Mathematics and Computer Science
2025; 10(1): 19-25
http://www.sciencepublishinggroup.com/j/mcs
doi: 10.11648/j.mcs.20251001.13
ISSN: 2575-6036 (Print); ISSN: 2575-6028 (Online)

Graph Entropy Based on Wiener Polarity Index Under Four Kinds of Graph Operations

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To cite this article:
Chen Yang, Chongmin Li. (2025). Graph Entropy Based on Wiener Polarity Index Under Four Kinds of Graph Operations. *Mathematics and Computer Science, 10*(1), 19-25. https://doi.org/10.11648/j.mcs.20251001.13

Received: 14 December 2024; **Accepted:** 25 December 2024; **Published:** 17 February 2025

Abstract: The Wiener polarity index of a graph G, is the number of unordered pairs of vertices that are at distance 3 in G. This index can reflect the specific distance relation between vertices in the graph, and provides a new way for the study of graph structure. In this paper, the graph entropy based on Wiener polarity index defined. Based on the above definition of graph entropy, it compares the graph entropy of path and balanced double star graphs based on Wiener polarity index. The expressions of graph entropy based on Wiener polarity index for trees with diameter $d \geq 3$ are studied under four graph operations: tensor product, strong product, Cartesian product and composite graph.

Keywords: Wiener Polarity Index, Tree, Graph Entropy, Graph Operation

1. Introduction

The concept of graph entropy was introduced by Shannon in 1948 [1]. With the research on graph entropy conducted by Dehmer, Mowshowitz, and others [2, 3], it has gradually evolved into an independent research field. Graph distance entropy is a significant research direction at the intersection of graph theory and information theory. In graph theory, distance entropy is used to quantify the uncertainty or complexity of the distance distribution between vertices or vertex pairs, studying the structural characteristics of graphs by calculating the entropy value based on the probability distribution of distances within the graph. Cao et al. [4] were the first to propose graph entropy based on vertex degree powers and investigated the bounds of graph entropy for various graph classes (such as tree graphs, unicyclic graphs, bicyclic graphs, etc.), as well as the corresponding extremal graphs. Lu et al. [5] obtained new bounds for graph entropy using Jensen's inequality and the minimum-maximum degree. Chen et al. [6] defined distance entropy based on the number of vertices with a distance of k from a given vertex and studied the extremal graphs for the entropy of trees. Dehmer et al. [7] provided an upper bound for distance entropy. Dong et al. [8] defined the Wiener entropy of graphs and obtained related results. With in-depth research on graph entropy, its concept has also been extended to hypergraphs [9, 10]. Other graph invariants, namely the number of vertices, degree sequence and distance, are also used in the graph entropy measure, and the research results can be referred to [11-15]. Graph arithmetic is widely used not only in computer science, but also in many topological index literature (such as Wiener index, Wiener polarity index, Zagreb index, Randić index, etc.).

Currently, there are relatively few research results related to graph distance entropy. This paper primarily focuses on the calculation of graph distance entropy. Specifically, this paper investigates graph entropy based on the Wiener polarity index. Through calculations, it obtains the graph entropy based on the Wiener polarity index for path graphs and balanced double-star graphs, as well as the expressions for graph entropy based on the Wiener polarity index for trees with a diameter of ≥ 3 under four graph operations: tensor product, strong product, Cartesian product, and composite graphs.

2. Preliminary Knowledge

Let $G = (V, E)$ is a simple connected graph with a set of vertices $v \in V(G)$ and a set of edges $e \in E(G)$. All vertices adjacent to vertex u are called neighbors of u. The neighborhood of u is the set of the neighbors of u. The number of edges adjacent to vertex u is the degree of u, denoted by $d(u)$. Vertices of degrees 0 and 1 are said to be isolated and pendent vertices, respectively. The distance between two vertices u and v, denoted by $d(u,v)$, is the length of the shortest path from u to v. The diameter $diam(G)$ of G is the maximum eccentricity among all vertices of G. The radius $rad(G)$ is the minimum eccentricity among all vertices of G. For a vertex $v \in V$, the $j-sphere$ of v is the set of vertices at distance j to v denoted by $S_j(v,G)$. For vertices $u \in V(G)$, the degree in which the vertex u is in G is represented by $d_G(u)$, record briefly as $d(u)$. For two simple graphs G and graphs H, Their tensor product, strong product, Cartesian product and composite graph are respectively expressed as $G \odot H$, $G \otimes H$, $G \times H$, $G[H]$. The Zagreb index [6] is defined as $M_1(G) = \sum_{v \in V(G)} d_G(v)^2 = \sum_{uv \in E(G)} (d_G(u) + d_G(v))$, $M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v)$, where $M_1(G)$ and $M_2(G)$ are the first and second Zagreb indices, respectively. All the logarithmic functions in this paper have base 2 as their based.

Definition 2.1. *[2] Let $G = (V,E)$ be the simple connected graph with n vertices. Define the probability function of the vertex $v_i \in V(G)$ as $p(v_i) = \frac{f(v_i)}{\sum_{j=1}^n f(v_j)}$. Then*

$$
\begin{aligned}
I_f(G) &= -\sum_{i=1}^n p(v_i) \log p(v_i) \\
&= -\sum_{i=1}^n \frac{f(v_i)}{\sum_{j=1}^n f(v_j)} \log \frac{f(v_i)}{\sum_{j=1}^n f(v_j)}. \qquad (1)
\end{aligned}
$$

As distance is an important graph invariant, a new distance-based graph entropy is defined below.

Definition 2.2. *[6] Let $G = (V,E)$ be the simple connected graph with n vertices. For vertex $v_i \in V$, $n_k(v_i)$ represents the number of vertices v_i with a distance of k between the vertex and all other vertices. Then the general form of distance entropy is*

$$
\begin{aligned}
I_k(G) &= -\sum_{i=1}^n \frac{n_k(v_i)}{\sum_{j=1}^n n_k(v_i)} \log \frac{n_k(v_i)}{\sum_{j=1}^n n_k(v_i)} \\
&= \log[\sum_{i=1}^n n_k(v_i)] - \frac{\sum_{i=1}^n n_k(v_i) \log n_k(v_i)}{\sum_{j=1}^n n_k(v_i)}, \qquad (2)
\end{aligned}
$$

where $n_k(v_i) = |S_k(v_i,G)| = |u : d(u,v_i) = k, u \in V(G)|$, k is an integer and $1 \le k \le D(G)$.

The graph entropy based on the Wiener polarity index can be obtained.

Definition 2.3. *[8] Let $G = (V,E)$ be the simple connected graph with n vertices. For vertex $v_i \in V$, $n_k(v_i)$ represents the number of vertices v_i with a distance of k between the vertex and all other vertices, the graph entropy based on the Wiener polarity index is*

$$
\begin{aligned}
I_3(G) &= \log[2W_3(G)] \\
&\quad - \frac{1}{2W_3(G)} \cdot \sum_{i=1}^n n_3(v_i) \log n_3(v_i), \qquad (3)
\end{aligned}
$$

where $n_3(v_i) = |S_k(v_i,G)| = |u : d(u,v_i) = 3, u \in V(G)|$, $\sum_{i=1}^n n_3(v_i) = 2W_3(G)$.

Definition 2.4. *[6] Let $T = (V,E)$ be a tree with n vertices. Since $W_3(T) = \sum_{uv \in E(T)} (d(u)-1)(d(v)-1)$, then*

$$
\begin{aligned}
I_3(T) &= \log(2 \sum_{uv \in E(T)} (d(u)-1)(d(v)-1)) \\
&\quad - \frac{\sum_{i=1}^n n_3(v_i) \log n_3(v_i)}{2 \sum_{uv \in E(T)} (d(u)-1)(d(v)-1)}. \qquad (4)
\end{aligned}
$$

If $T \cong S_n$, then $I_3(S_n) = -\infty$;

If $T \cong P_n$, then $I_3(P_n) = \log(n-3) + \frac{3}{n-3}, n \ge 7$.

Let $S_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ be the balanced double star graph with n vertices. It can be obtained by simple calculation $I_3(S_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) = \log(2\sqrt{n_1 n_2})$, where n_1 and n_2 be the pendent vertices on the left and right sides of the balanced double star graph $S_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, and $n_1 + n_2 + 2 = n$.

Theorem 2.1. *Let P_n, $S_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ be the path graph and the balanced double star graph with n vertices, respectively. Then*

$$
I_3(P_n) > I_3(S_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}).
$$

Proof.

$$
\begin{aligned}
&I_3(P_n) - I_3(S_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) \\
&= \log(n-3) + \frac{3}{n-3} - \log(2\sqrt{n_1 n_2}) \\
&= \log(\frac{n_1 + n_2 - 1}{2\sqrt{n_1 n_2}}) + \frac{3}{n_1 + n_2 - 1} > 0,
\end{aligned}
$$

then $I_3(P_n) > I_3(S_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})$.

Lemma 2.1. *[6] Let T_1 and T_2 be the trees with the number of vertices m and n and the number of edges p and q, respectively. The graph obtained after tensor product operation is $T_1 \otimes T_2$, where $|V(T_1 \otimes T_2)| = |V(T_1)||V(T_2)| = mn$, $E(T_1 \otimes T_2) = 2|E(T_1)||E(T_2)| = 2pq$, then for $T_1 \otimes T_2$, the degree of the vertex (u_i, v_j) is expressed as $d_{T_1 \otimes T_2}((u_i, v_j)) = d_{T_1}(u_i) d_{T_2}(v_j)$.*

Lemma 2.2. *[6] Let T_1 and T_2 be the trees with the number of vertices m and n and the number of edges p and q, respectively. The graph obtained after strong product operation is $T_1 \odot T_2$, where $|V(T_1 \odot T_2)| = |V(T_1)||V(T_2)| = mn$, $E(T_1 \odot T_2) = |V(T_1)||E(T_2)| + |E(T_1)||V(T_2)| + 2|E(T_1)||E(T_2)| = mq + np + 2pq$, then for $T_1 \odot T_2$, the degree of the vertex (u_i, v_j) is expressed as $d_{T_1 \odot T_2}((u_i, v_j)) = d_{T_1}(u_i) + d_{T_2}(v_j) + d_{T_1}(u_i) d_{T_2}(v_j)$.*

Lemma 2.3. *[6] Let T_1 and T_2 be the trees with the number of vertices m and n and the number of edges p and q, respectively. The graph obtained after Cartesian product* *operation is $T_1 \times T_2$,, where $|V(T_1 \times T_2)| = |V(T_1)||V(T_2)| = mn$, $E(T_1 \times T_2) = |E(T_1)||E(T_2)| + |E(T_1)||V(T_2)| = pq + pn$, then for $T_1 \times T_2$, the degree of the vertex (u_i, v_j) is expressed as $d_{T_1 \times T_2}((u_i, v_j)) = d_{T_1}(u_i) + d_{T_2}(v_j)$.*

Lemma 2.4. *[6] Let T_1 and T_2 be the trees with the number of vertices m and n and the number of edges p and q, respectively. The graph obtained after tensor product operation is $T_1[T_2]$, where $|V(T_1[T_2])| = |V(T_1)||V(T_2)| = mn$, $E(T_1[T_2]) = |E(T_1)||E(T_2)| + |E(T_1)||V(T_2)|^2 = pq + pn^2$, then for $T_1[T_2]$, the degree of the vertex (u_i, v_j) is expressed as $d_{T_1[T_2]}((u_i, v_j)) = nd_{T_1}(u_i) + d_{T_2}(v_j)$.*

3. Main Results

In this section, T_1 and T_2 are trees with diameter $d \geq 3$ respectively. The graph entropy based on distance equal to 3 under four graph operations is obtained mainly through calculation.

Theorem 3.1. *[6] Let T_1 and T_2 be the trees with the number of vertices m and n and the number of edges p and q, respectively. The graph obtained after tensor product operation is $T_1 \otimes T_2$. Then*

$$I_3(T_1 \otimes T_2) = \log(4M_2(T_1)M_2(T_2) - 32pq + 4mn) - \frac{1}{4M_2(T_1)M_2(T_2) - 32pq + 4mn} \cdot \sum_{w=1}^{m}\sum_{r=1}^{n}\Big(\sum_{N^2_{T_1}(u_w), N^2_{T_2}(v_r)} d_{T_1}(u_w)d_{T_2}(v_r) - 1\Big)\log\Big(\sum_{N^2_{T_1}(u_w), N^2_{T_2}(v_r)} d_{T_1}(u_w)d_{T_2}(v_r) - 1\Big),$$

where $N^2_{T_1}(u_w) = \{u_i \in V(T_1)|d(u_i, u_w) = 2\}$, $N^2_{T_2}(v_r) = \{v_j \in V(T_2)|d(v_j, v_r) = 2\}$,
$M_2(T_1) = \sum_{uv \in E(T_1)} d_{T_1}(u)d_{T_1}(v)$, $M_2(T_2) = \sum_{uv \in E(T_2)} d_{T_2}(u)d_{T_2}(v)$.

Proof.

By Lemma 2.1, we have

$$\begin{aligned} W_3(T_1 \otimes T_2) &= \sum_{(u_i,v_j)(u_k,v_r) \in E(T_1 \otimes T_2)} (d_{T_1 \otimes T_2}(u_i, v_j) - 1)(d_{T_1 \otimes T_2}(u_k, v_r) - 1) \\ &= 2\sum_{u_iu_k \in E(T_1)}\sum_{v_jv_r \in E(T_2)} (d_{T_1}(u_i)d_{T_2}(v_j) - 1)(d_{T_1}(u_k)d_{T_2}(v_r) - 1) = 2M_2(T_1)M_2(T_2) - 16pq + 2mn. \end{aligned}$$

Since $\sum_{i=1}^{n} n_3(v_i)\log n_3(v_i) = \sum_{j=1}^{n}(\sum_{N^2_T(w_j)}(d_T(w_j) - 1))\log(\sum_{N^2_T(w_j)}(d_T(w_j) - 1))$,
where $N^2_T(w_j) = \{u \in V(T)|d(u, w_j) = 2\}$. Then

$$\sum_{w=1}^{m}\sum_{r=1}^{n}\Big(\sum_{N^2_{T_1}(u_w), N^2_{T_2}(v_r)} d_{T_1}(u_w)d_{T_2}(v_r) - 1\Big)\log\Big(\sum_{N^2_{T_1}(u_w), N^2_{T_2}(v_r)} d_{T_1}(u_w)d_{T_2}(v_r) - 1\Big).$$

By calculation

$$I_3(T_1 \otimes T_2) = \log(4M_2(T_1)M_2(T_2) - 32pq + 4mn) - \frac{1}{4M_2(T_1)M_2(T_2) - 32pq + 4mn} \cdot \sum_{w=1}^{m}\sum_{r=1}^{n}\Big(\sum_{N^2_{T_1}(u_w), N^2_{T_2}(v_r)} d_{T_1}(u_w)d_{T_2}(v_r) - 1\Big)\log\Big(\sum_{N^2_{T_1}(u_w), N^2_{T_2}(v_r)} d_{T_1}(u_w)d_{T_2}(v_r) - 1\Big).$$

Corollary 3.1. Let P_{n_1} and P_{n_2} are connected paths with diameter $d \geq 3$ respectively. P_{n_1} and P_{n_2} after the tensor product operation to produce the $P_{n_1} \otimes P_{n_2}$. Then

$$I_3(P_{n_1} \otimes P_{n_2}) = \log(4(249n_1n_2 - 216n_1 - 216n_2 + 188) - \frac{10 + 5\log 5 + 3(n_1 + n_2 - 16)\log(n_1 + n_2 - 16)}{4(249n_1n_2 - 216n_1 - 216n_2 + 188)}.$$

Proof.

By Theorem 3.1, we have

$$W_3(P_{n_1} \otimes P_{n_2}) = 4(249n_1n_2 - 216n_1 - 216n_2 + 188,$$
$$\sum_{i=1}^{n} n_3(v_i)\log n_3(v_i) = 10 + 5\log 5 + 3(n_1 + n_2 - 16)\log(n_1 + n_2 - 16).$$

Theorem 3.2. *[6] Let T_1 and T_2 be the trees with the number of vertices m and n and the number of edges p and q, respectively. The graph obtained after strong product operation is $T_1 \odot T_2$. Then*

$$I_3(T_1 \odot T_2) = \log(M) - \frac{1}{M} \cdot \sum_{w=1}^{m}\sum_{r=1}^{n}(\sum_{N_{T_1}^2(u_w), N_{T_2}^2(v_r)} (d_{T_1}(u_w) + d_{T_2}(v_r) + d_{T_1}(u_w)d_{T_2}(v_r) - 1)$$
$$\cdot \log(\sum_{N_{T_1}^2(u_w), N_{T_2}^2(v_r)} d_{T_1}(u_w) + d_{T_2}(v_r) + d_{T_1}(u_w)d_{T_2}(v_r))),$$

where $N_{T_1}^2(u_w) = \{u_i \in V(T_1) | d(u_i, u_w) = 2\}$, $N_{T_2}^2(v_r) = \{v_j \in V(T_2) | d(v_j, v_r) = 2\}$,
$M_2(T_1) = \sum_{uv \in E(T_1)} d_{T_1}(u)d_{T_1}(v)$, $M_2(T_2) = \sum_{uv \in E(T_2)} d_{T_2}(u)d_{T_2}(v)$, $M_1(T_1) = \sum_{v \in V(T_1)} d_{T_1}(v)^2$,
$M_1(T_2) = \sum_{v \in V(T_2)} d_{T_2}(v)^2$, $M = 2W_3(T_1 \odot T_2) = 4M_2(T_1)M_2(T_2) + 20qM_2(T_1) + 20pM_2(T_2) - 28pq$.
Proof.
By Lemma 2.2, we have

$$\begin{aligned} W_3(T_1 \odot T_2) &= \sum_{(u_i,v_j)(u_k,v_r) \in E(T_1 \odot T_2)} (d_{T_1 \odot T_2}(u_i, v_j) - 1)(d_{T_1 \odot T_2}(u_k, v_r) - 1) \\ &= 2 \sum_{u_i u_k \in E(T_1)} \sum_{v_j v_r \in E(T_2)} (d_{T_1}(u_i)d_{T_2}(v_j) + d_{T_1}(u_i) + d_{T_2}(v_j) - 1) \\ &\cdot (d_{T_1}(u_k)d_{T_2}(v_r) + d_{T_1}(u_k) + d_{T_2}(v_r) - 1) \\ &= 2M_2(T_1)M_2(T_2) + 10qM_2(T_1) + 10pM_2(T_2) - 14pq. \end{aligned}$$

Since $\sum_{i=1}^{n} n_3(v_i) \log n_3(v_i) = \sum_{j=1}^{n}(\sum_{N_T^2(w_j)}(d_T(w_j) - 1)) \log(\sum_{N_T^2(w_j)}(d_T(w_j) - 1))$,
where $N_T^2(w_j) = \{u \in V(T) | d(u, w_j) = 2\}$. *Then*

$$\sum_{w=1}^{m}\sum_{r=1}^{n}(\sum_{N_{T_1}^2(u_w), N_{T_2}^2(v_r)} d_{T_1}(u_w)d_{T_2}(v_r) + d_{T_1}(u_w) + d_{T_2}(v_r) - 1)$$
$$\cdot \log(\sum_{N_{T_1}^2(u_w), N_{T_2}^2(v_r)} d_{T_1}(u_w)d_{T_2}(v_r) + d_{T_1}(u_w) + d_{T_2}(v_r) - 1).$$

By calculation

$$I_3(T_1 \odot T_2) = \log(M) - \frac{1}{M} \cdot \sum_{w=1}^{m}\sum_{r=1}^{n}(\sum_{N_{T_1}^2(u_w), N_{T_2}^2(v_r)} (d_{T_1}(u_w) + d_{T_2}(v_r) + d_{T_1}(u_w)d_{T_2}(v_r) - 1)$$
$$\cdot \log(\sum_{N_{T_1}^2(u_w), N_{T_2}^2(v_r)} d_{T_1}(u_w) + d_{T_2}(v_r) + d_{T_1}(u_w)d_{T_2}(v_r) - 1).$$

Let $X = 4(409n_1n_2 - 295n_1 - 295n_2 + 203), Y = [n_1n_2(1 + n_1)(1 + n_2)(158 + 108 \log 3 + 5 \log 5 + 2(n_1 + n_2 - 14))]$, the following corollary is obtained

Corollary 3.2. Let P_{n_1} and P_{n_2} are connected paths with diameter $d \geq 3$ respectively. P_{n_1} and P_{n_2} after the strong product operation to produce the $P_{n_1} \odot P_{n_2}$. Then

$$I_3(P_{n_1} \odot P_{n_2}) = \log X - \frac{Y}{X}.$$

Proof.
By Theorem 3.2, we have

$$W_3(P_{n_1} \odot P_{n_2}) = 4(409n_1n_2 - 295n_1 - 295n_2 + 203),$$
$$\sum_{i=1}^{n} n_3(v_i) \log n_3(v_i) = [n_1n_2(1 + n_1)(1 + n_2)(158 + 108 \log 3 + 5 \log 5 + 2(n_1 + n_2 - 14))].$$

Theorem 3.3. *[6] Let T_1 and T_2 be the trees with the number of vertices m and n and the number of edges p and q, respectively. The graph obtained after Cartesian product operation is $T_1 \times T_2$. Then*

$$I_3(T_1 \times T_2) = \log(2qM_2(T_1) + 2pM_2(T_2) + 2pq) - \frac{1}{2qM_2(T_1) + 2pM_2(T_2) + 2pq} \cdot \sum_{w=1}^{m}\sum_{r=1}^{n}\Big(\sum_{N^2_{T_1}(u_w),N^2_{T_2}(v_r)} d_{T_1}(u_w) + d_{T_2}(v_r) - 1\Big)\log\Big(\sum_{N^2_{T_1}(u_w),N^2_{T_2}(v_r)} d_{T_1}(u_w) + d_{T_2}(v_r) - 1\Big),$$

where $N^2_{T_1}(u_w) = \{u_i \in V(T_1)|d(u_i,u_w) = 2\}$, $N^2_{T_2}(v_r) = \{v_j \in V(T_2)|d(v_j,v_r) = 2\}$,
$M_2(T_1) = \sum_{uv\in E(T_1)} d_{T_1}(u)d_{T_1}(v)$, $M_2(T_2) = \sum_{uv\in E(T_2)} d_{T_2}(u)d_{T_2}(v)$.
Proof.
By Lemma 2.3, we have

$$\begin{aligned}W_3(T_1 \times T_2) &= \sum_{(u_i,v_j)(u_k,v_r)\in E(T_1\times T_2)} (d_{T_1\times T_2}(u_i,v_j) - 1)(d_{T_1\times T_2}(u_k,v_r) - 1)\\ &= \sum_{u_iu_k\in E(T_1)}\sum_{v_jv_r\in E(T_2)} (d_{T_1}(u_i) + d_{T_2}(v_j) - 1)(d_{T_1}(u_k) + d_{T_2}(v_r) - 1)\\ &= qM_2(T_1) + pM_2(T_2) + pq.\end{aligned}$$

Since $\sum_{i=1}^{n} n_3(v_i)\log n_3(v_i) = \sum_{j=1}^{n}(\sum_{N^2_T(w_j)}(d_T(w_j) - 1))\log(\sum_{N^2_T(w_j)}(d_T(w_j) - 1))$,
where $N^2_T(w_j) = \{u \in V(T)|d(u,w_j) = 2\}$.
Then $\sum_{h=1}^{mn}(\sum_{N^2_{T_1}(u_w),N^2_{T_2}(v_r)} d_{T_1}(u_w) + d_{T_2}(v_r) - 1)\log(\sum_{N^2_{T_1}(u_w),N^2_{T_2}(v_r)} d_{T_1}(u_w) + d_{T_2}(v_r) - 1)$.
By calculation

$$I_3(T_1 \times T_2) = \log(2qM_2(T_1) + 2pM_2(T_2) + 2pq) - \frac{1}{2qM_2(T_1) + 2pM_2(T_2) + 2pq} \cdot \sum_{w=1}^{m}\sum_{r=1}^{n}\Big(\sum_{N^2_{T_1}(u_w),N^2_{T_2}(v_r)} d_{T_1}(u_w) + d_{T_2}(v_r) - 1\Big)\log\Big(\sum_{N^2_{T_1}(u_w),N^2_{T_2}(v_r)} d_{T_1}(u_w) + d_{T_2}(v_r) - 1\Big).$$

Corollary 3.3. Let P_{n_1} and P_{n_2} are connected paths with diameter $d \geq 3$ respectively. P_{n_1} and P_{n_2} after the Cartesian product operation to produce the $P_{n_1} \times P_{n_2}$. Then

$$I_3(P_{n_1} \times P_{n_2}) = \log(7n_1n_2 - 16) - \frac{16 + 6(n_1n_2 - 4)\log 3}{7n_1n_2 - 16}.$$

Proof.
By Theorem 3.3, we have

$$\begin{aligned}&W_3(P_{n_1} \times P_{n_2}) = 7n_1n_2 - 16,\\ &\sum_{i=1}^{n} n_3(v_i)\log n_3(v_i) = 16 + 6(n_1n_2 - 4)\log 3.\end{aligned}$$

Theorem 3.4. *[6] Let T_1 and T_2 be the trees with the number of vertices m and n and the number of edges p and q, respectively. The graph obtained after composite graph operation is $T_1[T_2]$. Then*

$$I_3(T_1[T_2]) = \log(2n^2qM_2(T_1) + 2pM_2(T_2) + 10npq - 6pq) - \frac{1}{2n^2qM_2(T_1) + 2pM_2(T_2) + 10npq - 6pq} \cdot \sum_{w=1}^{m}\sum_{r=1}^{n}\Big(\sum_{N^2_{T_1}(u_w),N^2_{T_2}(v_r)} nd_{T_1}(u_w) + d_{T_2}(v_r) - 1\Big)\log\Big(\sum_{N^2_{T_1}(u_w),N^2_{T_2}(v_r)} nd_{T_1}(u_w) + d_{T_2}(v_r) - 1\Big),$$

where $N^2_{T_1}(u_w) = \{u_i \in V(T_1)|d(u_i,u_w) = 2\}$, $N^2_{T_2}(v_r) = \{v_j \in V(T_2)|d(v_j,v_r) = 2\}$,
$M_2(T_1) = \sum_{uv\in E(T_1)} d_{T_1}(u)d_{T_1}(v)$, $M_2(T_2) = \sum_{uv\in E(T_2)} d_{T_2}(u)d_{T_2}(v)$.
Proof.

By Lemma 2.4, we have

$$\begin{aligned} W_3(T_1[T_2]) &= \sum_{(u_i,v_j)(u_k,v_r)\in E(T_1[T_2])} (d_{T_1[T_2]}(u_i,v_j)-1)(d_{T_1[T_2]}(u_k,v_r)-1) \\ &= \sum_{u_iu_k\in E(T_1)}\sum_{v_jv_r\in E(T_2)} (nd_{T_1}(u_i)+d_{T_2}(v_j)-1)(nd_{T_1}(u_k)+d_{T_2}(v_r)-1) \\ &= n^2qM_2(T_1)+pM_2(T_2)+5npq-3pq. \end{aligned}$$

Since $\sum_{i=1}^{n} n_3(v_i)\log n_3(v_i) = \sum_{j=1}^{n}(\sum_{N_T^2(w_j)}(d_T(w_j)-1))\log(\sum_{N_T^2(w_j)}(d_T(w_j)-1))$, *where* $N_T^2(w_j)=\{u\in V(T)|d(u,w_j)=2\}$. *Then*

$$\sum_{w=1}^{m}\sum_{r=1}^{n}(\sum_{N_{T_1}^2(u_w),N_{T_2}^2(v_r)} nd_{T_1}(u_w)+d_{T_2}(v_r)-1)\log(\sum_{N_{T_1}^2(u_w),N_{T_2}^2(v_r)} nd_{T_1}(u_w)+d_{T_2}(v_r)-1).$$

By calculation

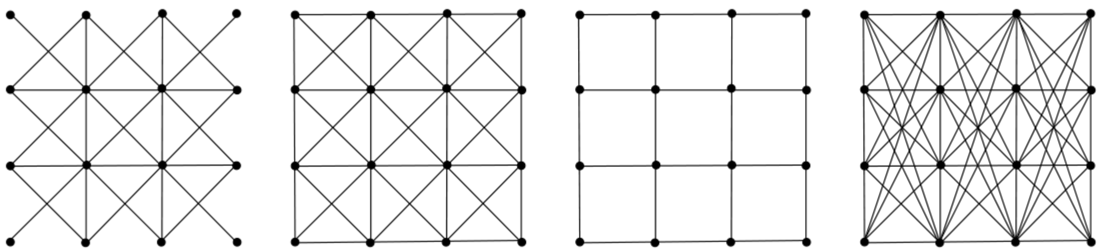
$$\begin{aligned} I_3(T_1[T_2]) &= \log(2n^2qM_2(T_1)+2pM_2(T_2)+10npq-6pq)-\frac{1}{2n^2qM_2(T_1)+2pM_2(T_2)+10npq-6pq} \\ &\cdot\sum_{w=1}^{m}\sum_{r=1}^{n}(\sum_{N_{T_1}^2(u_w),N_{T_2}^2(v_r)} nd_{T_1}(u_w)+d_{T_2}(v_r)-1)\log(\sum_{N_{T_1}^2(u_w),N_{T_2}^2(v_r)} nd_{T_1}(u_w)+d_{T_2}(v_r)-1). \end{aligned}$$

Corollary 3.4. Let P_{n_1} and P_{n_2} are connected paths with diameter $d\geq 3$ respectively. P_{n_1} and P_{n_2} after the composite graph operation to produce the $P_{n_1}[P_{n_2}]$. Then

$$\begin{aligned} I_3(P_{n_1}[P_{n_2}]) = \log[(n_1-1)(n_2-1)(4n_2^2+5n_2+1)] &- \frac{1}{(n_1-1)(n_2-1)(4n_2^2+5n_2+1)} \\ .[12(2n_2+1)\log(2n_2+1)+4(3n_2-2)\log(3n_2-2)+ \\ (n_1+n_2-16)(4n_2-3)\log(4n_2-3)]. \end{aligned}$$

Proof.
By Theorem 3.4, we have

$$W_p(P_{n_1}[P_{n_2}]) = (n_1-1)(n_2-1)(4n_2^2+5n_2+1),$$
$$\sum_{i=1}^{n} n_3(v_i)\log n_3(v_i) = 12(2n_2+1)\log(2n_2+1)+4(3n_2-2)\log(3n_2-2)+(n_1+n_2-16)(4n_2-3)\log(4n_2-3).$$



(a) Figure2 $P_4\otimes P_4$ (b) Figure3 $P_4\odot P_4$ (c) Figure4 $P_4\times P_4$ (d) Figure5 $P_4[P_4]$

Figure 1. *The illustration of the graph P? under four types of graph operations.*

4. Conclusion

Currently, the research findings on the distance entropy of graphs are relatively scant, particularly in the realms of computational methods and practical applications, where a multitude of unresolved questions persist. This paper primarily focuses on computing graph entropy based on the Wiener polarity index within the context of graph operations. Future research avenues could involve a deeper exploration of the distance entropy across a broader spectrum of graph classes and their associated computational challenges, an investigation into the influence of various graph operations on distance entropy, an analysis of the patterns exhibited by entropy changes during graph operations, and the provision of theoretical underpinnings for graph structural optimization and algorithm design.

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