
Existence Results for ψ -Caputo Fractional Integro-Differential Equations with Strip Boundary Conditions

Karthikeyan Panjaiyan*, Manikandan Ambigapathi

Department of Mathematics, Sri Vasavi College, Erode, India

Email address:

srimanierode@gmail.com (Karthikeyan Panjaiyan), pkarthisvc@gmail.com (Manikandan Ambigapathi)

To cite this article:

Karthikeyan Panjaiyan, Manikandan Ambigapathi. (2026). Existence Results for ψ -Caputo Fractional Integro-Differential Equations with Strip Boundary Conditions. *Applied and Computational Mathematics*, 15(5), 185-197. <https://doi.org/10.11648/j.iotcc.20261505.14>

Received: 28 May 2026; **Accepted:** 27 June 2026 **Published:** 13 October 2026

Abstract: In this paper, we discuss the existence and uniqueness of solutions for nonlinear fractional integro-differential equations with the ψ -Caputo fractional derivative under slit-strip integral boundary conditions. Fractional differential equations have been the focus of intense interest, since they can express memory and hereditary properties of various materials and processes in science and engineering. The ψ -Caputo fractional derivative which extends some known fractional operators is a very fruitful instrument for modeling multidimensional dynamical systems and thus applicable to a wide variety of problems. The everything aim of current paper is to get certain sufficient conditions that ensures the multi- solvability as well as uniqueness of the boundary value problem. The study is conducted on the pre-determined space which is a Banach space by using nonlinear functional analysis. In particular, suitable estimates for the nonlinearities involved in the equation are obtained by Hölders inequality and the Mönch fixed-point theorem is the main tool for proving existence of solution. In addition, further conditions are required for ensuring the uniqueness of the obtained solution. The present work contributes to the theory of fractional integro-differential equations with nonlocal boundary conditions and generalizes and improves many results in the literature. The addition of slit-strip integral boundary conditions and the ψ -Caputo fractional derivative results in a more generalized- setting for the study of such types of problems. Finally, a numerical example is provided to illustrate the feasibility and effectiveness of the theoretical results and to check the correctness of the theoretical results.

Keywords: Fractional Differential Equations, ψ -Caputo Fractional Derivative, Fractional ψ -integral, Boundary Value Problem, Monch's Fixed Point Theorem

1. Introduction

In recent years, fractional calculus has emerged as a commanding and effective tool in various fields of engineering and science. Fractional differential equations have been broadly applied to model numerous real-world phenomena in mathematics, chemistry, biochemistry, biomedical sciences, economics, material science, signal processing, electromagnetic theory, control theory, earthquake modeling, finance, and engineering applications (see [1, 2, 5-8, 13, 16-18]). In specific, fractional calculus plays a significant role in the medical field, especially in cancer treatment and in the pharmaceutical manufacturing process for developing effective medicines and maintaining appropriate dosage levels.

Moreover, according to complex real-life system fractional-order models are often used to find the accuracy instead of using classical integer-order models.

Fractional calculus has been developing since its origin in 1695, during which many definitions of fractional integrals and derivatives have been proposed. As research progressed, new fractional operators with different kernel functions were introduced, resulting in a broad range of definitions. Although several formulations of fractional integrals and derivatives exist, the Riemann-Liouville(RL), Caputo, Hilfer fractional derivatives are among the most prevalent in the literature.

Nonlinear fractional boundary value problems play a major role in diverse fields such as economics, financial

mathematics and other areas of applied science. Recently, considerable attention has been given to the investigation of the existence results are obtained for a nonlocal-boundary conditions associated with a ψ -Caputo fractional integro-differential equation (ψ -CFIDE). These studies typically employ techniques from nonlinear analysis, including the Banach fixed point theorem, Leray-Schauder principle, and similar approaches cited in (see [1, 2, 5, 6, 9, 10, 13, 16-18]).

The measure of non compactness (MNC) quantifies how far a bounded set in a metric space is from being compact, assigning a value of zero to sets that are relatively compact. Introduced by Kuratowski in 1930, this concept has become a fundamental tool in nonlinear analysis.

Among the commonly used measures are the Kuratowski and Hausdorff measures, with the latter applied extensively in various branches of nonlinear analysis. In this work, measures of non compactness are employed as a central method, which has contributed significantly to the study and development of different types of integral equations. For additional examples and applications [1, 6, 12].

The Caputo fractional derivative (CFD) is a fundamental operator in fractional calculus and is widely applied in the physical and engineering phenomena that involve memory effects. This derivative was introduced by Michele Caputo in 1967 in the context of geophysical studies on dissipation processes, with the objective of extending classical calculus while retaining standard integer-order initial conditions. Such a feature makes the Caputo derivative particularly suitable for real-world applications and provides it with a clear physical interpretation. Building on this classical framework, the ψ -Caputo fractional derivative was later proposed by Ricardo Almeida in 2017 (see [3]) as an extension of the Caputo derivative, in which the differentiation is performed with respect to an auxiliary increasing function ψ . This extension allows the fractional operator to capture dynamics on non-uniform or nonlinear time scales, thereby offering greater flexibility in modeling complex systems. While the Caputo derivative depends on the conventional time variable and employs a standard power-law kernel, the ψ -Caputo derivative modifies this kernel through the function ψ , leading to a broader and more adaptable formulation. Notably, the classical Caputo derivative can be recovered as a special case of the ψ -Caputo derivative when the function $\psi(t) = t$ indicating that the ψ -Caputo approach serves as a natural and comprehensive generalization rather than a replacement.

In [3], Almeida et al. investigated models for population growth and gross domestic product using the ψ -Caputo derivative, proving existence and uniqueness results through a fixed-point theorem. Their method is highly versatile, making it well-suited for analyzing and developing complex dynamical systems that exhibit non-local properties and memory effects. Compared to the classical Caputo derivative, the ψ -Caputo derivative provides greater flexibility and improved accuracy, especially for higher orders. This derivative has attracted significant attention across many scientific and engineering domains.

Karthikeyan et al. [15] discussed the existence and

uniqueness of fractional integro differential equations with boundary value conditions of the form

$${}^C D_{a+}^{\alpha} v(t) = f(t, v(t), (Rv)(t)), t \in J = [0, 1], \alpha \in (0, 1), \quad (1)$$

$$a\delta(0) + b\delta(h) = c, \quad (2)$$

where ${}^C D_{a+}^{\alpha}$ is the Caputo fractional derivative of order α and a, b, c are real constant and R is a nonlinear integral operator given by $(Rv)(t) = \int_0^t h(\sigma, \wp) \delta(\wp) d\wp$, where $h \in \mathbb{C}(\mathbb{J} \times \mathbb{J}, \mathbb{R}^+)$.

Sun et al. [22] studied the existence and uniqueness result for boundary value problem associated with a Caputo fractional derivative of the form

$${}^C D_{a+}^{\alpha} v(t) = f(t, v(t)), t \in [a, b] \quad (3)$$

$$v^k(a) = v_k, v^{n-1}(b) = y_b, \quad (4)$$

where ${}^C D_{a+}^{\alpha}$ is the Caputo fractional derivative of order $n-1 < \alpha \leq n$, f is the continuous functions and $v_0, v_1, v_3, \dots, v_{n-2}, v_b$ are the real constants.

Almeida et al. [4] analyzed the existence and uniqueness solution of ψ -Caputo fractional differential equation with boundary condition.

$${}^C D_{a+}^{\alpha; \psi} v(t) = f(t, v(t)), t \in [a, b], \quad (5)$$

$$v(a) = v_a, v_{\psi}^k(a) = v_a^k, k = 1, 2, 3, \dots, n-1, \quad (6)$$

where ${}^C D_{a+}^{\alpha; \psi}$ is the ψ -Caputo fractional derivative of order $n-1 < \alpha < n$, $v_a, v_a^k \in \mathbb{R}$, $v \in C^{n-1}[a, b]$.

Boutiara et al. [9] presented the existence and uniqueness of Implicit ψ -Caputo fractional differential equation with boundary condition.

$${}^C D_{a+}^{\alpha; \psi} u(t) = f(t, u(t), {}^C D_{a+}^{\alpha; \psi} u(t)), t \in J; = [0, T], \quad (7)$$

$$u(T) = \lambda u(\eta), \quad (8)$$

where ${}^C D_{a+}^{\alpha; \psi}$ is the ψ -Caputo fractional derivative of order α , $0 < \alpha \leq 1$, f is the continuous function. λ is a real constant and $\eta \in (a, T)$.

Inspired by the above works, we consider the ψ -Caputo fractional integro-differential equation with two boundary condition of the form

$${}^C D_{a+}^{\alpha; \psi} x(t) = f(t, x(t), (Rx)(t)), 1 < \alpha \leq 2, t \in J; = [0, 1], \quad (9)$$

$$x(0) = h(x), \quad (10)$$

$$x(\eta) = a \int_0^{\beta} x(p) dp + b \int_{\gamma}^1 x(p) dp, 0 < \beta < \eta < \gamma < 1 \quad (11)$$

where ${}^C D_{a+}^{\alpha; \psi}$ is the Caputo fractional derivative of order α , $f : J \times X \times X \rightarrow X$, $h(x)$ are continuous functions and a, b are real constants and R is a nonlinear integral operator given by $(Rx)(t) = \int_0^t k(t, s, x(s)) ds$, where $k \in C(J \times J \times X, X)$.

The integral boundary condition given in equation (1.1) can be interpreted as representing the combined influence of several finite strips with different lengths. The total effect of these strips is assumed to be proportional to the value of the unknown function at a chosen point inside the slit, which lies outside the two strips. This shows that the condition is nonlocal, since the value at one point depends on the function values at other locations.

The nonlocal term $h(x)$ in equation (1.1) can be expressed as a weighted sum of the function evaluated at specific

points: $h(x) = \kappa_1 x(t_1) + \kappa_2 x(t_2) + \dots + \kappa_p x(t_p)$ where $\kappa_1, \kappa_2, \dots, \kappa_n$ are given constants, and the points satisfy $0 < t_1 < t_2 < \dots < t_p \leq 1$. Such boundary conditions arise in situations where the state at a particular position is determined by the cumulative effect of the function values at several other points in the domain

This paper has the following structure: Section 2 provides, the basic definitions and fundamental results. Section 3, we study the existence result by using the MÃ¶nch fixed-point theorem via Hausdorff's measure of noncompactness. In Section 4, we establish the second result by another Hausdorff's measure of noncompactness. The application of our theoretical results is given in section 5.

2. Preliminaries

Notation, definitions, and preliminary results are given in this section.

Let $C(J, R)$ the continuous functions with the norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in J\}.$$

Let $L^1(J, R)$ be the Banach space of Lebesgue integrable functions $u : J \rightarrow R$, equipped with the norm

$$\|x\|_{L^1} = \int_J |x(t)| dt.$$

Definition 1. [23] The Riemann-Liouville fractional integral of order $\alpha > 0$ and $\alpha \in R$ of a the function x is defined by

$$I_{a+}^\alpha x(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds. \quad (12)$$

Definition 2. [23] Let $x:(0, \infty) \rightarrow R$ be the Caputo fractional, then the Riemann-Liouville fractional derivative of order $\alpha > 0$, $n = [\alpha] + 1$, $[\alpha]$ denote the integer part of α and x defined by

$$(D_{a+}^\alpha x)(t) = \left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} x(s) ds. \quad (13)$$

Definition 3. [23] Let $x:(0, \infty) \rightarrow R$ be the Caputo fractional, then the Caputo fractional derivative of order $\alpha > 0$, $n = [\alpha] + 1$, $[\alpha]$ denote the integer part of α and x defined by

$$(D_{a+}^\alpha x)(t) = \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} x^{(n)}(s) ds. \quad (14)$$

Definition 4. [26] The left-sided ψ -Riemann-Liouville fractional integral of order $\alpha \geq 0$ for the integral function defined as

$$I_{a+}^{\alpha;\psi} x(t) = \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} x(s) ds. \quad (15)$$

Definition 5. [9] The left-sided ψ Riemann-Liouville fractional derivative of order $\alpha \geq 0$, $n = [\alpha] + 1$ is defined

by

$$\begin{aligned} D_{a+}^{\alpha;\psi} x(t) &= \beta_\psi^{(n)} I_{a+}^{n-\alpha;\psi} x(t), \\ &= \beta_\psi^{(n)} \int_a^t \psi'(s) \frac{(\psi(t) - \psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} x(s) ds, \end{aligned} \quad (16)$$

where $\beta_\psi^{(n)} = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n$.

Definition 6. [6] The left-sided ψ -Caputo fractional derivative of x of order α , $n = [\alpha] + 1$ for $\alpha \notin N$, $n = \alpha$ for $\alpha \in N$ is defined by

$${}^C D_{a+}^{\alpha;\psi} x(t) = I_{a+}^{n-\alpha;\psi} \beta_\psi^{(n)} x(t), \quad (17)$$

where $\beta_\psi^{(n)} = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n$.

The definition, clearly indicates that

$${}^C D_{a+}^{\alpha;\psi} x(t) = \begin{cases} \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} x_\psi^{[n]}(s) ds, & \text{if } \alpha \notin N, \\ u_\psi^{[n]}(t), & \text{if } \alpha \in N. \end{cases} \quad (18)$$

Here α is the order of ψ -Caputo fractional derivative, if $x \in C^n(J, R)$ then

$${}^C D_{a+}^{\alpha;\psi} x(t) = D_{a+}^{\alpha;\psi} \left[x(t) - \sum_{z=0}^{n-1} (\psi(t) - \psi(a))^z \frac{\beta_\psi^{(z)}(a)}{z!} \right]. \quad (19)$$

Lemma 1. [9] Let $\alpha > 0$, then

$$\begin{aligned} I_{a+}^{y;\psi} {}^C D_{a+}^{y;\psi} x(t) &= x(t) + c_0 + c_1(\psi(t) - \psi(0)) \\ &\quad + c_2(\psi(t) - \psi(0))^2 + \dots \\ &\quad + c_{n-1}(\psi(t) - \psi(0))^{n-1}, \end{aligned} \quad (20)$$

where $n = [\alpha] + 1$.

Lemma 2. [9] Let $y, z > 0$, and $x \in L^1(J, R)$. Then

$$I_{a+}^{y;\psi} I_{a+}^{z;\psi} x(t) = I_{a+}^{y+z;\psi} x(t), \text{ a.e. } t \in J, \quad (21)$$

specifically, if $x \in C(J, R)$, then

$$I_{a+}^{y;\psi} I_{a+}^{z;\psi} x(t) = I_{a+}^{y+z;\psi} x(t), t \in J \quad (22)$$

Lemma 3. [9] Let $y > 0$, the subsequent holds: If $x \in C(J, R)$, then

$${}^C D_{a+}^{y;\psi} I_{a+}^{y;\psi} x(t) = x(t), t \in J, \quad (23)$$

suppose $n-1 < y < n$, $x \in C^n(J, R)$, then

$$I_{a+}^{y;\psi} {}^C D_{a+}^{y;\psi} x(t) = x(t) - \sum_{z=0}^{n-1} (\psi(t) - \psi(a))^z \frac{\beta_\psi^{(z)}(a)}{z!}, \quad t \in J. \quad (24)$$

Lemma 4. [23] Let $t > a$, $y, z \geq 0$, and by assuming $(Q_b(t)) = \psi(t) - \psi(a)$. Then

$$(i) \quad I_{a+}^{y;\psi} (Q_b(t))^{z-1} = \frac{\Gamma(z)}{\Gamma(z+y)} (Q_b(t))^{z+y-1}. \quad (25)$$

$$(ii) \quad {}^C D_{a+}^{y;\psi} (Q_b(t))^{z-1} = \frac{\Gamma(z)}{\Gamma(z-y)} (Q_b(t))^{z-y-1}. \quad (26)$$

$$(iii) \quad {}^C D_{a+}^{y;\psi} (Q_b(t))^k = 0, \text{ for everyone } k \in \{0, \dots, n-1\}. \quad (27)$$

Definition 7. [9] Let D^+ be the non negative cone of an ordered Banach space $(D, \leq)$. A function $\mathfrak{R}$ defined on the

X with value in D^+ is called a Measure of Noncompactness (MNC) on X iff $\mathfrak{R}(\overline{\text{conv}}(\Omega)) = \mathfrak{R}(\Omega)$ for all $\Omega \in X$.

The MNC is considered as:

(i) Monotone iff $(\Omega_1 \subseteq \Omega_2) \Rightarrow \mathfrak{R}(\Omega_1) \leq \mathfrak{R}(\Omega_2)$ for every Ω_1, Ω_2 of X ,

(ii) Non singular if $\mathfrak{R}((a) \cup \Omega) = \mathfrak{R}(\Omega)$ for every non empty subset $\Omega \subseteq X$, $\forall a \in X$,

(iii) Regular if $\mathfrak{R}(\Omega) = 0 \Leftrightarrow \Omega$ is relative compact in X .

The Hausdorff MNC of Φ is given by $\Phi(\Omega) = \inf r > 0$: Ω be the finite number of balls with radii r .

It is established that MNC Φ possess the aforementioned properties and additional properties for all bounded subsets $\Omega, \Omega_1, \Omega_2$ of X .

(iv) $\Phi(\Omega_1 + \Omega_2) \leq \Phi(\Omega_1) + \Phi(\Omega_2)$, where $\Omega_1 + \Omega_2 = (x + y; x \in \Omega_1, y \in \Omega_2)$,

(v) $\Phi(\Omega_1) \cup (\Omega_2) \leq \max(\Phi(\Omega_1), \Phi(\Omega_2))$,

(vi) $\Phi(\lambda\Omega) = |\lambda| \Phi(\Omega)$ $\lambda \in \mathbb{R}$,

(vii) If the map $U : D(U) \subseteq X \rightarrow Y$ Lipschitz continuous, k is constant then $\Phi_z(U(\Omega)) \leq k\Phi(\Omega)$ for any bounded subset $\Omega \subseteq D(U)$, as Y is the Banach space.

Definition 9. [9] *Dominated convergent theorem*

Let (X, A, μ) be a measure space. Suppose $\{f_n\}$ is a sequence of measurable functions that converges pointwise to a function f almost everywhere on X . $\exists$ an integrable function $g : X \rightarrow \mathbb{R}$ such that

$$|f_n(x)| \leq g(x) \text{ for all } x \in X \quad (28)$$

then,

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu. \quad (29)$$

Lemma 5. [9] Let $k(W(t))$ is continuous for $t \in [a, b]$, if $W \subseteq C([a, b], X)$ is bounded and equicontinuous

$$k(W) = \sup(k(W(t)), t \in [a, b]), \quad (30)$$

where

$$W(t) = x(t); x \in W \subseteq X.$$

Lemma 6. [9] The sequence $\{g_m\}_{m=1}^\infty$ is a Bochner integrable function from $J \rightarrow X$ with the estimation $\|g_m(t)\| \leq \mu(t)$ for almost all $t \in J$ and every $m \geq 1$, where $\mu \in L^1(J, \mathbb{R})$ then the function $\psi(t) = k(g_m(t) : m \geq 1)$ belongs to $L^1(J, \mathbb{R})$ and satisfies

$$k\left(\int_0^t g_m(\tau) d\tau : m \geq 1\right) \leq 2 \int_0^t \psi(\tau) d\tau. \quad (31)$$

Lemma 7. [9] *Monch fixed point theorem* Let H be a closed convex subset of a Banach space X and $0 \in H$, the continuous map $G : H \rightarrow X$, satisfy Monch's condition. i.e., if $M \subseteq H$ is denumerable and $M \subseteq \text{Conv}(0 \cup G(M)) \Rightarrow \bar{M}$ is compact. Then G has a fixed point in D .

Lemma 8. [26] For any $y \in C([0, 1], \mathbb{R})$, the unique solution

of the linear boundary value problem

$${}^C D^{\alpha; \psi} x(t) = y(t), 1 < \alpha \leq 2, t \in [0, 1] \quad (32)$$

$$x(0) = h(x), \quad (33)$$

$$x(\mu) = a \int_0^\beta x(p) dp + b \int_\gamma^1 x(p) dp, 0 < \beta < \eta < \gamma < 1 \quad (34)$$

has the unique solution is given by

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t \psi_t y(s) ds \\ & + \frac{\Psi}{Q_\psi} \left[a \int_0^\beta \int_0^p \frac{\psi_p}{\Gamma(\alpha)} y(s) ds dp \right. \\ & + b \int_\gamma^1 \int_0^p \frac{\psi_p}{\Gamma(\alpha)} y(s) ds dp \\ & \left. - \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} y(s) ds \right] \\ & + h(x) \left[1 - \frac{\Psi}{Q_\psi} (1 - a\beta - b(1 - \gamma)) \right], \end{aligned} \quad (35)$$

where

$$\psi_t = [\psi(t) - \psi(s)]^{\alpha-1} \psi'(s)$$

$$\psi_p = [\psi(t) - \psi(s)]^{\alpha-1} \psi'(s)$$

$$\psi_\mu = [\psi(t) - \psi(s)]^{\alpha-1} \psi'(s)$$

$$\begin{aligned} Q_\psi = & \left[\psi(\mu) - \psi(0) - a \int_0^\beta (\psi(p) - \psi(0)) dp \right. \\ & \left. - b \int_\gamma^1 (\psi(p) - \psi(0)) dp \right] \end{aligned}$$

$$\Psi = \psi(t) - \psi(0)$$

$$M = \psi(1) - \psi(0).$$

Proof.

Let x satisfies equation (30), then lemma (18) implies that

$$x(t) = \int_0^t \frac{\psi_t}{\Gamma(\alpha)} y(s) ds + c_0 + c_1(\Psi),$$

where $c_0, c_1 \in \mathbb{R}$. By applying the boundary conditions: $x(0) = h(x)$ we get $c_0 = h(x)$

$$x(t) = \int_a^t \frac{\psi_t}{\Gamma(\alpha)} y(s) ds + h(x) + c_1(\Psi).$$

By applying the boundary conditions: $x(\mu) = a \int_0^\beta x(p) dp + b \int_\gamma^1 x(p) dp$. we get

$$\begin{aligned} c_1 = & \frac{1}{Q_\psi} \left[\int_0^\beta \int_0^p \frac{a\psi_p}{\Gamma(\alpha)} y(s) ds dp \right. \\ & + \int_\gamma^1 \int_0^p \frac{b\psi_p}{\Gamma(\alpha)} y(s) ds dp \\ & \left. - \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} y(s) ds \right] \\ & - [1 - a\beta - b(1 - \gamma)] h(x) \end{aligned}$$

Finally we obtain the solution in the equation (32).

3. Main Result

To prove the main results, we enumerate the subsequent hypotheses;

(C1) *Carathéodory Condition.* The function f and k is satisfies the caratheodory-type conditions.

(C2) *Growth Condition.*

There exist a constants $L_f, L_k > 0$, ensuring that

$$\|f(t, x(t), Rx(t))\| \leq L_f(1 + \|x(t)\| + \|Rx(t)\|),$$

$$\|k(t, s, x(s))\| \leq L_k(1 + \|x(s)\|),$$

for every $s \in [0, t], t \in J$ and all $x \in X$.

(C3) *Measure of Noncompactness Condition.*

There exist a constant $\alpha \in (0, 1)$ and real valued functions $n_1(t), n_2(t) \in L^{\frac{1}{\alpha}}(J, R)$ ensuring that

$$\Phi(f(t, E, F)) \leq n_1(t)(\Phi(E) + \Phi(F)),$$

$$\Phi(k(t, s, F)) \leq n_2(t)\Phi(C),$$

for each bounded sets $E, F \subset X$ a.e $t \in J$.

To be brief, Let $N = \|n_1 + 2Tn_1n_2\|_{L^{\frac{1}{\alpha}}(\mathfrak{I}, R)}$

(C4) *Boundedness Condition.*

There exists a positive constants L_f, L_k , we have

$$\frac{PL_f(1 + L_kT)M^\alpha}{\Gamma(\alpha + 1)} \leq 1.$$

Where

$$P = \left(1 + \frac{(a\beta + b(1 - \gamma) + 1)M}{Q_\psi}\right)$$

Now, we present our main result.

Theorem 3.1.

Assume that hypotheses $(C_1) - (C_4)$ are holds. If

$$E = \left[2N \left[e_1 + \frac{\Psi}{Q_\psi}(e_2 + e_3 + e_4)\right]\right] < 1,$$

where

$$\begin{aligned} e_1 &= \frac{1}{\Gamma(\alpha)} \left(\frac{1 - \alpha_1}{\alpha - \alpha_1}\right)^{1 - \alpha_1} [\Psi]^{\alpha - \alpha_1} \\ e_2 &= \frac{a}{\Gamma(\alpha + 1)} \beta^{\alpha_1} [M]^\alpha \\ e_3 &= \frac{b}{\Gamma(\alpha + 1)} (1 - \gamma)^{\alpha_1} [M]^\alpha \\ e_4 &= \frac{1}{\Gamma(\alpha)} \left(\frac{1 - \alpha_1}{\alpha - \alpha_1}\right)^{1 - \alpha_1} \\ &\quad \left[(\Psi)^{\frac{\alpha - \alpha_1}{1 - \alpha_1}} - (\psi(t) - \psi(\mu))^{\frac{\alpha - \alpha_1}{1 - \alpha_1}}\right]^{1 - \alpha_1}. \end{aligned}$$

$$\psi_t = [\psi(t) - \psi(s)]^{\alpha - 1} \psi'(s)$$

$$\psi_p = [\psi(t) - \psi(s)]^{\alpha - 1} \psi'(s)$$

$$\psi_\mu = [\psi(t) - \psi(s)]^{\alpha - 1} \psi'(s)$$

Then, the equation (9) has at least one solution on J .

Proof: First we have to prove the existence solution of the problem (9) transform it into a fixed point problem.

Define the mapping $G : C(J, X) \rightarrow C(J, X)$ as follows;

$$\begin{aligned} Gx(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \psi_t f(s, x(s), Rx(s)) ds \\ &+ \frac{\Psi}{Q_\psi} \left[a \int_0^\beta \int_0^p \frac{\psi_p}{\Gamma(\alpha)} f(s, x(s), Rx(s)) ds dp \right. \\ &+ b \int_\gamma^1 \int_0^p \frac{\psi_p}{\Gamma(\alpha)} f(s, x(s), Rx(s)) ds dp \\ &\left. - \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} f(s, x(s), Rx(s)) ds \right] \\ &+ \mathfrak{h}(x) \left[1 - \frac{\Psi}{Q_\psi} (1 - a\beta - b(1 - \gamma)) \right] \end{aligned}$$

Let $W_r = \{x \in C(J, X), \|x\|_\infty \leq r \text{ for some } r > 0\}$. We shall show that G fulfills the assumptions of Lemma 8. The proof will be the following steps.

Step:1

There exist, $r > 0$ ensuring that $G(W_r) \subseteq W_r$. If this is false, then for each nonnegative number r , there exists a function $p^r \in W_r$. But $G(p^r) \notin W_r$. Here $\Omega = f(s, p^r(s), Rp^r(s))$. ie., $\|G(p^r)\| > r$ for some $t \in J$ using (C_3) . We have

$$\begin{aligned} r &< \|G(p^r)(t)\| \\ &= \left\| \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) [\psi(t) - \psi(s)]^{\alpha - 1} \Omega ds \right. \\ &+ \frac{\Psi}{Q_\psi} \left[a \int_0^\beta \int_0^p \frac{\psi'(s) [\psi(p) - \psi(s)]^{\alpha - 1}}{\Gamma(\alpha)} \Omega ds dp \right. \\ &+ b \int_\gamma^1 \int_0^p \frac{\psi'(s) [\psi(p) - \psi(s)]^{\alpha - 1}}{\Gamma(\alpha)} \Omega ds dp \\ &\left. - \int_0^\mu \frac{\psi'(s) [\psi(\mu) - \psi(s)]^{\alpha - 1}}{\Gamma(\alpha)} \Omega ds \right] \\ &+ \mathfrak{h}(x) \left[1 - \frac{\Psi}{Q_\psi} (1 - a\beta - b(1 - \gamma)) \right] \Big\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) [\psi(t) - \psi(s)]^{\alpha - 1} \|\Omega\| ds \\ &+ \frac{\Psi}{|Q_\psi|} \left[a \int_0^\beta \int_0^p \frac{\psi'(s) [\psi(p) - \psi(s)]^{\alpha - 1}}{\Gamma(\alpha)} \|\Omega\| ds dp \right. \\ &+ b \int_\gamma^1 \int_0^p \frac{\psi'(s) [\psi(p) - \psi(s)]^{\alpha - 1}}{\Gamma(\alpha)} \|\Omega\| ds dp \\ &\left. - \int_0^\mu \frac{\psi'(s) [\psi(\mu) - \psi(s)]^{\alpha - 1}}{\Gamma(\alpha)} \|\Omega\| ds \right] \end{aligned}$$

$$\begin{aligned}
& + \|\mathfrak{h}(x)\| \left[1 - \frac{\Psi}{|Q_\psi|} (1 - a\beta - b(1 - \gamma)) \right] \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi_t L_f(1 + \|p^r(s)\| + \|(Rp^r)(s)\|) ds \\
& + \frac{\Psi}{Q_\psi} \left[\int_0^\beta \int_0^p \frac{a\psi_p}{\Gamma(\alpha)} L_f(1 + \|p^r(s)\| + \|(Rp^r)(s)\|) ds dp \right. \\
& + b \int_\gamma^1 \int_0^p \frac{\psi_p}{\Gamma(\alpha)} L_f(1 + \|p^r(s)\| + \|(Rp^r)(s)\|) ds dp \\
& \left. - \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} L_f(1 + \|p^r(s)\| + \|(Rp^r)(s)\|) ds \right] \\
& + \|\mathfrak{h}(x)\| \left[1 - \frac{\Psi}{Q_\psi} (1 - a\beta - b(1 - \gamma)) \right] \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi_t L_f(1 + \|p^r\|_\infty + L_k(1 + \|p^r\|_\infty)) T ds \\
& + \frac{\Psi}{Q_\psi} \left[\int_0^\beta \int_0^p \frac{a\psi_p}{\Gamma(\alpha)} L_f(1 + \|p^r\|_\infty \right. \\
& + L_k(1 + \|p^r\|_\infty)) T ds dp + b \int_\gamma^1 \int_0^p \frac{\psi_p}{\Gamma(\alpha)} L_f(1 + \|p^r\|_\infty \\
& + L_k(1 + \|p^r\|_\infty)) T ds dp \\
& \left. + \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} L_f(1 + \|p^r\|_\infty + L_k(1 + \|p^r\|_\infty)) T ds \right] \\
& + \|\mathfrak{h}(x)\| \left[1 - \frac{\Psi}{Q_\psi} (1 - a\beta - b(1 - \gamma)) \right] \\
& \leq \frac{L_f(1+r)(1+L_kT)}{\Gamma(\alpha+1)} [M]^\alpha + \frac{\Psi}{Q_\psi} \left[\frac{L_f(1+r)(1+L_kT)}{\Gamma(\alpha+1)} \right. \\
& \quad \left. [M]^\alpha (a\beta + b(1-\gamma) + 1) \right] \\
& + J_0 \left(1 + \frac{M}{Q_\psi} (1 - a\beta - b(1 - \gamma)) \right) \\
& \leq \frac{L_f(1+r)(1+L_kT)}{\Gamma(\alpha+1)} [M]^\alpha + \left[\frac{L_f(1+r)(1+L_kT)}{Q_\psi \Gamma(\alpha+1)} \right. \\
& \quad \left. [M]^{\alpha+1} (a\beta + b(1-\gamma) + 1) \right] \\
& + J_0 \left(1 + \frac{M}{Q_\psi} (1 - a\beta - b(1 - \gamma)) \right) \\
& \leq \frac{L_f(1+r)(1+L_kT) [\psi(1) - \psi(0)]^\alpha}{\Gamma(\alpha+1)} \\
& \quad \left(1 + \frac{(a\beta + b(1-\gamma) + 1)(\psi(1) - \psi(0))}{Q_\psi} \right) \\
& \quad + J_0 \left(1 + \frac{M}{Q_\psi} (1 - a\beta - b(1 - \gamma)) \right)
\end{aligned}$$

while dividing the above equation by ' r ',

$$\begin{aligned}
1 & < \frac{L_f(\frac{1}{r} + 1)(1 + L_kT)M^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{(a\beta + b(1-\gamma) + 1)M}{Q_\psi} \right) \\
& + \frac{J_0}{r} \left(1 + \frac{M}{Q_\psi} (1 - a\beta - b(1 - \gamma)) \right)
\end{aligned}$$

Now taking lim as $r \rightarrow \infty$,

$$1 < \frac{PL_f(1 + L_kT)M^\alpha}{\Gamma(\alpha+1)}$$

which goes against the hypothesis (C_4) . Hence for some positive number r .

$$G(W_r) \subseteq W_r \text{ that is } G : W_r \rightarrow W_r.$$

Step:2

G is continuous operator on W_r .

Let x_n be sequence ensuring that $x_n \rightarrow x$ in W_r as $n \rightarrow \infty$.

Given that f fulfills (C_1) for almost every $t \in J$, we get

$$f(t, x_n(t), (Rx_n)(t)) \rightarrow f(t, x(t), (Rx)(t)).$$

Here assume

$$\Lambda = f(t, x(t), (Rx)(t)) \text{ and } \Lambda_n = f(t, x_n(t), (Rx_n)(t)).$$

By the Definition [9], for each $t \in J$,

$$\begin{aligned}
& \|G(x_n)(t) - G(x)(t)\| \\
& \leq \left\| \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) [\psi(t) - \psi(s)]^{\alpha-1} \Lambda_n ds \right. \\
& + \frac{\Psi}{Q_\psi} \left[a \int_0^\beta \int_0^p \frac{\psi'(s) [\psi(p) - \psi(s)]^{\alpha-1}}{\Gamma(\alpha)} \Lambda_n ds dp \right. \\
& + b \int_\gamma^1 \int_0^p \frac{\psi'(s) [\psi(p) - \psi(s)]^{\alpha-1}}{\Gamma(\alpha)} \Lambda_n ds dp \\
& \left. - \int_0^\mu \frac{\psi'(s) [\psi(\mu) - \psi(s)]^{\alpha-1}}{\Gamma(\alpha)} \Lambda_n ds \right] \\
& + \|\mathfrak{h}(x)\| \left[1 - \frac{\Psi}{Q_\psi} (1 - a\beta - b(1 - \gamma)) \right] \\
& - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) [\psi(t) - \psi(s)]^{\alpha-1} \Lambda ds \\
& - \frac{\Psi}{Q_\psi} \left[a \int_0^\beta \int_0^p \frac{\psi'(s) [\psi(p) - \psi(s)]^{\alpha-1}}{\Gamma(\alpha)} \Lambda ds dp \right. \\
& - b \int_\gamma^1 \int_0^p \frac{\psi'(s) [\psi(p) - \psi(s)]^{\alpha-1}}{\Gamma(\alpha)} \Lambda ds dp \\
& \left. + \int_0^\mu \frac{\psi'(s) [\psi(\mu) - \psi(s)]^{\alpha-1}}{\Gamma(\alpha)} \Lambda ds \right]
\end{aligned}$$

$$\begin{aligned}
 & -\mathfrak{h}(x) \left[1 - \frac{\Psi}{Q_\psi} (1 - a\beta - b(1 - \gamma)) \right] \| \\
 & \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) [\psi(t) - \psi(s)]^{\alpha-1} \|\Lambda_n - \Lambda\| ds \\
 & + \frac{\Psi}{Q_\psi} a \int_0^\beta \int_0^p \frac{\psi'(s) [\psi(p) - \psi(s)]^{\alpha-1}}{\Gamma(\alpha)} \|\Lambda_n - \Lambda\| ds dp \\
 & + b \int_\gamma^1 \int_0^p \frac{\psi'(s) [\psi(p) - \psi(s)]^{\alpha-1}}{\Gamma(\alpha)} \|\Lambda_n - \Lambda\| ds dp \\
 & + \int_0^\mu \frac{\psi'(s) [\psi(\mu) - \psi(s)]^{\alpha-1}}{\Gamma(\alpha)} \|\Lambda_n - \Lambda\| ds \\
 & \leq \frac{M^\alpha}{\Gamma(\alpha+1)} \|f(\cdot, x_n(\cdot), R(x_n)(\cdot)) - f(\cdot, x(\cdot), R(x)(\cdot))\|_\infty \\
 & + \frac{M^{\alpha+1}(a\beta + b(1 - \gamma) + 1)}{Q_\psi \Gamma(\alpha+1)} \\
 & \quad \|f(\cdot, x_n(\cdot), R(x_n)(\cdot)) - f(\cdot, x(\cdot), R(x)(\cdot))\|_\infty
 \end{aligned}$$

So by taking supremum, to obtain

$$\begin{aligned}
 & \|Gx_n - Gp\|_\infty \\
 & \leq \frac{M^\alpha}{\Gamma(\alpha+1)} \|f(\cdot, x_n(\cdot), R(x_n)(\cdot)) - f(\cdot, x(\cdot), R(x)(\cdot))\|_\infty \\
 & + \frac{M^{\alpha+1}(a\beta + b(1 - \gamma) + 1)}{Q_\psi \Gamma(\alpha+1)} \\
 & \quad \times \|f(\cdot, x_n(\cdot), R(x_n)(\cdot)) - f(\cdot, x(\cdot), R(x)(\cdot))\|_\infty
 \end{aligned}$$

Therefore,

$$\|Gx_n - Gx\|_\infty \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Step:3

$G(W_r)$ is equicontinuous on J . Indeed for $0 \leq t_1 \leq t_2 \leq T$, $x \in W_r$ and (C_2) , we have

$$\begin{aligned}
 & \|G(x)(t_2) - G(x)(t_1)\| = \\
 & \left\| \int_0^{t_1} \frac{\psi'(s) [\psi(t_2) - \psi(s)]^{\alpha-1} - \psi'(s) [\psi(t_1) - \psi(s)]^{\alpha-1}}{\Gamma(\alpha)} \Lambda ds \right. \\
 & + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s) [\psi(t_2) - \psi(s)]^{\alpha-1} \Lambda ds \\
 & + \frac{(\psi(t_2) - \psi(0)) - (\psi(t_1) - \psi(0))}{Q_\psi} \left[\int_0^\beta \int_0^p \frac{a\psi_p}{\Gamma(\alpha)} \Lambda ds dp \right. \\
 & + b \int_\gamma^1 \int_0^p \frac{\psi_p}{\Gamma(\alpha)} \Lambda ds dp - \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} \Lambda ds \left. \right] \\
 & + \mathfrak{h}(x) \left[1 - \frac{(\psi(t_2) - \psi(0)) - (\psi(t_1) - \psi(0))}{Q_\psi} \right. \\
 & \quad \left. (1 - a\beta - b(1 - \gamma)) \right] \\
 & \leq L_f(1+r)(1+L_kT) \left[\frac{1}{\Gamma(\alpha+1)} (\psi(t_2) - \psi(t_0))^\alpha \right.
 \end{aligned}$$

$$\begin{aligned}
 & + (\psi(t_2) - \psi(t_1))^\alpha + (\psi(t_1) - \psi(t_0))^\alpha \\
 & + \frac{(\psi(t_2) - \psi(t_1))^\alpha}{\Gamma(\alpha+1)} \left. \right] + \frac{L_f(1+r)(1+L_kT)M^{\alpha+1}}{Q_\psi \Gamma(\alpha+1)} \\
 & (\psi(t_2) - \psi(t_1))(a\beta + b(1 - \gamma) + 1)
 \end{aligned}$$

Since $t_2 \rightarrow t_1$ the aforementioned inequality's right hand side tends to 0.

Since x is an arbitrary in W_r . We infer that $G(W_r)$ is equicontinuous.

Step:4

The Monch condition (lemma [7]) holds.

Assuming $Z \subseteq W_r$ is denumerable and $Z \subseteq \text{conv}(0 \cup G(Z))$ To prove that $\Phi(Z) = 0$, where Φ is Hausdorff Measure of noncompactness. Suppose that $Z = (p_k)_{k=1}^\infty$. To show that $\Phi(Z(t))$ is relatively compact in X , $\forall t \in J$,

Let $\Pi = f(s, (p_k(s))_{k=1}^\infty, (R(p_k(s))_{k=1}^\infty))$.

By using lemma (4) and (5), hypotheses (C_3) and Holder inequality, we have

$$\begin{aligned}
 & \Phi(G(p_k(s))_{k=1}^\infty(t)) = \Phi \left(\int_0^t \frac{\psi'(s) [\psi(t) - \psi(s)]^{\alpha-1}}{\Gamma(\alpha)} \Pi ds \right. \\
 & + \frac{\Psi}{Q_\psi} \left[a \int_0^\beta \int_0^p \frac{\psi'(s) [\psi(p) - \psi(s)]^{\alpha-1}}{\Gamma(\alpha)} \Pi ds dp \right. \\
 & + b \int_\gamma^1 \int_0^p \frac{\psi'(s) [\psi(p) - \psi(s)]^{\alpha-1}}{\Gamma(\alpha)} \Pi ds dp \\
 & + \left. \int_0^\mu \frac{\psi'(s) [\psi(\mu) - \psi(s)]^{\alpha-1}}{\Gamma(\alpha)} \Pi ds \right] \\
 & + \left. \mathfrak{h}(x) \left[1 - \frac{\Psi}{Q_\psi} (1 - a\beta - b(1 - \gamma)) \right] \right) \\
 & \leq \frac{2}{\Gamma(\alpha)} \int_0^t \psi_t n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \\
 & + \Phi(R((p_k(s))_{k=1}^\infty))] ds \\
 & + \frac{\Psi}{Q_\psi} \left[\int_0^\beta \int_0^p \frac{2a\psi_p}{\Gamma(\alpha)} n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \right. \\
 & + \Phi(R((p_k(s))_{k=1}^\infty))] ds dp \\
 & + 2b \int_\gamma^1 \int_0^p \frac{\psi_p}{\Gamma(\alpha)} n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \\
 & + \Phi(R((p_k(s))_{k=1}^\infty))] ds dp \\
 & + 2 \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \\
 & + \Phi(R((p_k(s))_{k=1}^\infty))] ds \left. \right]
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{2}{\Gamma(\alpha)} \int_0^t \psi_t n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \\
&+ \Phi(\int_0^t k((s, \theta, (p_k(\theta))_{k=1}^\infty) d\theta))] ds \\
&+ \frac{\Psi}{Q_\psi} \left[2a \int_0^\beta \int_0^p \frac{\psi_p}{\Gamma(\alpha)} n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \right. \\
&+ \Phi(\int_0^t k((s, \theta, (p_k(\theta))_{k=1}^\infty) d\theta))] ds dp \\
&+ 2b \int_\gamma^1 \int_0^p \frac{\psi_p}{\Gamma(\alpha)} n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \\
&+ \Phi(\int_0^t k((s, \theta, (p_k(\theta))_{k=1}^\infty) d\theta))] ds dp \\
&+ 2 \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \\
&+ \Phi(\int_0^t k((s, \theta, (p_k(\theta))_{k=1}^\infty) d\theta))] ds \Big] \\
&\leq \frac{2}{\Gamma(\alpha)} \int_0^t \psi_t n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \\
&+ 2 \int_0^t \Phi(k(s, \theta, (p_k(\theta))_{k=1}^\infty) d\theta)] ds \\
&+ \frac{\Psi}{Q_\psi} \left[2a \int_0^\beta \int_0^p \frac{\psi_p}{\Gamma(\alpha)} n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \right. \\
&+ 2 \int_0^t \Phi(k(s, \theta, (p_k(\theta))_{k=1}^\infty) d\theta)] ds dp \\
&+ 2b \int_\gamma^1 \int_0^p \frac{\psi_p}{\Gamma(\alpha)} n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \\
&+ 2 \int_0^t \Phi(k(s, \theta, (p_k(\theta))_{k=1}^\infty) d\theta)] ds dp \\
&+ 2 \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \\
&+ 2 \int_0^t \Phi(k(s, \theta, (p_k(\theta))_{k=1}^\infty) d\theta)] ds \Big] \\
&\leq \frac{2}{\Gamma(\alpha)} \int_0^t \psi_t n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \\
&+ 2 \int_0^t n_2(s) \Phi((p_k(\theta))_{k=1}^\infty) d\theta] ds \\
&+ \frac{\Psi}{Q_\psi} \left[2a \int_0^\beta \int_0^p \frac{\psi_p}{\Gamma(\alpha)} n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \right. \\
&+ 2 \int_0^t n_2(s) \Phi((p_k(\theta))_{k=1}^\infty) d\theta] ds dp \\
&+ 2b \int_\gamma^1 \int_0^p \frac{\psi_p}{\Gamma(\alpha)} n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \\
&+ 2 \int_0^t n_2(s) \Phi((p_k(\theta))_{k=1}^\infty) d\theta] ds dp \\
&+ 2 \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \\
&+ 2 \int_0^t n_2(s) \Phi((p_k(\theta))_{k=1}^\infty) d\theta] ds \Big] \\
&+ 2 \int_0^t n_2(s) \Phi((p_k(\theta))_{k=1}^\infty) d\theta \\
&+ 2 \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \\
&+ 2 \int_0^t n_2(s) \Phi((p_k(\theta))_{k=1}^\infty) d\theta] ds \Big] \\
&\leq \frac{2}{\Gamma(\alpha)} \int_0^t \psi_t n_1(s) (\Phi(Z) \\
&+ 2Tn_2(s)\Phi(Z)) ds \\
&+ \frac{\Psi}{Q_\psi} \left[2a \int_0^\beta \int_0^p \frac{\psi_p}{\Gamma(\alpha)} n_1(s) (\Phi(Z) \right. \\
&+ 2Tn_2(s)\Phi(Z)) ds dp \\
&+ 2b \int_\gamma^1 \int_0^p \frac{\psi_p}{\Gamma(\alpha)} n_1(s) (\Phi(Z) \\
&+ 2Tn_2(s)\Phi(Z)) ds dp \\
&+ 2 \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} n_1(s) (\Phi(Z) + 2Tn_2(s)\Phi(Z)) ds \Big] \\
&\leq \left[\frac{2}{\Gamma(\alpha)} \int_0^t \psi_t ds \left(\int_a^t (n_1(s) \right. \right. \\
&+ 2Tn_1(s)n_2(s)) ds) \\
&+ \frac{\Psi}{Q_\psi} \left[2a \int_0^\beta \int_0^p \frac{\psi_p}{\Gamma(\alpha)} ds \left(\int_a^t (n_1(s) \right. \right. \\
&+ 2Tn_1(s)n_2(s)) ds) dp \\
&+ 2b \int_\gamma^1 \int_0^p \frac{\psi_p}{\Gamma(\alpha)} ds \left(\int_a^t (n_1(s) + 2Tn_1(s)n_2(s)) ds) dp \right. \\
&+ 2 \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} ds \left(\int_a^t (n_1(s) + 2Tn_1(s)n_2(s)) ds) \right) \Big] \Phi(Z) \\
&\leq \left[\frac{2}{\Gamma(\alpha)} \int_0^t \psi_t ds \left(\int_a^t (n_1(s) + 2Tn_1(s)n_2(s)) ds) \right. \right. \\
&+ \frac{\Psi}{Q_\psi} \left[2a \int_0^\beta \int_0^p \frac{\psi_p}{\Gamma(\alpha)} ds \right. \\
&+ 2Tn_1(s)n_2(s)) ds) dp \\
&+ 2b \int_\gamma^1 \int_0^p \frac{\psi_p}{\Gamma(\alpha)} ds \left(\int_a^t (n_1(s) + 2Tn_1(s)n_2(s)) ds) dp \right. \\
&+ 2 \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} ds \left(\int_a^t (n_1(s) + 2Tn_1(s)n_2(s)) ds) \right) \Big] \Phi(Z) \\
&\leq \left[\frac{2}{\Gamma(\alpha)} \left(\int_0^t \psi'(s) [\psi(t) - \psi(s)]^{\frac{\alpha-1}{1-\alpha_1}} ds \right)^{1-\alpha_1} \right. \\
&\left(\int_0^t (n_1(s) + 2Tn_1(s)n_2(s))^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1} \\
&+ \frac{\Psi}{Q_\psi} \left[\frac{2a}{\Gamma(\alpha+1)} \int_0^\beta \left([\psi(p) - \psi(0)]^{\frac{\alpha}{\alpha_1}} dp \right)^{\alpha_1} \right.
\end{aligned}$$

$$\begin{aligned}
 & \left(\int_0^t (n_1(s) + 2Tn_1(s)n_2(s))^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1} \\
 & + \frac{2b}{\Gamma(\alpha+1)} \int_\gamma^1 \left([\psi(p) - \psi(0)]^{\frac{\alpha}{\alpha_1}} dp \right)^{\alpha_1} \\
 & \left(\int_0^t (n_1(s) + 2Tn_1(s)n_2(s))^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1} \\
 & + \frac{2}{\Gamma(\alpha)} \left(\int_0^\mu \psi'(s) [\psi(t) - \psi(s)]^{\frac{\alpha-1}{1-\alpha_1}} ds \right)^{1-\alpha_1} \\
 & \left(\int_0^t (n_1(s) + 2Tn_1(s)n_2(s))^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1} \Big] \Phi(Z) \\
 & \leq \left[2N \left[C_1 + \frac{\Psi}{Q_\psi} (C_2 + C_3 + C_4) \right] \right] \Phi Z \\
 & \Phi(\phi(Z)) \leq E\Phi(Z).
 \end{aligned}$$

Thus from Monch condition, we have

$$\Phi(Z) < \Phi(CONV(0 \cup (G(Z)))) = \Phi(G(Z)) \leq E\Phi(Z),$$

which implice that $\Phi(Z) = 0$. Hence Z is relatively compact.

Therefore, according to Lemma (8), $\exists$ a fixed point x of the operator G on W_r .

Which is a solution of the equation (9).

4. The Second Existence Result

In this section, we will prove the result of theorem 3.1 in the case there is not restricted hypothesis (C_5) via another MNC

Let $\mathfrak{R}$ be the MNC in the Banach space $C(J, X)$ determined by

$$\mathfrak{R}(\Omega) = \max_{D \in \Delta(\Omega)} (x(D) \text{mod}_e(D)),$$

for all bounded subsets Ω of $C(J, X)$, where $\Delta(\Omega)$ is the set of denumerable subsets of Ω , x is the real MNC define by

$$\mathfrak{F}(D) = \sup_{L \in [0, b]} e^{-Lt} \Phi(D(t)),$$

$D(t) = (x(t) : x \in D)$, $t \in J$, L is a constant that we shall choose conveniently and $\text{mod}_c(D)$ is the modulus of equi continuity of the function set D given by the formula,

$$\text{mod}_c(D) = (\lim_{\delta \rightarrow 0}) (\sup_{x \in D} (\max_{|t_2 - t_1| \leq \delta} \|x(t_2) - x(t_1)\|),$$

that $\mathfrak{R}$ is properly defined (i.e., there is $D_0 \in \Delta(\Omega)$ which achieves the maximum in (2.1) and is nonsingular, monotone and regular MNC.

Now we are willing to prove the second existences result.

Theorem 4.1. Assume that $(C_1) - (C_3)$ holds, then the problem (9) has at least one solution on J .

Proof:

It is sufficient to prove that the function $G : C(J, X) \rightarrow C(J, X)$ and $G(x(t))$ satisfies the Monch condition.

Let $Z \in W_r$ be denumerable set and $Z \subseteq \text{conv}((0) \cup G(Z))$, by the regularity of $\mathfrak{R}$, Z is relatively compact. It sufficient to prove that $\mathfrak{R}(Z) = (0, 0)$. Since $\mathfrak{R}(G(Z))$ is a maximum.

Let $(y_k)_{k=1}^\infty \subseteq G(Z)$ be the denumerable set which arrive its maximum.

Obviously, there exists a set $(p_k)_{k=1}^\infty \subseteq \mathbb{Z}$, such that $y_k = (Gp_k)(t)$ for all $t \in J$, $n \geq 1$.

Now we need to give the value for $\mathfrak{F}((x_k)_{k=1}^\infty)$,

$$\begin{aligned}
 \Phi((y_k)_{k=1}^\infty) &= \Phi(((Gp_k)(t))_{k=1}^\infty) \\
 &\leq \frac{2}{\Gamma(\alpha)} \int_0^t \psi_t n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \\
 &+ 2 \int_0^t n_2(s) \Phi((p_k(\theta))_{k=1}^\infty) d\theta] ds \\
 &+ \frac{\Psi}{Q_\psi} \left[2a \int_0^\beta \int_0^p \frac{\psi_p}{\Gamma(\alpha)} n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \right. \\
 &+ 2 \int_0^t n_2(s) \Phi((p_k(\theta))_{k=1}^\infty) d\theta] ds dp \\
 &+ 2b \int_\gamma^1 \int_0^p \frac{\psi_p}{\Gamma(\alpha)} n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \\
 &+ 2 \int_0^t n_2(s) \Phi((p_k(\theta))_{k=1}^\infty) d\theta] ds dp \\
 &+ 2 \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} n_1(s) [\Phi((p_k(s))_{k=1}^\infty) \\
 &+ 2 \int_0^t n_2(s) \Phi((p_k(\theta))_{k=1}^\infty) d\theta] ds \Big] \\
 &\leq \frac{2}{\Gamma(\alpha)} \int_0^t \psi_t n_1(s) \left[\sup_{t \in (0,1)} \Phi((p_k(s))_{k=1}^\infty) \right. \\
 &+ 2Tn_2(s) \sup_{t \in (0,1)} \Phi((p_k(\theta))_{k=1}^\infty) \Big] ds \\
 &+ \frac{\Psi}{Q_\psi} \left[2a \int_0^\beta \frac{[\psi(p) - \psi(s)]^\alpha}{\Gamma(\alpha+1)} n_1(s) \left[\sup_{t \in (0,1)} \Phi((p_k(s))_{k=1}^\infty) \right. \right. \\
 &+ 2Tn_2(s) \sup_{t \in (0,1)} \Phi((p_k(\theta))_{k=1}^\infty) \Big] ds dp \\
 &+ 2b \int_\gamma^1 \frac{[\psi(p) - \psi(0)]^\alpha}{\Gamma(\alpha+1)} n_1(s) \left[\sup_{t \in (0,1)} \Phi((p_k(s))_{k=1}^\infty) \right. \\
 &+ 2Tn_2(s) \sup_{t \in (0,1)} \Phi((p_k(\theta))_{k=1}^\infty) \Big] ds dp \\
 &+ 2 \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} n_1(s) \left[\sup_{t \in (0,1)} \Phi((p_k(s))_{k=1}^\infty) \right.
 \end{aligned}$$

$$\begin{aligned}
& + 2Tn_2(s) \sup_{t \in (0,1)} \Phi((p_k(\theta))_{k=1}^\infty) \Big] ds \Big] \\
& \leq \frac{2}{\Gamma(\alpha)} \int_0^t \psi_t e^{Ls} n_1(s) \left[\sup_{t \in (0,1)} e^{-Lt} \Phi((p_k(s))_{k=1}^\infty) \right. \\
& \quad \left. + 2Tn_2(s) \sup_{t \in (0,1)} e^{-Lt} \Phi((p_k(\theta))_{k=1}^\infty) \right] ds \\
& \quad + \frac{\Psi}{Q_\psi} \left[\int_0^\beta \frac{2a[\psi(p) - \psi(s)]^\alpha}{\Gamma(\alpha+1)} \right. \\
& \quad \left. e^{Ls} n_1(s) \left[\sup_{t \in (0,1)} e^{-Lt} \Phi((p_k(s))_{k=1}^\infty) \right. \right. \\
& \quad \left. \left. + 2Tn_2(s) \sup_{t \in (0,1)} e^{-Lt} \Phi((p_k(\theta))_{k=1}^\infty) \right] ds dp \right. \\
& \quad \left. + 2b \int_\gamma^1 \frac{[\psi(p) - \psi(0)]^\alpha}{\Gamma(\alpha+1)} \right. \\
& \quad \left. e^{Ls} n_1(s) \left[\sup_{t \in (0,1)} e^{-Lt} \Phi((p_k(s))_{k=1}^\infty) \right. \right. \\
& \quad \left. \left. + 2Tn_2(s) \sup_{t \in (0,1)} e^{-Lt} \Phi((p_k(\theta))_{k=1}^\infty) \right] ds dp \right. \\
& \quad \left. + 2 \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} e^{Ls} n_1(s) \left[\sup_{t \in (0,1)} e^{-Lt} \Phi((p_k(s))_{k=1}^\infty) \right. \right. \\
& \quad \left. \left. + 2Tn_2(s) \sup_{t \in (0,1)} e^{-Lt} \Phi((p_k(\theta))_{k=1}^\infty) \right] ds \right] \\
& \leq \frac{2\mathfrak{F}(p_k(s))_{k=1}^\infty}{\Gamma(\alpha)} \int_0^t \psi_t [e^{Ls} n_1(s) + 2Te^{Ls} n_1(s) n_2(s)] ds \\
& \quad + \frac{\Psi}{Q_\psi} \left[2a\mathfrak{F}(p_k(s))_{k=1}^\infty \int_0^\beta \frac{[\psi(p) - \psi(s)]^\alpha}{\Gamma(\alpha+1)} [e^{Ls} n_1(s) \right. \\
& \quad \left. + 2Te^{Ls} n_1(s) n_2(s)] ds dp \right. \\
& \quad \left. + 2b\mathfrak{F}(p_k(s))_{k=1}^\infty \int_\gamma^1 \frac{[\psi(p) - \psi(0)]^\alpha}{\Gamma(\alpha+1)} [e^{Ls} n_1(s) \right. \\
& \quad \left. + 2Te^{Ls} n_1(s) n_2(s)] ds dp \right. \\
& \quad \left. + 2\mathfrak{F}(p_k(s))_{k=1}^\infty \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} [e^{Ls} n_1(s) + 2Te^{Ls} n_1(s) n_2(s)] ds \right] \\
& \quad + 2b\mathfrak{F}(p_k(s))_{k=1}^\infty \int_\gamma^1 \frac{[\psi(p) - \psi(0)]^\alpha}{\Gamma(\alpha+1)} [e^{Ls} n_1(s) \\
& \quad + 2Te^{Ls} n_1(s) n_2(s)] ds dp \\
& \quad + 2\mathfrak{F}(p_k(s))_{k=1}^\infty \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} [e^{Ls} n_1(s) + 2Te^{Ls} n_1(s) n_2(s)] ds \Big] \\
& \leq \left[\frac{2}{\Gamma(\alpha)} \int_0^t \psi_t [e^{Ls} n_1(s) + 2Te^{Ls} n_1(s) n_2(s)] ds \right.
\end{aligned}$$

$$\begin{aligned}
& \left. + \frac{\Psi}{Q_\psi} \left[2a \int_0^\beta \frac{[\psi(p) - \psi(s)]^\alpha}{\Gamma(\alpha+1)} [e^{Ls} n_1(s) \right. \right. \\
& \quad \left. \left. + 2Te^{Ls} n_1(s) n_2(s)] ds dp \right. \right. \\
& \quad \left. + 2b \int_\gamma^1 \frac{[\psi(p) - \psi(0)]^\alpha}{\Gamma(\alpha+1)} [e^{Ls} n_1(s) \right. \\
& \quad \left. + 2Te^{Ls} n_1(s) n_2(s)] ds dp \right. \\
& \quad \left. + 2 \int_0^\mu \frac{\psi'(s) [\psi(\mu) - \psi(s)]^{\alpha-1}}{\Gamma(\alpha)} [e^{Ls} n_1(s) \right. \\
& \quad \left. + 2Te^{Ls} n_1(s) n_2(s)] ds \right] \Big] \mathfrak{F}(p_k(s))_{k=1}^\infty.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \mathfrak{F}((y_k)_{k=1}^\infty) \\
& \leq \sup_{0 \in (a,1)} e^{Lt} \left[\frac{2}{\Gamma(\alpha)} \int_0^t \psi_t [e^{Ls} n_1(s) \right. \\
& \quad \left. + 2Te^{Ls} n_1(s) n_2(s)] ds \right. \\
& \quad \left. + \frac{\Psi}{Q_\psi} \left[2a \int_0^\beta \frac{[\psi(p) - \psi(s)]^\alpha}{\Gamma(\alpha+1)} [e^{Ls} n_1(s) \right. \right. \\
& \quad \left. \left. + 2Te^{Ls} n_1(s) n_2(s)] ds dp \right. \right. \\
& \quad \left. + 2b \int_\gamma^1 \frac{[\psi(p) - \psi(0)]^\alpha}{\Gamma(\alpha+1)} [e^{Ls} n_1(s) \right. \\
& \quad \left. + 2Te^{Ls} n_1(s) n_2(s)] ds dp \right. \\
& \quad \left. + 2 \int_0^\mu \frac{\psi_\mu}{\Gamma(\alpha)} [e^{Ls} n_1(s) \right. \\
& \quad \left. + 2Te^{Ls} n_1(s) n_2(s)] ds \right] \Big] \mathfrak{F}((p_k(s))_{k=1}^\infty)
\end{aligned}$$

set

$$\begin{aligned}
L' & = \sup_{0 \in (a,1)} e^{Lt} \left[\frac{2}{\Gamma(\alpha)} \int_0^t \psi_t [e^{Ls} n_1(s) \right. \\
& \quad \left. + 2Te^{Ls} n_1(s) n_2(s)] ds \right. \\
& \quad \left. + \frac{\Psi}{Q_\psi} \left[2a \int_0^\beta \frac{[\psi(p) - \psi(s)]^\alpha}{\Gamma(\alpha+1)} [e^{Ls} n_1(s) \right. \right. \\
& \quad \left. \left. + 2Te^{Ls} n_1(s) n_2(s)] ds dp \right. \right. \\
& \quad \left. + 2b \int_\gamma^1 \frac{[\psi(p) - \psi(0)]^\alpha}{\Gamma(\alpha+1)} [e^{Ls} n_1(s) \right. \\
& \quad \left. + 2Te^{Ls} n_1(s) n_2(s)] ds dp \right. \\
& \quad \left. + 2 \int_0^\mu \frac{\psi'(s) [\psi(\mu) - \psi(s)]^{\alpha-1}}{\Gamma(\alpha)} [e^{Ls} n_1(s) \right. \\
& \quad \left. + 2Te^{Ls} n_1(s) n_2(s)] ds \right] \Big]
\end{aligned}$$

Then, $\mathfrak{F}(y_k)_{k=1}^\infty \leq L' \mathfrak{F}(p_k(s))_{k=1}^\infty$. It is not difficult to choose a suitable constant $0 < L' < 1$.

Therefore,

$$\begin{aligned}\mathfrak{F}((p_k(s))_{k=1}^\infty) &\leq \mathfrak{F}(Z) \leq \mathfrak{F}(\text{conv}(0 \cup G(Z))) \\ &= \mathfrak{F}((y_n)_{k=1}^\infty) \leq L' \mathfrak{F}((p_k(s))_{k=1}^\infty)\end{aligned}$$

$\implies$ that $\mathfrak{F}((p_k(s))_{k=1}^\infty) = 0$ and hence $\mathfrak{F}((y_k)_{k=1}^\infty) = 0$, by the theorem (3.1), the set $(y_k)_{k=1}^\infty$ is equicontinuous on J . Thus, by definition of $\mathfrak{R}$, we can see that $\mathfrak{R}((y_k)_{k=1}^\infty) = (0, 0)$. Since $\mathfrak{R}$ is monotone, nonsingular and regular measures of noncompactness.

Hence, $\mathfrak{R}(Z) \leq \mathfrak{R}(\text{conv}(0 \cup G(Z))) \leq \mathfrak{R}(G(Z))$, where $\mathfrak{R}(G(Z)) = \mathfrak{R}((y_k)_{k=1}^\infty) = (0, 0)$.

As a result, Z is relatively compact. This completes the proof.

5. Example

Consider the following ψ -Caputo fractional integrodifferential equation,

$${}^C D_{a+}^{\frac{3}{2}; \psi} x(t) = \left(\frac{e^{-t}}{9 + e^t} \frac{|x(t)|}{1 + |x(t)|} \int_0^t (e^{-\frac{1}{4}})^t dt \right), t \in J; = [0, \hbar], \quad (36)$$

$$x(0) = \frac{1}{4} + \frac{|x(t_0)|}{12(1 + |x(t_0)|)}, t_0 \in (0, 1)$$

$$x\left(\frac{1}{2}\right) = \frac{1}{2} \int_0^{\frac{3}{10}} x(p) dp + \frac{1}{2} \int_{\frac{7}{10}}^1 x(p) dp, 0 < \beta < s < \gamma < 1$$

Set

$$f(t, x(t), (Rx)(t)) = f\left(\frac{e^{-t}}{9 + e^t}, \frac{|x(t)|}{1 + |x(t)|}, \int_0^t (e^{-\frac{1}{4}})^t dt\right),$$

where $\alpha = \frac{3}{2}$, $n_1(t) = \frac{e^{-t}}{9 + e^t}$, $n_2(t) = e^{-\frac{1}{4}}$, $t = \frac{1}{2}$, $\psi(t) = t$, $T = 1$, $\hbar = 1$

for each bounded sets and $t \in J$, we get

$$N = (n_1(t) + 2Tn_1(t)n_2(t))$$

$$N \approx 0.157$$

Hence $L_f = \frac{3}{10}$, $L_k = \frac{5}{10}$, $T=1$, $\alpha = \frac{3}{2}$, $P = 4.7143$ then,

$$\frac{PL_f(1 + L_k T)}{\Gamma(\alpha + 1)} [\psi(1) - \psi(0)]^\alpha \approx 0.63 \leq 1.$$

which is the existence result for equation (35).

Hence the hypotheses $(C_1) - (C_4)$ holds, here $e_1 = 0.399$, $e_2 = 0.206$, $e_3 = 0.206$, $e_4 = 0.798$ and $Q_\psi = 0.35$ then

$$E = \left[2N \left[e_1 + \frac{\Psi}{Q_\psi} (e_2 + e_3 + e_4) \right] \right] \approx 0.67 < 1$$

Therefore the above condition fulfilled by hypothesis $(C_1) - (C_4)$ and theorem (4.1) and show that the problem (35) has atleast one solution.

6. Conclusions

Abstract In this paper, we consider the existence and uniqueness of solution of a class of nonlinear fractional integro-differential equations with the ψ -Caputo fractional derivative under slit-strip integral boundary conditions. By reducing the considered BVP into an equivalent integral equation, some sufficient conditions on the existence of solutions are presented. The result is established by using nonlinear functional analysis methods, in particular Mönchs fixed-point principle and the Hausdorff measure of noncompactness. The conditions introduced in this paper give good estimates for the nonlinear terms and the integral operators appearing in the considered problem. Many of the obtained results generalize some known results in the literature on from fractional boundary value problems by considering generalized ψ -Caputo fractional derivative without and with nonlocal slit-strip integral boundary conditions. Also, the uniqueness of solutions is established under some additional appropriate hypotheses. In fact, the theoretical results of this paper can be applied to many other types of nonlinear fractional systems such as nonlocal boundary value problems, fractional dynamic models and mathematical models with memory and hereditary properties. The development of fractional calculus has offered a new way to study complex systems, especially the fractional operators in Banach spaces considered here. An example is given to illustrate and verify the effectiveness of the obtained results. From the example, we can see that the assumptions presented here are feasible, and the theoretical results are valid. In the future, one could consider generalizing these results to more general fractional operators, impulsive systems and fractional models in higher dimensions.

Abbreviations

BVP	Boundary Value Problem
RL	Riemann-Liouville
CFD	Caputo Fractional Derivatives
ψ -CFIDE	ψ -Caputo Fractional Integro-Differential Equation
MNC	Measure of Non Compactness

Author Contributions

Manikandan Ambigapathi: Conceptualization, Data curation, Formal Analysis, Investigation, Methodology, Project administration, Writing – original draft

Karthikeyan Panjaiyan: Conceptualization, Software, Supervision, Validation, Writing – review & editing

Conflicts of Interest

The authors declare no conflicts of interest.

References

- [1] M. S. Abdo, S. K. Panchal, A. M. Saeed, Fractional boundary value problem with ψ -Caputo fractional derivative, *Proceeding Mathematical Science*. (2019), 129, 1–14. <https://doi.org/10.1007/s12044-019-0514-8>
- [2] M. S. Abdo, A. G. Ibrahim, S. K. Panchal, Nonlinear implicit fractional differential equation involving ψ -Caputo fractional derivative, *Proceedings of the Jangjeon Mathematical Society*. (2019), 22(3), 387–400. <http://dx.doi.org/10.17777/pjms2019.22.3.387xt>
- [3] R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Communications in Nonlinear Science and Numerical Simulation*, (2017), 44, 460–481. <https://doi.org/10.48550/arXiv.1609.04775t>
- [4] Almeida R, Malinowska A. B, Monteiro M. T, Fractional differential equations with a Caputo derivative with respect to a kernel function and their application, *Mathematical Methods in the Applied Sciences*, (2018), 41(1) 336–352.
- [5] R. Arul, P. Karthikeyan, On nonlinear ψ -Caputo fractional integro differential equations involving non-instantaneous conditions, *Symmetry*, (2023), 15(5) 1–12. <https://doi.org/10.3390/sym15010005t>
- [6] Z. Baitichea, C. Derbazia, M. Benchohrab, ψ -Caputo fractional differential equations with multi-point boundary conditions by topological degree theory, *Results in Nonlinear Analysis*, (2020), 3(4), 167–178.
- [7] T. Zajac, The Concept of Measures of Noncompactness in Banach Spaces, *Symmetry*, 17(8), 1248, 2025.
- [8] A. Boutiara, M. Benbachir, Measure of noncompactness for nonlinear Hilfer fractional differential equation with mixed fractional integral boundary conditions in Banach spaces, *Journal of Innovative Applied Mathematics and Computational Sciences*, 2(1) (2021), 1–17. <https://doi.org/10.1155/2020/2875152>
- [9] A. Boutiara, M. Benbachir, Implicit fractional differential equation involving ψ -Caputo with boundary conditions, *Bulletin of the Institute of Mathematics*, 16(1), (2021), 1–19. <https://doi.org/10.1155/2020/8821471>
- [10] D. B. Dhaigude, V. S. Gore, P. D. Kundgar, Existence and uniqueness of solution of nonlinear boundary value problems for ψ -Caputo fractional differential equations, *Malaya Journal of Matematik*, 9(1), (2021), 112–117. <https://doi.org/10.26637/MJM0901/0019>
- [11] Y. Dimitrov, S. Georgiev, V. Todorov, Approximation of Caputo fractional derivative and numerical solutions of fractional differential equations, *Fractal and Fractional*, (2023), 7(10), 1–26. <https://doi.org/10.20944/preprints202309.0337.v1>
- [12] P. Karthikeyan, R. Arul, Integral boundary value problems for implicit fractional differential equations involving Hadamard and Caputo fractional derivatives, *Kragujevac Journal of Mathematics*, (2021), 45(3), 331–341. <https://doi.org/10.46793/KgJMat2103.331K>
- [13] A. EL. Mfadel, S. Melliani, M. Elomari, Existence and uniqueness results of boundary value problems for nonlinear fractional equations involving ψ -Caputo type fractional derivatives, *Acta Mathematica Universitatis Comenianae*, (2023), XCII(1), 23–33.
- [14] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific, Singapore, 2000, 87–130. <https://doi.org/10.1142/3779>
- [15] K. Karthikeyan, J. J. Trujillo, Existence and uniqueness results for fractional integrodifferential equations with boundary value conditions, *Communications in Nonlinear Science and Numerical Simulation*, 17(11), (2012), 4037–4043. <https://doi.org/10.1016/j.cnsns.2011.11.036>
- [16] P. Karthikeyan, A. Manikandan, Mohammed Rabi, M. Sathiskumar, On the Existence of Solutions for ψ -Caputo Fractional Integro-Differential Boundary Value Problems, *Fractal and Fractional*, (2026), 10(348). <https://doi.org/10.3390/fractalfract10050348>
- [17] P. Karthikeyan, A. Manikandan, D. Vijay, Some results on ψ -Caputo fractional integro differential equations, *Journal of Fractional Calculus and Applications*, (2025), 15(25), 1–16. <https://doi.org/10.21608/jfca.2025.319322.1132>
- [18] P. Karthikeyan, A. Manikandan, D. Vijay, Existence results for fractional integro differential equations involving two boundary conditions with ψ -Caputo fractional derivative, *Palestine Journal of Mathematics*, (2025), 14(4), 352–366.
- [19] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006. [https://doi.org/10.1016/S0304-0208\(06\)80001-0](https://doi.org/10.1016/S0304-0208(06)80001-0)
- [20] K. S. Miller, B. Ross, *An introduction to the fractional calculus and differential equations*, Wiley, New York, 1993. <https://doi.org/10.12691/ajma-3-3-1>
- [21] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.
- [22] Y. Sun, Z. Zeng, J. Song, Existence and uniqueness for the boundary value problems of nonlinear fractional differential equation, *Applied Mathematics*, (2017), 8(3), 312–323. <https://doi.org/10.4236/am.2017.83026>
- [23] T. M. S. Thabet, M. B. Dhakne, On abstract fractional integro-differential equations via measure of noncompactness, *Advances in Fixed Point Theory*, (2016), 6(2), 175–193.

- [24] T. M. S. Thabet, M. B. Dhakne, On nonlinear fractional integro-differential equations with two boundary conditions, *Advanced Studies in Contemporary Mathematics*, (2016), 26(3), 513–526.
- [25] H. A. Wahasha, S. K. Panchal, M. S. Abdo, Existence and stability of a nonlinear fractional differential equation involving a ψ -Caputo operator, *Advances in the Theory of Nonlinear Analysis and its Applications*, (2020), 4(4), 266–278. <https://doi.org/10.31197/atnaa.664534>
- [26] F. Wang, Z. Liu, Anti-periodic fractional boundary value problems for nonlinear differential equations of fractional order, *Advances in Difference Equations*, (2012), 2012(116), 1–13. <https://doi.org/10.1186/1687-1847-2012-116>
- [27] J. R. Wang, L. Lv, Y. Zhou, Boundary value problems for fractional differential equations involving Caputo derivative in Banach spaces, *Journal of Applied Mathematics and Computing*, (2012), 38(1), 209–224. <https://doi.org/10.1007/s12190-011-0474-3>