$$

This model is known as GARCH(1, 1) and may be expanded to a GARCH(*p*, *q*) formulation, where the present conditional variance is parameterised to rely on *q* lags of the squared error and *p* lags of the conditional variance

$$\sigma^2 = \frac{\alpha_0}{1 - \sum_{i=1}^{q} \alpha_i - \sum_{i=1}^{p} \beta_j} \tag{47}$$

3.10.4. Parameter Estimation Using Method of Moments

Consider the model

$$r_t = \phi + \beta r_{t-1} + \phi_t Z_t \tag{48}$$

$$\sigma_{t-1} = x + \alpha\sigma_{t-2} + \varepsilon_{t-1} \tag{49}$$

From *equation*(3.48) we get

$$\sigma_{t-1} Z_t = r_t - \alpha - \beta r_{t-1} \tag{50}$$

Taking the expectations of both sides, we have

$$E(\sigma_{t-1} Z_t) = E(r_t - \alpha - \beta r_{t-1}) \tag{51}$$

Since any expectation can be written in the form of conditional expectation and σ_{t-1} is F_{t-1} measurable, then

$$E(\sigma_{t-1} Z_t) = E[E(\sigma_{t-1} Z_t) \mid F_{t-1}] = E[\sigma_{t-1} E(Z_t \mid F_{t-1})] = 0 \tag{52}$$

Thus,

$$E(r_t - \alpha - \beta r_{t-1}) = 0 \tag{53}$$

If we take the square of both sides in *equation* (51) and then take the expectation of both sides we get

$$E(\sigma_{t-1}^2 Z_t^2) = E[(r_t - \alpha - \beta r_{t-1})^2] \tag{54}$$

since

$$E(\sigma_{t-1}^2 Z_t^2) = E[E(\sigma_{t-1}^2 Z_t^2) \mid F_{t-1}] = E[\sigma_{t-1}^2 E(Z_t^2 \mid F_{t-1})] = E(\sigma_{t-1}^2) \tag{55}$$

then,

$$E[(r_t - \alpha - \beta r_{t-1})^2] = E(\sigma_{t-1}^2) \tag{56}$$

Remark 3.10.3. Let Z be a random variable with standard normal distribution, i.e. $Z \sim N(0,1)$, then

$$E(Z^+) = \begin{cases} (t-1)(t-3) & \text{for } t \text{ even} \\ 0 & \text{for } t \text{ odd} \end{cases} \tag{57}$$

Using the above remark, and taking expectations of *equation* (51) we get

$$E[(r_{(t)} - \alpha - \beta r_{t-1})^3] = 0 \tag{58}$$

$$E[(r_{(t)} - \alpha - \beta r_{t-1})^4] = 3E(\sigma_{t-1}^4) \tag{59}$$

$E(\sigma^2)$ and $E(\sigma^4)$ are the second and fourth moments of σ respectively. Thus, we can calculate these moments using the moment generating function for σ. σ has the normal distribution with mean $M = \dfrac{x}{1-\alpha}$ and variance $\sum^2 = \dfrac{\varepsilon^2}{1-\alpha^2}$. Therefore the moment generating function for α is

$$v(t) = exp\left\{ Mt + \frac{\varepsilon^2 t^2}{2} \right\} \tag{60}$$

From this function, we can find that

$$E(\sigma^2) = v^{(2)}(0) = M^2 + \varepsilon^2 \tag{61}$$

and

$$E(\sigma^4) = v^{(4)}(0) = M^4 + 3\varepsilon^4 + 6\varepsilon^2 M^2 \tag{62}$$

Thus we have the following system of equations

$$\begin{aligned} &E[(r_{(t)} - \alpha - \beta r_{t-1})] = 0 \\ &E[(r_{(t)} - \alpha - \beta r_{t-1})^2] = M^2 + \varepsilon^2 \\ &E[(r_{(t)} - \alpha - \beta r_{t-1})^3] = 0 \\ &E[(r_{(t)} - \alpha - \beta r_{t-1})^4] = 3(M^4 + 3\varepsilon^4 + 6\varepsilon^2 M^2) \end{aligned} \tag{63}$$

Remark 3.10.4. The solutions of the system of *equation* (63) are the estimates of our model's parameters. The MATLAB function "fmincon" was used in Chapter Four to estimate the parameters.

3.10.5. Data Analysis Procedure

The following steps were adopted in the analysis of our data.

Step 1: We apply Ljung-Box Q-statistic (LBQ) and ARCH Test to check whether the data have autocorrelation or not.

Step 2: We use *equation* (63) to estimate the parameters of the short rate models (α, β, the mean M, and variance ε).

Step 3: ARCH(1) and GARCH(1,1) models are established.

Step 4: Finally, the out of sample performance of our stochastic volatility model—against ARCH and GARCH, the models were tested.

4. Results and Discussion

4.1. Autocorrelation Test

First, we determine if the data has autocorrelation or not. For this, we used the Ljung-Box Q-statistic, and the results are shown in this section:

The table below displays the short rate models empirically examined in this study.

Table 2. Ljung-Box Q-statistics.

	H	P-value	Qstat	Critical value
LBQ test	1	0	L4400e + 004	31.4104

In this test, the null hypothesis is H_0, which means there is no autocorrelation. Since the results gave $H = 1$, we reject the null hypothesis.

We also performed the ARCH Test to the data, and the results are reported as follows:

Table 3. ARCH Test.

	H	P-value	Qstat	Critical value
ARCH Test	1	0	775.1894	3.8415

Since the results give $H = 1$, then we reject the null hypothesis (H_0 = there is no ARCH effect) of the ARCH Test.

Remark 4.1.1. From the above tests, we conclude that the data have autocorrelation, so it is convenient to work on.

4.2. Results

The findings of applying the ARCH and GARCH models to our data are shown in this section. We provide the Bayesian Information Criteria (BIC) and Akaike Information Criteria (AIC) values for a few ARCH processes in the table that follows. AIC and BIC are mathematical techniques for assessing how well a model fits the data from which it was created [8].

Table 4. AIC and BIC for ARCH models.

	ARCH(1)	ARCH(2)	ARCH(3)	ARCH(4)
AIC	5187.8	5188.6	5190.5	5192.5
BIC	5201.7	5207.2	5213.8	5223.2

Both AIC and BIC were employed in the study to choose the model order. Models that minimise the criterion are preferable when employing either AIC or BIC. For our data, ARCH(1) and ARCH models are supported by AIC and BIC. The following table lists the AIC and BIC values for a few GARCH models. For our data, AIC and BIC support GARCH(1,1) among GARCH models.

Table 5. AIC and BIC for GARCH models.

	G(1,1)	G(2,1)	G(1,2)	G(2,2)	G(1,3)	G(2,3)	G(3,3)
AIC	5188.5	5190.5	5190.5	5192.2	5192.5	5194.4	5196.5
BIC	5207.2	5213.9	5213.9	5220.5	5220.5	5227.2	5233.8

We can determine that ARCH(1) is the best model for our data by comparing the AIC and BIC values for the ARCH and GARCH models. Using the ARCH(1) and GARCH(1,1) models, we can obtain the following parameters for our data.

Table 6. Parameter value for ARCH(1) and GARCH(1,1) models.

	α_0	α_1	β_1
ARCH(1)	0.2657	1	
GARCH(1,1)	0.2107	0.8948	0.1052

So the ARCH(1) model is

$$\sigma_t^2 = 0.2657 + \varepsilon_{t-1}^2$$

and the GARCH(1,1) model is

$$\sigma_t^2 = 0.2107 + 0.8948\varepsilon_{t-1}^2 + 0.1052\sigma_{t-1}^2$$

4.3. Testing the Performance of Our Model

The results of our stochastic volatility model's sample performance were compared to the ARCH and GARCH models in this section. We make advantage of the models' 10-day volatility estimates. The next table displays the model predictions as well as the observed volatility (we used innovations as the volatility):

Table 7. *Volatility Forecast.*

	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
SV	18.9445	18.8877	18.8311	18.7747	18.7185	18.6625
ARCH(1)	18.9445	18.9515	18.9585	18.9655	18.9725	18.9795
GARCH(1,1)	18.9445	18.9753	18.9808	18.9863	18.9918	18.9973
OBSERVED	18.9445	18.7956	18.1177	18.2731	18.4042	18.5750
	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$	
SV	18.6066	18.5510	18.4956	18.4403	18.3853	
ARCH(1)	18.9865	18.9935	19.0005	19.0075	19.0145	
GARCH(1,1)	19.0028	19.0083	19.0138	19.0193	19.0248	
OBSERVED	18.7002	18.8284	18.8387	18.8492	18.6372	

The sum of squares of the deviations from the observed volatility for each model's forecasts are as follows:

Table 8. *Sum of Squared Deviations.*

	SV	**ARCH**	**GARCH(1,1)**
$\sum_{t=1}^{10}(\sigma_{\text{observed}}-\sigma_{\text{model}})^2$	1.3095	2.0001	2.1433

Remark 4.3.1. Findings from our results show that the stochastic volatility model gives closer predictions to the observed volatility than ARCH and GARCH models for our data in the studied time interval.

5. Conclusion and Recommendation

5.1. Conclusion

In this work, we examined the stochastic volatility of short rate models. We introduced ARCH processes and presented several parameters estimate methods. Using chosen short rate models, we built a stochastic volatility model and then worked on it. We utilised the moment approach to estimate the parameters. In the application section, we used interest rate data from FGN Bond, NTB, and OMO. We determined the parameters for our stochastic volatility model as well as the ARCH and GARCH models. Then we evaluated our model's out-of-sample performance against the ARCH and GARCH models. The findings demonstrated that in the tested time frame, our stochastic volatility model provides closer forecasts of the stock market than ARCH and GARCH models.

5.2. Recommendation

From the results of the ARCH, GARCH and the observed stochastic volatility, the use of observed volatility over ARCH and GARCH models is recommended for forecasting and analysis of interest rate models.

5.3. Contribution to Knowledge

Findings from the study show that higher volatility indicates a larger risk of a declining market, and lower volatility indicates a higher probability of a rising market. Investors can use this information on long-term stock market volatility to align their portfolios with the predicted returns. Furthermore, by allowing prices to fluctuate, stochastic volatility models enhanced the accuracy of calculations and forecasts.

Abbreviations

ARCH	Autoregressive Conditional Heteroskedasticity
BDT	Black-Derman-Toy
CBN	Central Bank of Nigeria
CIR	Cox-Ingersoll-Ross
GARCH	Generalised Autoregressive Conditional Heteroskedasticity
HJM	Heath-Jarrow-Morton
LIBOR	London Interbank Offering Rate
SABR	Stochastic Alpha Beta Rho
Q-statistic	Ljung-Box Q-statistic

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Bernard Ojonugwa Anthony: Funding acquisition, Investigation, Project administration, Resources, Validation, Writing – review & editing.

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Conflicts of Interest

The authors declare no conflicts of interest.

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