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Research Article

Geometric Thinking of Prospective Mathematics Teachers: Assessing the Foundation Built by University Undergraduate Education in Ghana

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Abstract

This study investigates the geometric thinking levels of final year prospective mathematics teachers in Ghana, utilizing the van Hiele model to evaluate their proficiency. The main purpose was to assess whether university undergraduate mathematics education provides a sufficiently strong foundation for teaching senior high school geometry. A descriptive survey design was employed, involving 1,255 prospective mathematics teachers from three universities: University of Education Winneba (UEW), University of Cape Coast (UCC), and Akenten Appiah-Menka University of Skills Training and Entrepreneurial Development (AAMUSTED). The van Hiele Geometry Test (VHGT) was administered to measure participants' levels of geometric thinking. The results revealed that 8.8% of participants attained van Hiele Level 1 (visualization), 30.0% reached Level 2 (analysis), and 32.4% achieved Level 3 (abstraction). However, only 15.9% and 12.9% of prospective teachers reached Levels 4 (deduction) and 5 (rigor), respectively. These findings indicate a significant gap between the current geometric thinking skills of prospective teachers and the expectations of the Ghanaian mathematics curriculum, which anticipates higher-order thinking skills. The study concludes that the current undergraduate mathematics education programs in Ghanaian universities may not be adequately preparing future teachers to teach senior high school geometry effectively. It is recommended that these programs be revised to include more focus on developing higher-order geometric thinking skills, with an emphasis on deductive reasoning, formal proof-based learning and rigor in geometry thinking. Enhancing the curriculum and teaching methods could narrow this gap and improve the overall quality of geometry education in Ghana.

Keywords

Geometric Thinking, van Hiele Levels, Prospective Mathematics Teachers, Undergraduate Education, Ghana

1. Introduction

Recent curriculum reform agendas consistently maintain that learners' geometric thinking is an essential requirement which deserves significant attention in the teaching of mathematics [19, 22]. Geometric thinking, which is more than the ability to perform geometry tasks, refers to learners' approaches to reasoning about shapes and other geometric ideas [28]. Studies have shown that interest in students' geometric thinking abilities is topical and the significance of geometry in school curriculum remains widely recognised in pedagogical literature [11, 25, 30]. For example, Alex and Mammen con-

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tended that it is the language of geometry conceptualized and analysed in physical and spatial environments that helps learners to develop the skills of critical thinking, deductive reasoning and problem-solving [3].

Geometry plays a crucial role in modeling our surroundings. For instance, the shapes of roofs often include triangles, trapeziums, and squares, while the design of dresses incorporates symmetry. Additionally, tiling patterns on pavements, walls, and floors commonly feature pentagons, triangles, and rectangles. In school mathematics curriculum, shapes and space are taught to foster the learning of higher mathematics such as mechanics, vectors and mensuration [21]. Apart from the significant role geometry plays in school mathematics curriculum and the rich connection it has with other areas in mathematics, geometry also plays a key role in advancing engineering, computer technologies, physics, chemistry, geology, architecture and mathematics education [12, 17, 30]. Geometrical ideas are required in real life activities such as building a house, designing an electronic circuit board, an airport, a bookshelf, or even a newspaper page [29]. Given the pervasive role geometry plays in stimulating students' mathematics learning and highly skilled individuals, most countries are concerned about how teachers teach and how students learn various aspects of geometry [4, 6, 20].

In Ghana, geometry is taught at all levels of education with its different proportions of knowledge ascribed in the mathematics curriculum from basic, secondary, through to college of education and university level. Geometry forms a substantial amount of the senior high school core mathematics curriculum as it is treated in two of the seven content domains of core mathematics curriculum as *Plane Geometry* and *Mensuration* occupying approximately 29% of the core mathematics teaching syllabus. Plane geometry covers angles of polygons, Pythagoras' theorem and its application and circle theorems including tangents. Mensuration on the other hand covers perimeters and areas of plane shapes, surface areas, volumes of solid shapes and the earth as a sphere [18]. The rationale for treating Plane geometry and Mensuration is to assist students develop the skills of visualization, critical thinking, deductive reasoning, logical argument and proof and to give them the capacity to "organise and use spatial relationships in two or three dimensions, particularly in solving problems [18]". Though these skills present their own instructional challenges, they have life-long values beyond the geometry classroom.

In the context of geometry instruction, van Hiele [28] posits that the quality of teaching is one of the most significant factors influencing students' acquisition of geometric knowledge in mathematics classes. He asserts that students' progression from one geometric understanding level to the next is more dependent on instructional quality than on other factors like biological maturation or age. Additionally, various factors such as teachers' knowledge, gender, task difficulty, learning environment, and curriculum also play crucial roles in student achievement and motivation within the mathematics classroom [6, 13]. Despite this, the quality of geometry instruction stands out as a particularly influential element.

Teachers' mathematical and pedagogical content knowledge are pivotal in positively impacting students' motivation and learning in geometry. [1] further emphasize that teachers' content knowledge is crucial for students' performance. They suggest that the inadequate geometry knowledge among prospective teachers is a significant factor contributing to students' poor performance in geometry. This observation aligns with the arguments made by [5, 13], who both noted that insufficient content knowledge in geometry among prospective mathematics teachers may lead to subpar performance in geometry examinations by students. Thus, while multiple factors influence student outcomes in geometry, the overarching theme is that the quality of instruction, underpinned by robust content and pedagogical knowledge among teachers, is paramount in fostering better student performance and motivation in geometry. Existing literature and preliminary observations indicate potential gaps in the geometric understanding of prospective mathematics teachers in Ghana. This study may help provide empirical evidence to confirm or refute these observations, thereby informing policy and practice in undergraduate mathematics teacher education.

1.1. Problem Statement

In Ghana, the education system has undergone various reforms aimed at improving the quality of teaching and learning mathematics in schools. Despite these efforts, concerns remain regarding the effectiveness of teacher training programs, particularly in mathematics education. Several studies have highlighted deficiencies in the geometrical competencies of both in-service and pre-service teachers, raising questions about the adequacy of teacher preparation programs [2, 5]. Geometry, being a fundamental component of the mathematics curriculum, often poses significant challenges to students and teachers alike. The abstract nature of geometric concepts requires a solid foundation and a progressive development of geometric thinking. However, anecdotal evidence and preliminary studies suggest that many Ghanaian mathematics teachers may not possess the necessary geometric understanding to facilitate effective learning among their students [2, 5, 6].

Research indicates that students who have not reached van Hiele Level 4 before enrolling in tertiary-level geometry courses face a significantly lower likelihood of success [5, 7, 16, 23]. Consequently, it is crucial for students to achieve level 4 by the end of their Senior High School education. In this sense, it is anticipated that prospective mathematics teachers should reach van Hiele level 5 of geometric thinking before completing their undergraduate mathematics programme. It follows logically that for senior high school students to attain these necessary levels of geometric understanding, their teachers must possess a geometric thinking level at or above these benchmarks. This competency allows teachers to provide appropriate scaffolding and learning experiences, essential for guiding students through the complexities of geometric concepts. To effectively support students in reaching these levels, it is imperative to assess the van Hiele levels of prospective mathematics teachers. This approach is grounded in the van Hiele theory, which has significantly contributed to the resurgence of interest in geometry education. Moreover, by ensuring that prospective mathematics teachers attain a high level of geometric thinking, we can better prepare them to facilitate student learning and improve outcomes in geometry. This approach underscores the importance of aligning teacher preparation with the cognitive demands of the subject, ultimately fostering a more robust understanding of geometry among students.

1.2. Purpose of Study and Research Question

The foundation of mathematical education, particularly in geometry, is critical for the development of logical reasoning and spatial understanding, skills essential for success in numerous fields. Geometry is not only a cornerstone of mathematics but also a subject that fosters critical thinking and problem-solving abilities. In the context of Ghana, where education is seen as a pivotal driver for socio-economic development, ensuring that prospective mathematics teachers possess a strong geometric foundation is of paramount importance. This study aims to examine whether the current undergraduate mathematics education in Ghanaian public teacher training universities effectively prepares future mathematics teachers in terms of their geometric thinking capabilities. In pursuance of this purpose, the following research question was formulated to guide the study: Which stages of van Hiele Levels of geometric thinking do Prospective Ghanaian Mathematics teachers reach in their study of geometry just before leaving the university undergraduate level?

1.3. Theoretical Framework

The van Hiele framework, developed by Dutch educators Dina van Hiele-Geldof and Pierre van Hiele, outlines five levels of geometric thinking: Visualization, Analysis, Abstraction, Deduction, and Rigor. This model serves as a robust framework for understanding how students learn geometry and progress through different stages of geometric thought. Each level represents a qualitative shift in thinking, emphasizing the need for education systems to facilitate transitions between these stages effectively. For prospective teachers, reaching higher levels of the van Hiele model is crucial as it equips them with the necessary depth of understanding to teach geometry effectively. The van Hiele Levels are described as follows:

Level 1: Visualization

At this initial level, students recognize shapes and objects based on their appearance, not their properties. They can identify and name figures but do not understand the relationships between them. For example, a student might recognize a square because it looks like a “box” without considering its defining properties such as equal sides and right angles [14, 26].

Level 2: Analysis

At the analysis level, students begin to identify properties and characteristics of shapes. They can recognize that a square has equal sides and four right angles. However, their understanding is still largely descriptive, and they do not yet grasp the relationships between different properties or figures. For instance, they might understand that all squares have four equal sides but may not connect this to the definition of a rectangle [4].

Level 3: Abstraction

This level marks the beginning of logical reasoning about geometric properties and relationships. Students can make informal arguments about the properties of shapes and understand the relationships between different figures. For example, they recognize that all squares are rectangles because they meet the criteria of having four right angles and opposite sides equal, but not all rectangles are squares [9].

Level 4: Deduction

At the deduction level, students can understand and construct formal proofs. They can follow and create logical sequences of statements to establish geometric truths. This level involves an understanding of the axiomatic structure of geometry, where students can work with definitions, theorems, and postulates systematically. For instance, they can prove that the base angles of an isosceles triangle are congruent using deductive reasoning [4].

Level 5: Rigor

The highest level, rigor, involves a deep understanding of the formal aspects of geometric systems. Students can compare different axiomatic systems and understand the role of undefined terms, definitions, and theorems within these systems. At this stage, they can work abstractly with various geometric concepts and appreciate the nuances of different geometric frameworks. This level is typically achieved in advanced mathematics courses at the university level [24].

The van Hiele model has significant implications for teaching geometry. One of its key insights is that instruction should be tailored to the student’s current level of understanding to facilitate progression to higher levels. This suggests a developmental approach to teaching geometry, where educators provide experiences and tasks appropriate to each level [10].

Critique and Extensions of the van Hiele Framework

While the van Hiele model has been widely praised for its insights into geometry learning, it has also faced some critiques. One critique is that the model may oversimplify the complexity of geometric thinking by categorizing it into discrete levels. Some researchers argue that students’ understanding of geometry is more fluid and context-dependent than the model suggests [15]. Additionally, recent research has explored how the van Hiele levels apply to other areas of mathematics beyond geometry. Extensions of the model have been proposed to understand how students develop algebraic thinking and other mathematical concepts [8].

2. Method

2.1. Research Design

The study employed a descriptive survey design. This design was chosen to gain a comprehensive and accurate understanding of the levels of geometric thinking attained by the prospective Ghanaian mathematics teachers as they approach the completion of their undergraduate education. The descriptive survey method is particularly well-suited for this type of research as it allows for the systematic collection and analysis of data from a large sample, providing valuable insights into the current state of geometric understanding among the target population. Descriptive surveys are particularly effective in obtaining a precise depiction of existing conditions, behaviors, or phenomena without manipulating the study environment. This aligns perfectly with the goal of understanding the van Hiele levels of geometric thinking in a natural educational setting.

2.2. Participants and Setting

Final year students from the 2022/2023 academic year in the Department of Mathematics Education at three distinct public teacher training universities namely, University of Education, Winneba (UEW), University of Cape Coast (UCC), and Akenten Appiah-Menka University of Skills Training and Entrepreneurial Development (AAMUSTED) were purposefully chosen for this study. These universities were deemed ideal for the research as they are officially mandated to provide teacher education across various subject areas, including mathematics. Furthermore, prospective mathematics teachers in these institutions are recruited from all 16 administrative regions of Ghana, enriching the sample with diverse abilities, cultural, and social backgrounds. Consequently, the sample effectively represents Ghanaian pre-service teachers nationwide who have completed at least two academic semesters studying undergraduate geometry. To select the participants, stratified random sampling was employed, resulting in a sample of 1,255 final-year students from the three universities. The participants included 81.27% males, 15.94% females, and 2.79% who did not disclose their gender identity. The majority of participants (65%) were in the early adulthood age range of 23 to 30 years. About 9% were under 23 years old, and 26% were over 30 years old. Additionally, approximately 53% of the participants entered university with diploma certificates in teaching, while close to 47% were admitted with senior high school certificates. This diversity indicates that the participants have varied background characteristics, with some having prior teaching experience at the basic school level in Ghana.

2.3. Overview of the Undergraduate Geometry Course

The undergraduate geometry content course at these teacher training universities, taught in the first and second semester of the first year, encompasses topics that require students to engage in advanced levels of geometric thinking. This approach is designed to help students construct knowledge and understanding through a structured process of exploration, analysis, and both inductive and deductive reasoning [13, 27]. The course includes topics that enable students to graph algebraic equations in two variables as lines and circles, calculate angles and distances between lines and circles, and define, prove, and construct Euclidean (plane) geometry, as well as perform measurements in geometrical shapes and solids. Other specific topics covered in the undergraduate geometry course include conic sections, the equation of a circle, the equation of a parabola, the equation of an ellipse, the equation of a hyperbola, asymptotes to a hyperbola, polar coordinates, relations between polar and cartesian coordinates, area of a sector, length of a curve, arc length, parametric equations, and polar equations.

These topics are not only fundamental to the mastery of geometry but also pivotal for fostering a deep understanding of mathematical principles that prospective teachers will eventually impart to their students. By engaging with these complex topics early in their academic journey, prospective mathematics teachers are better prepared to develop the analytical and reasoning skills necessary for teaching senior high school geometry effectively. This rigorous grounding in geometric concepts ensures that future teachers can approach the subject with confidence and inspire the same level of understanding and appreciation in their future classrooms.

2.4. Instrument

The van Hiele Geometry test (VHGT) was adopted and used to collect data in order to address the research question in this study. The VHGT, adapted from [26]'s Cognitive Development and Achievement in Secondary School Geometry (CDASSG) project, is a well-crafted 25-item multiple-choice test designed to assess varying levels of geometric understanding. Each set of five items targets a specific level of cognitive development in geometry. Here is how the VHGT is organized:

1. *Items 1-5 (Subtest 1)*: These items focus on the identification, naming, and comparison of geometric shapes such as squares, rectangles, and rhombi. They measure students' understanding at Level 1, where basic recognition and description of shapes are assessed.
2. *Items 6-10 (Subtest 2)*: These items deal with recognizing and naming the properties of geometric figures. They

evaluate students' understanding at Level 2, where students identify specific characteristics and properties of the shapes they recognize.

3. *Items 11-15 (Subtest 3)*: This section addresses the logical order of properties and the relationships between these properties of previously identified figures. It measures students' understanding at Level 3, where the focus is on comprehending the connections and hierarchies among geometric properties.
4. *Items 16-20 (Subtest 4)*: These items require students to understand the significance of deduction and the roles of postulates, axioms, theorems, and proofs. They assess students' understanding at Level 4, where formal logical reasoning and the construction of geometric arguments are key.
5. *Items 21-25 (Subtest 5)*: This final block deals with the formal aspects of deduction, measuring students' understanding at Level 5. At this level, students engage with advanced deductive reasoning, comparing and contrasting different geometric systems and their underlying axioms and theorems.

The VHGT's design ensures a comprehensive assessment of students' geometric thinking across different levels, providing insights into their cognitive development in geometry. By organizing the test sequentially according to the van Hiele levels, educators can diagnose specific areas where students may need further instruction or support. This structured approach not only helps in identifying students' current levels of understanding but also guides the development of targeted interventions to enhance their geometric reasoning skills.

The VHGT has been widely recognized and utilized in numerous studies, consistently demonstrating strong validity and reliability [11, 26]. This robust track record justifies its adoption in this study. By employing the VHGT, the researcher ensured that the assessment tool is both credible and capable of accurately measuring the levels of geometric thinking among prospective mathematics teachers. This choice not only enhances the reliability of the findings but also aligns the study with established methods in the field of mathematics education research.

2.5. Analysis of Data

The data collected from participants were meticulously coded and entered into SPSS for comprehensive processing and analysis. Descriptive statistical methods, including measures of central tendency, frequency counts, and percentages, were employed to analyze the data. The results were presented using tables and bar charts, providing a clear visual representation. These descriptive statistics were instrumental in understanding, interpreting, and describing the participants' experiences and their levels of geometric conceptualization.

3. Results

3.1. Prospective Mathematics Teachers' Performance in the VHGT

The objective of the study was to find out the van Hiele levels (VHLs) of geometric thinking reached by prospective mathematics teachers just before completing their university undergraduate programme. The geometric thinking skills of prospective mathematics teachers were defined according to the van Hiele theory, encompassing their abilities to visualize, analyze, abstract, deduce, and ability to display rigor in geometry thinking.

3.1.1. Visualization Skills: Performance on Subtest 1 (van Hiele Level 1)

Visualization skills, corresponding to VHL 1, imply students' ability to recognize and identify geometric shapes based on their overall appearance. In other words, it implies students' ability to name and categorize shapes such as squares, triangles, rectangles and parallelograms based on visual characteristics. Students' understanding at this stage is primarily visual and intuitive, and students do not yet comprehend the formal properties or relationships that connect different geometric figures. Table 1 presents the distribution of participants' correct responses to the five VHGT items designed to assess their visualization skills.

Table 1. *Distribution of Participants' Visualization Skills by Frequency count and Percentage (VHL 1).*

Level	Item Number	Geometry aspect examined	Correct response (n) (%)	
VHL1 Visualization	1	Recognition of squares	1205	96.02
	2	Recognition of triangles	1198	95.46
	3	Recognition of rectangles	1150	91.63
	4	Characterizing the Orientation of Squares	990	78.88
	5	Identifying Orientation and Class Inclusion of Parallelograms	886	70.60

Table 1 depicts that as prospective mathematics teachers approached the completion of their undergraduate education, more than 90% demonstrated proficient visual recognition and differentiation of squares, triangles, and rectangles. However, fewer than 80% of participants accurately identified the orientation of various squares and parallelograms based on class inclusivity principles. This indicates that while a significant majority of participants developed strong visualization skills during their four-year program, few challenges persisted in recognizing specific geometric orientations using inclusive properties.

3.1.2. Analysis Skills: Performance on Subtest 2 (van Hiele Level 2)

Analysis skills in geometry refer to the ability to examine and identify the properties and relationships of geometric figures systematically and logically. This includes understanding how these relationships contribute to a deeper understanding of geometric concepts. Table 2 illustrates the percentage distribution of participants who effectively applied their analytical skills to achieve correct responses on the VHGT test.

Table 2. *Distribution of Participants' Analysis Skills by Frequency count and Percentage (VHL 2).*

Level	Item Number	Geometry aspect examined	Correct response (n)	(%)
VHL2 Analysis	6	Relational properties of a square	1112	88.61
	7	Diagonal property of rectangle	1014	80.80
	8	Properties of rhombus	805	64.14
	9	Properties of isosceles triangle	1220	97.21
	10	Properties of kite	890	70.92

The table reveals that a significant majority of participants, over 80%, demonstrated accurate analysis in understanding the relationship properties of squares and the diagonal properties of rectangles. Specifically, 64.14% and 70.92% effectively analyzed the fundamental properties of rhombuses and kites, respectively. Notably, more than 97% successfully identified the basic properties of isosceles triangles. These findings underscore the participants' strong ability to analyze geometric properties and establish meaningful relationships among different shapes.

3.1.3. Abstraction Skills: Performance on Subtest 3 (van Hiele Level 3)

Abstraction and ordering skills encompass the capacity to arrange shapes logically, create abstract definitions, and discern essential from incidental properties within geometric contexts. Table 3 illustrates the distribution of participants who demonstrated proficient abstraction skills in the VHGT assessment.

Table 3. *Distribution of Participants' Abstraction Skills by Frequency count and Percentage (VHL 3).*

Level	Item Number	Geometry aspect examined	Correct response (n)	(%)
VHL3 Abstraction	11	Logical reasoning using verbal cues: rectangles and triangles	695	55.38
	12	Analytical reasoning based on triangle properties	798	63.59
	13	Abstraction through rectangular orientation	769	61.27
	14	Constructing logical arguments using inclusive properties	985	78.49
	15	Establishing logical connections among parallelograms	755	60.16

From Table 3, the percentages of participants (55.38%, 63.59%, 61.27%, 78.49%, and 60.16%) who effectively applied verbal logical reasoning and logical analysis to shape and space were somewhat satisfactory. However, despite nearly four years of studying university undergraduate mathematics, their demonstration of verbal and logical reasoning was not as robust as expected. Moreover, the proportion of participants who accurately abstracted concepts was also modest.

3.1.4. Deduction Skills: Performance on Subtest 4 (van Hiele Level 4)

Deduction skills refer to the ability to derive conclusions logically based on established principles, postulates, axioms and theorems within geometry. This involves applying deductive reasoning to make valid assertions about geometric shapes, properties, and relationships. Table 4 presents the distribution of the percentage of participants who applied deductive reasoning effectively to answer the VHGT test items.

Table 4. *Distribution of Participants' Deduction Skills by Frequency count and Percentage (VHL 4).*

Level	Item Number	Geometry aspect examined	Correct response (n)	(%)
VHL4 Deduction	16	Deriving figural structures through deductive reasoning	779	62.07
	17	Inferencing figural characteristics through deduction	619	49.32
	18	Proof	859	68.45
	19	Generalization	622	49.56
	20	Deduction	602	47.97

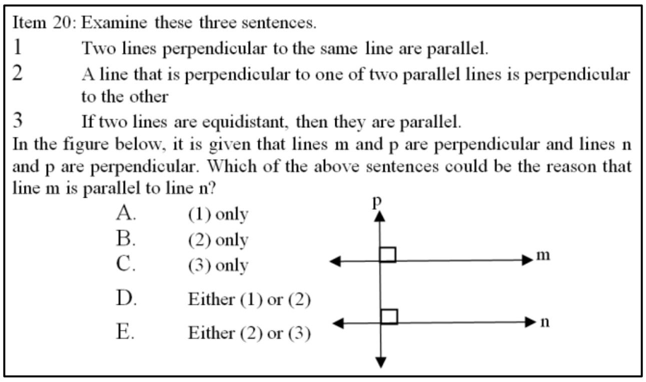

Figure 1. *Sample Item in Subtest 4.*

Table 4 reveals that 62.07% and 68.45% of participants effectively utilized deductive reasoning to derive geometric structures and conduct geometric proofs. However, fewer than 50% (49.32%, 49.56%, and 47.97%, respectively) correctly responded to items 17, 19, and 20, which required making deductions, generalizing from observations, and applying deductive reasoning. Figure 1 is an item from Subtest 4. The correct answer for this item is choice A. Table 4 shows that only 602 (47.97%) of prospective mathematics teachers had this question correct. These results indicate a notable portion of participants struggled with tasks involving higher-level deductive reasoning.

3.1.5. Rigor in Geometry: Performance on Subtest 5 (van Hiele Level 5)

Rigor in geometry thinking involves the capacity to compare axiomatic systems through formal theoretical approaches, independent of concrete models. It refers to the ability to engage in thorough and precise reasoning, applying formal mathematical methods such as proofs and logical arguments to analyze geometric concepts and properties systematically. The analysis of participants' responses to items 21 to 25 aimed to ascertain the extent to which they demonstrated rigorous geometric thinking in drawing conclusions about theories, axioms, or implicative statements. The findings are detailed in Table 5.

Table 5. *Distribution of Participants' Rigor in geometry thinking in shapes by Frequency count and Percentage (VHL 5).*

Level	Item Number	Geometry aspect examined	Correct response (n)	(%)
VHL5 Rigor	21	Applying deduction with rigor	563	44.86
	22	Conclusive deduction with rigor	456	36.33
	23	Applying deduction with rigor	496	39.52

Level	Item Number	Geometry aspect examined	Correct response (n) (%)	
	24	Applying deduction with rigor	416	33.15
	25	Implications in Geometry	403	32.11

Item 25: Suppose you have proved statements I and II.

I: If p, then q.

II: If s, then not q.

Which statement follows from statements I and II?

A. If p, then s.
B. If not p, then not q.
C. If p or q, then s.
D. If s, then not p.
E. If not s, then p.

Figure 2. *Sample Item in Subtest 5.*

Table 5 reveals that close to the end of their 4-year undergraduate mathematics education programme, participants' correct responses to items 21 to 25 were notably low, with each item yielding less than 50% correct response. Sample item from Subtest 5 is shown in Figure 2. The correct answer for this item is choice D. However, Table 5 shows that only 403 (32.11%) of prospective mathematics teachers had this question correct. This indicates significant challenges among participants in applying rigorous methods for geometric constructions and generalizing implicative statements.

3.2. Overall Scores of Prospective Mathematics Teachers in the VHGT

Table 6 illustrates the overall performance scores of prospective mathematics teachers on the VHGT. Scores ranged from a minimum of 32% to a maximum of 80%. The average score among these prospective teachers was 52.32%, with a standard deviation of 7.21%. This data indicates a moderate to high proficiency level, with the average score slightly above the midpoint of the scoring range. The standard deviation of 7.21 suggests some variability in performance, implying that while many scores were close to the mean, there was a notable spread in the participants' abilities.

Table 6. *Means, Standard Deviations, Minimum and Maximum VHGT Scores of Prospective Mathematics Teachers.*

N	Mean	Standard Deviation	Maximum	Minimum
1255	52.32	7.21	80	32

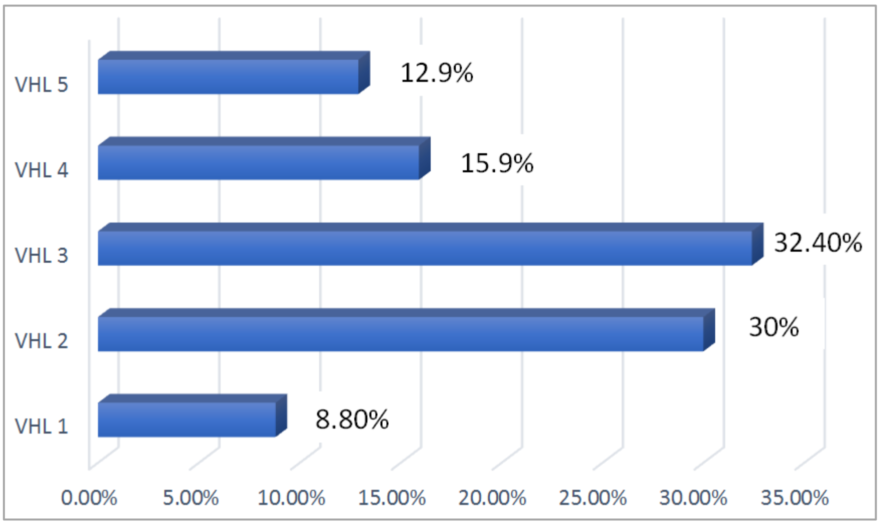

Figure 3. *Prospective Mathematics Teachers' van Hiele Levels of Geometric Thinking.*

3.3. Levels Reached Prospective Mathematics Teachers in the VHGT

This study aimed to determine the van Hiele Levels of geometric thinking achieved by Ghanaian prospective mathematics teachers nearing the completion of their 4-year undergraduate program, preparing them to teach at the senior high school level. The bar chart in Figure 3 visually represents the geometric thinking levels attained by these future educators. This graphical depiction offers a clear and immediate understanding of their proficiency, highlighting the distribution across different van Hiele levels. Analyzing these results can provide valuable insights into the effectiveness of the current educational program and identify areas needing improvement to better prepare teachers for their future roles.

Figure 3 shows that 8.8% of participants reached VHL 1, demonstrating strong visualization skills, while 30.0% achieved VHL 2, operating at the analysis level. Additionally, 32.4% of participants attained VHL 3, functioning at the abstraction level. However, only 15.9% and 12.9% reached VHL 4 and VHL 5, operating at the deduction and rigor levels, respectively.

4. Discussion

The study's purpose was to examine whether the current undergraduate mathematics education in Ghanaian public teacher training universities effectively prepares future mathematics teachers in terms of their geometric thinking capabilities. This was done by investigating the van Hiele levels of geometric thinking of prospective mathematics teachers involved in this study. These prospective mathematics teachers were in the final (fourth) year of their university undergraduate programme preparing to graduate to teach at senior high schools in Ghana. Structured around van Hiele's geometric thinking levels, this study examined participants abilities across visualization, analysis, abstraction, deduction, and rigor stages. The results highlight the varying degrees of geometric reasoning skills among prospective mathematics teachers, emphasizing the need for targeted interventions to enhance higher-level thinking skills.

The VHGT results indicated that 8.8% of prospective mathematics teachers reached van Hiele Level 1, 30.0% attained Level 2, and 32.4% achieved Level 3. Furthermore, 15.9% and 12.9% of the participants reached Levels 4 and 5, respectively. This indicates that the majority of prospective mathematics teachers are operating within the first three stages of geometric reasoning as defined by the van Hiele model, demonstrating foundational visualization, analytical, and abstract thinking skills. However, a smaller proportion of participants exhibit the advanced deductive and rigorous reasoning skills necessary for higher-level geometric thinking. In particular, the largest group of participants (32.40%) reached the abstraction level where they understand relationships between properties of shapes and can logically deduce theorems based on these properties. Overall, only 28.8% of participants reached the highest levels (i.e. level 4 and 5). In other words, only this small proportion of participants can understand and form formal proofs and comprehend the structure of axiomatic systems. Also, these were the few who understood working within different systems and understanding the implications of altering axioms. These findings suggest the need for enhanced focus on developing higher-order reasoning capabilities within the mathematics education curriculum.

The findings highlighted above align with earlier research conducted in Ghana [2, 7, 4] and underscore the ongoing concern about the methods of teaching and learning geometry in Ghanaian schools. The studies referenced indicate that both prospective mathematics teachers and senior high school students are performing at lower levels of geometric thinking than anticipated by the national mathematics curriculum. This discrepancy suggests potential gaps in instructional strategies and educational resources, highlighting the need for a comprehensive review of the geometry curriculum and teaching practices to ensure they effectively promote higher-order thinking skills including rigor and deductive reasoning skills. Addressing these gaps is crucial for aligning educational outcomes with curriculum standards and enhancing overall mathematical proficiency.

It is essential to emphasize that deductive reasoning underpins the comprehension of definitions, properties, axioms, postulates, and other geometric elements used in geometric proofs. Consequently, the absence of this critical reasoning skill indicates that prospective mathematics teachers will likely face challenges in explaining geometric concepts and applying their knowledge to related fields such as algebra, trigonometry, vectors and mechanics which are key components of both core and elective mathematics at the senior high school level. This gap in deductive reasoning not only hinders their understanding of geometry but also impacts their overall mathematical proficiency, making it imperative to enhance instructional approaches that foster strong deductive reasoning skills in students.

According to educational standards, senior high school students should reach van Hiele Level 4 in geometric thinking by the end of their secondary education [23]. Consequently, it is expected that prospective mathematics teachers achieve Levels 4 and 5 to effectively and confidently teach high school geometry upon completing their undergraduate studies. However, this study revealed that only 15.9% of these future teachers attained Level 4, and just 12.9% reached Level 5. This significant shortfall raises critical concerns about the effectiveness of the current undergraduate mathematics education programs in Ghanaian universities, especially in geometry.

5. Conclusion and Recommendations

The study concludes that majority of prospective mathematics teachers preparing to graduate to teach at the senior high school level in Ghana operate at the initial three levels of geometric thinking and only a small proportion of them operate at the higher levels. These results highlight a persistent gap between the current competencies of prospective mathematics teachers and the expectations of the Ghanaian mathematics curriculum. The deficiency in higher-order geometric thinking skills raises significant concerns about the adequacy of the current undergraduate mathematics education programs in Ghanaian universities. This gap suggests that many future teachers may struggle to effectively teach senior high school geometry, which requires a solid understanding of deductive reasoning and the ability to work with complex geometric concepts. Based on the findings of this study, the following recommendations are made:

1. Teacher training universities in Ghana should update their undergraduate geometry education curriculum to place greater emphasis on developing higher-order geometric thinking skills. They should include more content on formal proofs, logical reasoning, and the structure of mathematical systems.
2. Teaching strategies that focus on problem-solving and proof-based learning should be implemented in teacher training universities in Ghana, encouraging prospective mathematics teachers to engage in activities that require logical reasoning and the formulation of formal geometric proofs.
3. Teacher training universities in Ghana should provide ongoing professional development opportunities for prospective mathematics teachers to strengthen their geometric reasoning skills. Workshops, seminars, and advanced courses can help future teachers stay updated on best practices and new developments in geometry education.
4. Regular assessments should be conducted by lecturers in teacher training universities in Ghana to monitor the geometric thinking levels of prospective mathematics teachers. These assessments should be used to identify areas needing improvement and to provide targeted support.

6. Limitations

The study admits three main limitations that should be considered when interpreting the results. First, the use of the van Hiele Geometry Test (VHGT) as the sole assessment tool may have inherent biases or limitations in accurately capturing the full range of geometric thinking skills. Incorporating multiple assessment methods, including interviews and observational studies, could provide a more comprehensive evaluation. Second, the study's cross-sectional design provides a snapshot of geometric thinking levels at a specific point in time, without accounting for the potential development and progression of these skills over the course of the undergraduate program. Longitudinal studies tracking the same cohort over time could offer deeper insights into the development of geometric reasoning. Third, the study did not account for variations in educational contexts, such as differences in teaching quality, curriculum implementation, and resources available at different institutions. These factors could significantly influence the development of geometric thinking skills.

Abbreviations

AAMUSTED	Akenten Appiah-Menka University of Skills Training and Entrepreneurial Development
CDASSG	Cognitive Development and Achievement in Secondary School Geometry
UCC	University of Cape Coast
UEW	University of Education, Winneba
VHGT	Van Hiele Geometry Test
VHL	Van Hiele Level

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Author Contributions

Robert Benjamin Armah is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The author declares no conflicts of interest.

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