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Solving Controllability Problems of Backward Ordinary Differential Systems with Matlab

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Abstract: Controllability problems of differential equations appear in many situations or phenomena for which one is interested in finding a mechanism for bringing a given state into a desired one. Their resolutions often involve constrained minimization problems governed by differential equations systems. This paper is particularly interested in a null controllability problem for backward differential equations systems. One develop a numerical scheme by first approximating the control space by a space of piecewise continuous functions and by transforming the controllability problem into a classical minimization problem with constraints in finite dimension space. Next, one proceed to an adapted implementation of the numerical scheme in Matlab using some of its built-in functions. One then construct a sequence of codes written in Matlab allowing to robustly compute an approximation of the null control at a lower cost. To validate the numerical approach adopted in this paper, two numerical examples are presented. The first ones concerns the controllability of a backward ordinary differential quations system and the second, the controllability of a partial differential heat equation. In both cases, the numerical results obtained are very satisfactory and show that the numerical approach with Matlab developed in this paper leads to new insights for a large class of PDE control problems.

Keywords: Control of Ordinary Differential Systems, Null Control, Approximation Scheme, Implementation with Matlab

1. Introduction

With the increasing complexity of engineering applications in recent decades, the subject of the controllability of systems governed by partial differential equations has moved from theory to calculation. The numerical computation of the optimal control of PDEs has become a science in itself, which has given rise to a variety of numerical methods and theirs implementatiosn by scientific softwares [5, 6, 7, 9]. In numerical practices, the spatial semi-discretization of systems of partial differential equations often gives rise to systems of ordinary differential equations. Instead of discretizing the differential system obtained, certain particularly suitable codes, in the Matlab environment, can be used. The aim of this article is then to use Matlab capabilities to numerically determine null-controls of large scale differential systems resulting from the semi-discretisaton of the problems governed by PDEs [1, 10, 12].

This paper is mainly concerned with the above controllability problem. Consider non-null integers n and r, for given positive real T, find $\mathbf{v} \in L^2(0,T;\mathbb{R}^r)$ such that the corresponding solution of the following system

$$\frac{d}{dt}\mathbf{u}(t) = G(\mathbf{u}(t)) + \mathbb{B}\mathbf{v}(t), \quad t \in]0, T[\tag{1}$$

$$\mathbf{u}(T) = \mathbf{u}^0 \tag{2}$$

satisfies

$$\mathbf{u}(0) = \mathbf{0}_{\mathbb{R}^n} \tag{3}$$

Where $\mathbf{0}_{\mathbb{R}^n}$ is the null vector of $\mathbb{R}^n$ and $\mathbf{u}^0 \in \mathbb{R}^n$ is a given terminal solution. Here, $\mathbb{B}$ is a $r \times n$ matrix, and G is a given function from $\mathbb{R}^n$ to $\mathbb{R}^n$.

There exists a variety of papers in the literature dedicated to null controllability study of problems of type (1)-(3). One can refer to [2, 3, 8, 11]. Recently, it has been established the necessary and sufficient condition to guarantee that systems of type (1)-(2) is completely null controllable [13]. However, the numerical approximation of null controls for such a problem is a difficult issue. To this end, this paper develops programs in Matlab, a widely used scientific computing [10].

The remainder of this paper is organized as follows. Section 2 describes an approximation process for computing null controls and stablishes its convergence properties. Section 3 presents a numerical scheme for solving approximated null controls as well as theirs implementation in Matlab Environment. Section 4 is devoted to some numerical examples and, section 5 gives some remarks.

2. Approximation of Null Controls

This section starts by giving a result concerning the existence and a property of continuity with respect to the terminal condition and to the control of the system (1)-(2):

Proposition 2.1 Assume that G is α - Lipstchiz. Then for given $\mathbf{u}^0 \in \mathbb{R}^n$ and for $\mathbf{v} \in L^2(0, T; \mathbb{R}^r)$ there exists a unique $\mathbf{u} \in \mathcal{C}^1(0, T; \mathbb{R}^n)$ which satisfies the system (1)-(2). In addition,

$$\|\mathbf{u}(0)\|_{\mathbb{R}^n} \le \left(\|\mathbf{u}^0\|_{\mathbb{R}^n} + \|\mathbb{B}\| \int_0^T \|\mathbf{v}(T-s)\|_{\mathbb{R}^r} ds \right) e^{\alpha T}. \tag{4}$$

Proof. Existence and uniqueness property follow from the Cauchy- Picard-Lipschitz theorem[14]. To establish the estimate (4) one can first rewrite (1) as follows.

$$\frac{d}{dt}\mathbf{u}(T-t) = -G(\mathbf{u}(T-t)) - \mathbb{B}\mathbf{v}(T-t), \quad t \in (0,\ T) \tag{5}$$

Integrating this equation over $(0,\ t)$, one obtains

$$\mathbf{u}(T-t) = \mathbf{u}^0 - \int_0^t G(\mathbf{u}(T-s))ds - \int_0^t \mathbb{B}\mathbf{v}(T-s)ds. \tag{6}$$

Since G is α Lipschitz, one can deduce

$$\begin{aligned} \|\mathbf{u}(T-t)\|_{\mathbb{R}^n} &\le \|\mathbf{u}^0\|_{\mathbb{R}^n} + \int_0^t \|G(\mathbf{u})(T-s)\|_{\mathbb{R}^n} ds + \int_0^t \|\mathbb{B}\mathbf{v}(T-s)\|_{\mathbb{R}^r} ds \\ &\le \|\mathbf{u}^0\|_{\mathbb{R}^n} + \|\mathbb{B}\| \int_0^T \|\mathbf{v}(T-s)\|_{\mathbb{R}^r} ds + \alpha \int_0^t \|\mathbf{u}(T-s)\|_{\mathbb{R}^n} ds \end{aligned}$$

Thus from Gronwal's lemma it follows

$$\|\mathbf{u}(T-t)\|_{\mathbb{R}^n} \le \left(\|\mathbf{u}^0\|_{\mathbb{R}^n} + \|\mathbb{B}\| \int_0^T \|\mathbf{u}(T-s)\|_{\mathbb{R}^r} ds \right) e^{\alpha t} \tag{7}$$

Therefore, the estimate (4) is achieved if t is replaced by T. □

In the following, the operator G will be assumed α-lipschitz.

Similar to the approximation of null controllability of some PDEs, to approach null controls of (1) - (2) one consider, for given non negative β, the problem

$$\min_{\mathbf{v} \in L^2(0,T;\mathbb{R}^r)} J_\beta(\mathbf{v}) := \frac{1}{2} \int_0^T \mathbf{v}(t)^T \mathbb{C} \mathbf{v}(t) dt + \frac{1}{2\beta} \|\mathbf{u}(0)\|_{R^n}^2 \tag{8}$$

Subject to the equations system (1) - (2) where $\mathbb{C}$ is an r order defined symmetric matrix[4, 15]. One can then formulate the following result.

Proposition 2.2 For given $\mathbf{u}^0 \in \mathbb{R}^n$, there exists a unique $\mathbf{v}^\beta \in L^2(0, T; \mathbb{R}^r)$ solution of the minimisation problem (8). Thurtheremore $\mathbf{v}^\beta$ converges weakly to $\overline{\mathbf{v}}$ in $L^2(0, T; \mathbb{R}^r)$ as $\beta \to 0$, and the corresponding $\overline{\mathbf{u}}$ solution of the system (1)-(2) for the control $\overline{\mathbf{v}}$ satisfies $\overline{\mathbf{u}}(0) = \mathbf{0}_{\mathbb{R}^n}$.

Proof. First, to prove the existence and uniqueness of the minimization problem (8), one needs to establish that J_β is strictly convex, lower semi-continuous and coercive in $L^2(0, T; \mathbb{R}^r)$.

(i) The strict convexity of J_β. Since, by hypothesis, $\mathbb{C}$ is a symmetric positive definite matrix, it is obvious to see that the

application $\mathbf{v} \longmapsto \frac{1}{2}\int_0^T \mathbf{v}(t)^T \mathbb{C}\mathbf{v}(t)dt$ is strictly convex in $L^2(0,T;\mathbb{R}^r)$. Thus, to establish(i) it suffices to show that the application $\mathbf{v} \longmapsto \|\mathbf{u}(0)\|$ is strictly convex in $L^2(0,T;\mathbb{R}^r)$. This is a consequence of the continuity of the solution of the system (1) relative to the initial state.

(ii) The lower semi-continuity of J_β. Consider the sequence $(\mathbf{v}_k)$ in $L^2(0,T;\mathbb{R}^r)$ which weakly converges toward $\hat{\mathbf{v}}$, and let $\mathbf{u}_k$ and $\hat{\mathbf{u}}$ be solutions of the system (1)-(2) respectively to controls $\mathbf{v}_k$ and $\hat{\mathbf{v}}$. Thanks to Proposition 2.1, following estimates hold

$$\|\mathbf{u}_k(0)\|_{\mathbb{R}^n} \leq \left(\|\mathbf{u}^0\|_{\mathbb{R}^n} + \|\mathbb{B}\| \int_0^T \|\mathbf{v}_k(T-s)\|_{\mathbb{R}^r} ds\right) e^{\alpha T}. \tag{9}$$

and

$$\|\hat{\mathbf{u}}(0)\|_{\mathbb{R}^n} \leq \left(\|\mathbf{u}^0\|_{\mathbb{R}^n} + \|\mathbb{B}\| \int_0^T \|\hat{\mathbf{v}}(T-s)\|_{\mathbb{R}^r} ds\right) e^{\alpha T}. \tag{10}$$

As $\mathbf{v}_k$ weakly converges to $\hat{\mathbf{v}}$, then from (9) and (10) one can deduce

$$\begin{aligned}\liminf_{k\to+\infty} \|\mathbf{u}_k(0)\| &\geq \left(\|\mathbf{u}^0\|_{\mathbb{R}^n} + \|\mathbb{B}\| \int_0^T \|\hat{\mathbf{v}}(T-s)\|_{\mathbb{R}^r} ds\right) e^{\alpha T} \\ &\geq \|\hat{\mathbf{u}}(0)\|_{\mathbb{R}^n}.\end{aligned} \tag{11}$$

Then it follows

$$\begin{aligned}\liminf_{k\to+\infty} J_\beta(\mathbf{v}_k) &= \liminf_{k\to+\infty}\left(\frac{1}{2}\int_0^T \mathbf{v}_k(t)^T\mathbb{C}\mathbf{v}_k(t)dt + \frac{1}{2\beta}\|\mathbf{u}_k(0)\|^2_{R^n}\right) \\ &\geq \frac{1}{2}\int_0^T \hat{\mathbf{v}}(t)^T\mathbb{C}\hat{\mathbf{v}}(t)dt + \frac{1}{2\beta}\|\hat{\mathbf{u}}(0)\|^2_{R^n} \\ &\geq J_\beta(\hat{\mathbf{v}})\end{aligned} \tag{12}$$

Which establishes the lower semi-continuity of J_β.

(iii) The coercivity of J_β. Since $\mathbb{C}$ is a symmetric positive definite matrix, then there exists a strictly positive constant μ such that

$$\begin{aligned}J_\beta(\mathbf{v}) &= \frac{1}{2}\int_0^T \mathbf{v}(t)^T\mathbb{C}\mathbf{v}(t)dt + \frac{1}{2\beta}\|\mathbf{u}(0)\|^2_{R^n} \\ &\geq \frac{1}{2}\int_0^T \mathbf{v}(t)^T\mathbb{C}\mathbf{v}(t)dt \\ &\geq \frac{\mu}{2}\|\mathbf{v}\|^2_{L^2(0,T;\mathbb{R}^r)}.\end{aligned} \tag{13}$$

This establishes the coercivity of J_β in $L^2(0,T;\mathbb{R}^r)$.

Finally *(i)*, *(ii)* and *(iii)* clearly show the existence and the uniqueness of the solution of the minimization problem (8).

It remains to show that $\mathbf{v}^\beta$ converges weakly to $\overline{\mathbf{v}}$ in $L^2(0,T;\mathbb{R}^r)$ as $\beta \to 0$, and the corresponding $\overline{\mathbf{u}}$ solution of the system (1)-(2) for the control $\overline{\mathbf{v}}$ satisfies $\overline{\mathbf{u}}(0) = \mathbf{0}_{\mathbb{R}^n}$. It is not difficult to show that the set

$$\mathcal{U} = \left\{\mathbf{v} \in L^2(0,T;\mathbb{R}^r),\ \frac{d\mathbf{u}}{dt} = G(\mathbf{u}) + \mathbb{B}\mathbf{v},\ \ \mathbf{u}(T) = \mathbf{u}^0 \text{ and } \mathbf{u}(0) = 0_{\mathbb{R}^n}\right\} \tag{14}$$

is not empty. Consider now $\overline{\mathbf{v}}$ the solution of the following problem

$$\frac{1}{2}\int_0^T \overline{\mathbf{v}}(t)^T\mathbb{C}\overline{\mathbf{v}}(t)dt = \min_{\mathbf{v}\in\mathcal{U}} \frac{1}{2}\int_0^T \mathbf{v}(t)^T\mathbb{C}\mathbf{v}(t)dt \tag{15}$$

By the optimal argument one can then write

$$\frac{1}{2}\int_0^T \mathbf{v}^\beta(t)^T\mathbb{C}\mathbf{v}^\beta(t)dt + \frac{1}{2\beta}\|\mathbf{u}^\beta(0)\|^2_{R^n} \leq \frac{1}{2}\int_0^T \overline{\mathbf{v}}(t)^T\mathbb{C}\overline{\mathbf{v}}(t)dt \tag{16}$$

Where $\mathbf{u}^\beta$ is the corresponding solution of the system (1) - (2) for the control $\mathbf{u}^\beta$. This implies

$$\begin{aligned}\|\mathbf{v}^\beta\|_{L^2(0,T;\mathbb{R}^r)} &\leq \mu\int_0^T \mathbf{v}^\beta(t)^T\mathbb{C}\mathbf{v}^\beta(t)dt \\ &\leq \mu\int_0^T \overline{\mathbf{v}}(t)^T\mathbb{C}\overline{\mathbf{v}}(t)dt\end{aligned} \tag{17}$$

The sequence $(\mathbf{v}^{\beta})$ is then bounded and Thus, one can find a subsequence $(\mathbf{v}^{\beta_k})$ which weakly converges to $\tilde{\mathbf{v}}$. So it follows

$$\begin{aligned} \int_0^T \tilde{\mathbf{v}}(t)^T \mathbb{C}\tilde{\mathbf{v}}(t)dt &\leq \liminf_{\beta_k \to 0} \int_0^T \mathbf{v}^{\beta}(t)^T \mathbb{C}\mathbf{v}^{\beta}(t)dt \\ &\leq \int_0^T \overline{\mathbf{v}}(t)^T \mathbb{C}\overline{\mathbf{v}}(t)dt \end{aligned} \tag{18}$$

Also, from (16) one has

$$\|\mathbf{u}^{\beta}(0)\|_{R^n}^2 \leq \frac{\beta}{2}\int_0^T \overline{\mathbf{v}}(t)^T \mathbb{C}\overline{\mathbf{v}}(t)dt$$

and by passing to the limit as β tends to 0, thanks to the property of continuity, it follows that $\tilde{\mathbf{v}} \in \mathcal{U}$. Since $\overline{\mathbf{v}}$ is the solution of the minimisation problem (15), then $\int_0^T \tilde{\mathbf{v}}(t)^T \mathbb{C}\tilde{\mathbf{v}}(t)dt = \int_0^T \overline{\mathbf{v}}(t)^T \mathbb{C}\overline{\mathbf{v}}(t)dt$. Thus, one deduces $\tilde{\mathbf{v}} = \overline{\mathbf{v}}$. This completes the proof of the weak convergence of $\left(\mathbf{v}^{\beta}\right)$. □

Proposition 2.1 clearly shows that the null control of the problem (1)- (2) can be approximated by the solution of the minimisation problem (8) by taking β small enough.

Let fix m a non null integer. To approach any control function in $L^2(0, T; \mathbb{R}^r)$ let introduce the following space of functions

$$E^m = \left\{\mathbf{v} : [0,\ T] \to \mathbb{R}^r \text{ tel que } \mathbf{v}|_{[t_k,\ t_{k+1}[} \text{ is constant for } k = 0, ..., m-1\right\}. \tag{19}$$

One now consider the minimization problem of J_β over E^m under the same constraints. The following proposition can be easily proved.

Proposition 2.3 For each m, there exists a unique $\mathbf{v}^{\beta,m} \in E^m$ solution of the minimisation of J_β in E^m subject to constrain (1) - (2). Moreover

$$\left|\left|\mathbf{v}^{\beta} - \mathbf{v}^{\beta,m}\right|\right|_{L^2(0,T;\mathbb{R}^r)} \longrightarrow 0, \text{ as } m \to +\infty. \tag{20}$$

One shall need the the above result to construct numrical scheme for solution of the null control problem (3) by taking β small enough and the parameter m great anough.

3. Numerical Scheme and Implementation in MATLAB

Note that, looking for $\mathbf{v}(t) = (v_i(t))$ in E^m amounts to finding the matrix $W(t) = (W_{ij})$ of order $r \times m$ which satisfies

$$W_{ij} = v_i(t_j) \text{ for all } t \in [t_j,\ t_{j+1}[\tag{21}$$

Conversely, for given matrix W, the corresponding function $v(t)$ can be obtained by using the following Matlab script

```
function v=controlfun(w,m,T,t)
dt=T/m;
for k=1:m-1
    if t>=(k-1)*dt && t<k*dt
        v=w(:,k);
    end
end
if t>=(m-1)*dt && t<=m*dt
        v=w(:,m);
end
```

The minimisation problem (8) comes down to determining the matrix W that minimizes the functional

$$\hat{J}_\beta(W) := \frac{1}{2}\sum_{k=1}^{m} W(:,k)^T \mathbb{C} W(:,k) + \frac{1}{2\beta}\|\mathbf{u}(0)\|_{R^n}^2 \tag{22}$$

To calculate the solution of the system (1)-(2), the MATLAB function *ode45* which solves differential systems of medium size is used.. Firstly, one will need to write the following code

```
function Z=secondmember(t,u,A,B,w,m,T)
v=controlfun(w,m,T,T-t);
Z=-A*u-B*v;
```

Then the solution can be obtained using the following code line:

```
[t,Z]=ode45(@(t,y)secondmember(t,y,A,...B,w,m,T),[0 T],u0)
```

Below is given a MATLAB code written in this paper to implement the objective function (22).

```
function J=objective(v,A,B,C,UT,T,...beta,m)
w=reshape(v,length(v)/m,m);
[t,Z]=ode45(@(t,y)secondmember(t,y,...A,B,w,m,T),[0 T],UT);
U0=Z(end,:)';
J=U0'*U0/(2*beta);
for k=1:m
    J=J+w(:,k)'*C*w(:,k)/2;
end
```

Finally, thanks to the MATLAB built-in function *fminunc* which solves minimization problems, we propose the following main code to calculate an approximate solution of the minimisation problem (8).

```
function [t,Z,v]=optimalcontrol(A,...B,C,u0,T,beta,m)
t=linspace(0,T,m+1);
ns=(length(t)-1)*size(B,2);
OPTION = optimset('LargeScale','off');
problem = createOptimProblem(...'fminunc','objective',...@(v)objective(v,A,B,
        C,u0,T,...beta,m),'x0',ones(ns,1),...'options',OPTION);
ms = MultiStart;
[v,J] = run(ms,problem,4);
w=reshape(v,length(v)/m,m);
[t,Z]=ode45(@(t,y)secondmember(t,y,...A,B,w,m,T),[0 T],u0);
Z=Z(end:-1:1,:);
v=[ ];
for k=1:length(t)
    v=[v controlfun(w,m,T,t(k))];
end
v=v';
```

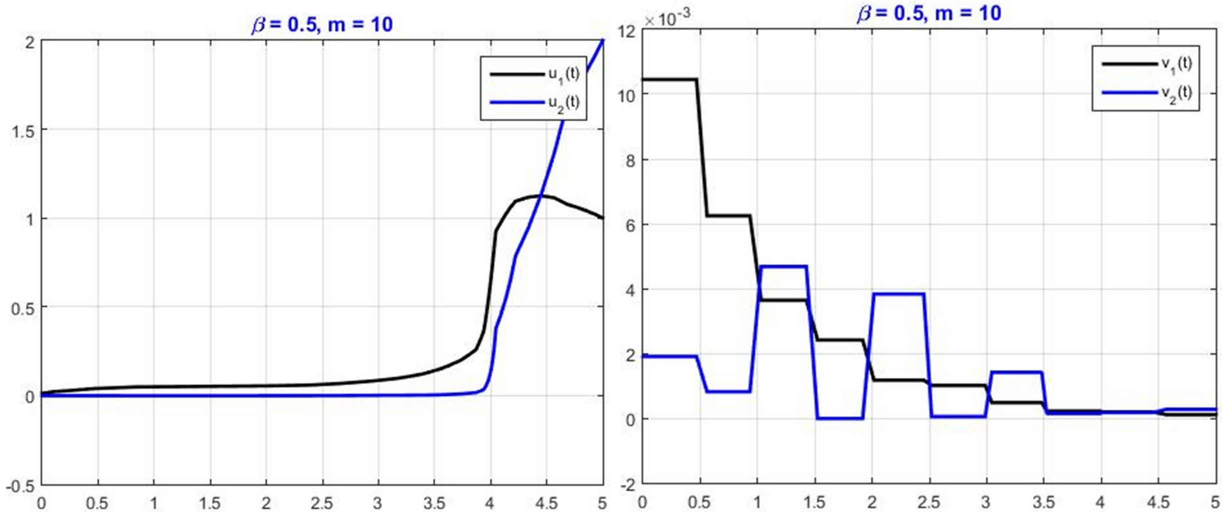

Figure 1. The controlled solution $(u_1(t), u_2(t))$ (left) and the approximated optimal control (right) for $\beta = 0.5$.

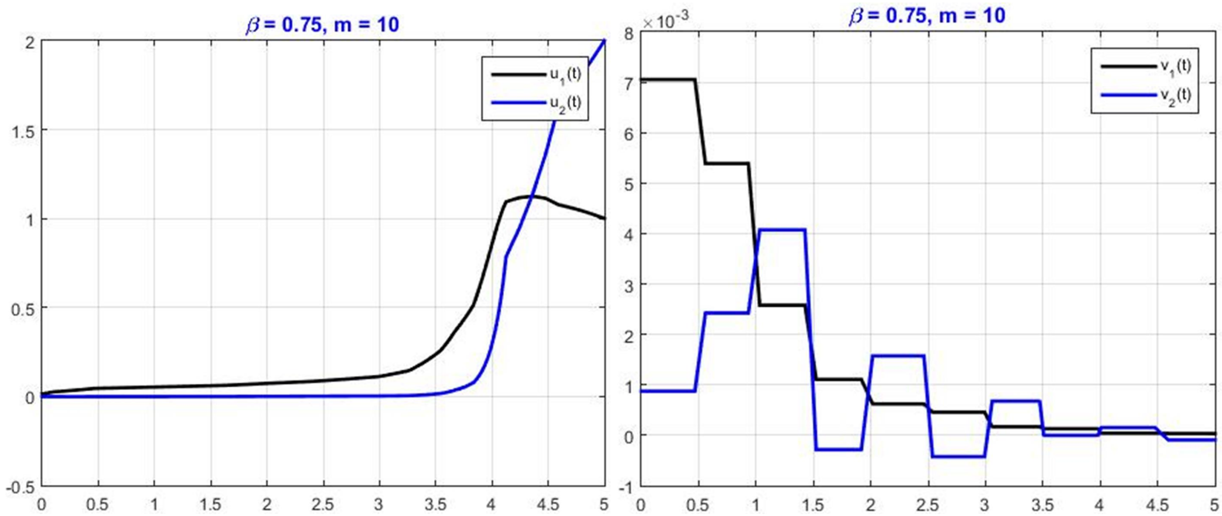

Figure 2. The controlled solution $(u_1(t), u_2(t))$ (left) and the approximated optimal control (right) for $\beta = 0.75$.

4. Numerical Experiments

4.1. Example 1

As a first example, consider the following system

$$\begin{cases} \dfrac{du_1}{dt} &= u_1 - u_2 + v_1,\ t \in [0, T] \\ \dfrac{du_2}{dt} &= 2u_2 + v_2,\ t \in [0, T] \\ u_1(T) &= 1 \\ u_2(T) &= 2 \end{cases} \tag{23}$$

And, determine an approximation of the control $\mathbf{v} = (v_1, v_2)$ such that the corresponding solution vanishes at time $t = 0$. As develpped in section 2, the objective function is defined by

$$J_\beta(v_1, v_2) = \frac{1}{2}\int_0^T \left(v_1^2(t) + v_2^2(t)\right) dt + \frac{1}{2\beta}\left(u_1^2(0) + u_2^2(0)\right) \tag{24}$$

The system (23) can be written in the form (1) by setting

$$G = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \text{ and } \mathbb{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The numerical experiment is conducted by taking $T = 5$ and $m = 10$. The approximate null controls for various values of β as well as theirs corresponding solutions are represented in Figures 1 and 2.

4.2. Example 2: Null Controllability of the 1d Heat Equation

Let set $\Omega =]0,\ 1[$ and $\omega \subset\subset \Omega$. Consider the problem: given any data $u_T \in L^2(\Omega)$, find a control $v \in L^2((0,T)\times\omega)$ such that the unique solution of the system

$$-u_t - u_{xx} = v\chi_\omega,\ (t,x) \in (0,T)\times\Omega \tag{25}$$

$$u(t,0) = u(t,1) = 0,\ t \in (0,T) \tag{26}$$

$$u(T,x) = u_T(x),\ x \in \Omega \tag{27}$$

satisfies

$$u(0,x) = 0,\ x \in \Omega. \tag{28}$$

This problem is transformed using spatial semi discretization. Let fix $N > 0$ and pose $x_k = \dfrac{k}{N+1}$, $k = 0, ..., N+1$. Discrete approximation via finite differences can be stated as follows.

$$\frac{d}{dt}(u_k) = \frac{-u_{k-1} + u_k - u_{k+1}}{h^2} - v_k\chi_\omega(x_k),\ k = 1, ..., N \tag{29}$$

$$u_0(t) = u_{N+1}(t) = 0,\ t \in (0,T) \tag{30}$$

$$u_k(T) = u_T(x_k),\ k = 1, ..., N \tag{31}$$

Setting $\mathbf{u}(t) = (u_1(t),\ u_2(t),\ ...,\ u_N(t))^T$ and $\mathbf{v}(t)$ the vector whose components are approximations of the control v at points (t, x_k) for $x_k \in \omega$, one can write the above discrete system in the form (1) where G is the N by N order matrix given by

$$G = \begin{pmatrix} 1 & -1 & & & \\ -1 & 1 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 1 & -1 \\ & & & -1 & 1 \end{pmatrix}$$

and $\mathbb{B}$ the N by r matrix, r being the number of discrete points in ω whose $(i,\ j)$th components are defined by

$$\mathbb{B}_{ij} = \begin{cases} 1 & \text{if } x_i \text{ is the } jth \text{ discrete point of } \omega \\ 0 & \text{otherwise} \end{cases}$$

The MATLBAB code for dysplaying optimal control and the corresponding solution of this problem is presented below

```
function testotpimcontrole2(u0,a,b,...beta,N,m)
x=linspace(0,1,N+2)'; h=1/(N+1);
A=toeplitz([2 -1 zeros(1,N-2)])/(h^2);
UT=u0(x(2:end-1)); I=find(x>=a & x<=b);
B=zeros(N,length(I));C=eye(size(B,2));
for j=1:length(I),    B(I(j),j)=1;end
[t,Z,v]=optimalcontrol(A,B,C,UT,...1.2,beta,m);
nt=length(t);nv=size(v,2);
Z=[zeros(1,nt);Z';zeros(1,nt)];
```

```
[tt,xx]=meshgrid(t',x); mesh(xx,tt,Z)
xlabel('x','fontsize',14)
ylabel('t','fontsize',14)
zlabel('u(t,x)','fontsize',14)
figure (2)
[ttt,xxx]=meshgrid(t',[0 a-.1 ...a-.02 linspace(a,b,nv)...b+.02 b+.1 1] );
v=[zeros(nt,3) v zeros(nt,3)]';
[tttt,xxxx]=meshgrid(linspace(0,...t(end),20),linspace(0,1,20));
vv=griddata(xxx,ttt,v,xxxx,tttt,...'cubic'); mesh(xxxx,tttt,vv)
```

This is illustrated in figures 3 and 4 via following commands

```
>> beta=0.002;N=10;m=18;
>> testotpimcontrole2(@(x)(sin(pi*x)),...
          .2,.5,beta,N,m);
```

and

```
>> beta=0.002;N=10;m=18;
>> testotpimcontrole2(@(x)(exp(-300*...
      (x-3/4).^2)),.2,.4,beta,N,m);
```

witch calculate null controls for problems with respectively the final states $u_0(x) = \sin(\pi x)$ and $u_0(x) = e^{-300(x-3/4)^2}$.

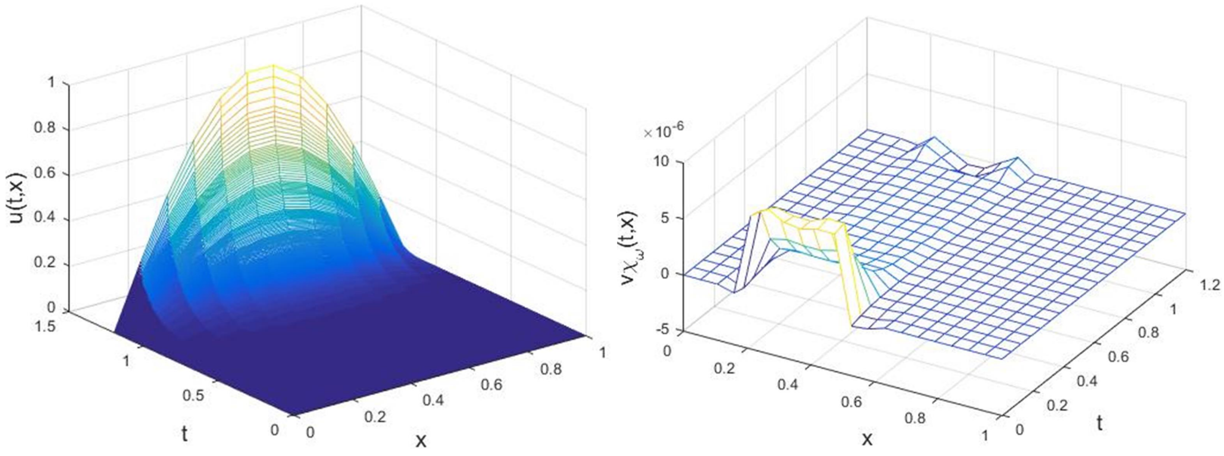

Figure 3. $\beta = 0.005$ - $N = 10$ - $m = 12$ Solution for optimal control of minimum norm with initial state $u_0(x) = \sin(\pi x)$). Control $v\chi_\omega$ with $\omega =]0.2,\ 0.5[$ (lower) and corresponding controlled solution (upper).

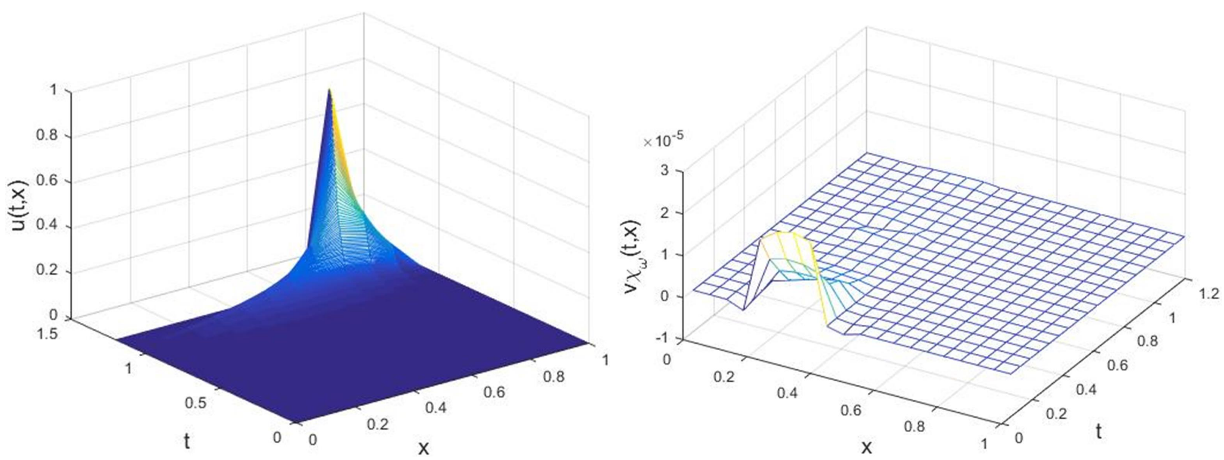

Figure 4. $\beta = 0.005$ - $N = 10$ - $m = 12$ Solution for optimal control of minimum norm with initial state $u_0(x) = e^{-300(x-3/4)^2}$. Control $v\chi_\omega$ with $\omega =]0.2,\ 0.5[$ (lower) and corresponding controlled solution (upper).

5. Concluding Remarks

In this paper, a new approach is developed for zero controllability problems for a more general Cauchy retrograde differential system and a suitable numerical scheme is proposed. It is shown how to implement this scheme in Matlab using some of its built-in functions. The proposed numerical experiment shows that the numerical approach developed using Matlab in this paper produces very satisfactory results. In particular, example 2 of the considered numerical experiment shows that this approach can be efficiently generalized to the computation of null control for more general PDE's systems.

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