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Construction of an Approximate-Analytical Solution for Boundary Value Problems of a Parabolic Equation

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Abstract: The article considers an approximate analytical solution (AAS) of a linear parabolic equation with initial and boundary conditions with one spatial variable. Many problems in engineering applications are reduced to solving an initial boundary value problem of a parabolic type. There are various analytical, approximate-analytical and numerical methods for solving such problems. Here we consider ways to obtain an AAS based on the movable node method. Three approaches to obtaining an AAS are considered. In all approaches, an arbitrary point (a movable node) is considered inside the area where the solution is being sought. In the first approach, the parabolic equation is approximated by a difference scheme with a movable node in both variables. As a result, the differential problem is reduced to one algebraic equation (in the case of a boundary condition of the first kind), solving which we obtain an AAS. If one of the boundary conditions is of the second or third kind, assuming the fulfillment of the difference equation up to the boundary, we determine the unknown value of the solution on the boundary. The second and third approaches are based on the idea of the method of lines for differential equations in combination with a movable node. In the second approach, in a parabolic equation, the time derivative is approximated by a difference relation with the node being moved, and as a result we obtain an ordinary second-order differential equation with boundary conditions. Solving the obtained ordinary differential equation, we have an AAS. In the third approach, the approximation of the parabolic equation is performed only by the spatial variable. As a result, we obtain an ordinary differential equation of the first order with an initial condition and the solution of which gives an AAS to the original problem. The simplicity of the described approach allows the use of its engineering calculations. Comparisons have been made.

Keywords: Parabolic Equation, Approximate-Analytical Solution, Moving Nodes, Method of Line

1. Introduction

Processes in hydrodynamics, heat transfer, boundary layer flow, elasticity, quantum mechanics and electromagnetic theory are modeled by differential equations. Only some of these equations can be solved analytically. But finding exact solutions when they exist is always necessary to better explain the phenomenon being modeled. The search for an analytical solution gives an advantage for the analysis of processes [1-3].

Analytical methods have a relatively low degree of universality for solving such problems. Approximate analytical methods (projection, variational methods, small parameter method, operational methods, various iterative methods) are more universal [4-7].

The proposed approach to solving initial-boundary value problems is based on the method of moving nodes [8-10]. The method combines the approximation of derivatives, appearing in the equation, by difference relations and obtaining an approximately analytical expression for solving the problem. In this case, we can obtain an approximate analytical solution of the problem, which is a hybrid of known methods. Note that obtaining an approximate-analytical solution of differential equations is based on numerical methods. The nature of numerical methods also makes it possible to obtain an approximately analytical expression for solving differential equations. For this, the socalled “movable node” is introduced [10].

The aim of the study is to develop a computing technology based on the method of moving nodes for a parabolic equation. Develop a parabolic equation for the shifted problem and give test examples.

2. Formulation of the Problem

Consider the parabolic equation

$$\frac{\partial u}{\partial t}=\sigma\frac{\partial^2 u}{\partial x^2}+f(x,t), \tag{1}$$

with initial

$$u(x,0)=\varphi(x) \tag{2}$$

and boundary conditions

$$u(0,t)=u_0(t),\ u(b,t)=u_b(t). \tag{3}$$

In (1) - (3) $u(x,t)$ the unknown function, $f(x,t),\ \varphi(x), u_0(t)$ and $u_b(t)$ the given functions, Equation (1) is considered in the area $0\le x\le b,\ 0\le t\le T,\ \sigma>0.$

In addition to the boundary condition of the first kind (3), there are boundary conditions of the second kind,

$$\frac{\partial u}{\partial x}(0,t)=u_0(t),\ \frac{\partial u}{\partial x}(b,t)=u_b(t). \tag{4}$$

and third kind

$$\begin{aligned}&\frac{\partial u}{\partial x}(0,t)=\alpha u(x,t)+u_0(t),\\&\frac{\partial u}{\partial x}(b,t)=\beta u(x,t)+u_b(t).\end{aligned} \tag{5}$$

To solve the problem, there are various methods: analytical and numerical. It is assumed that the initial and boundary conditions are such that the solution to the problem exists and is unique.

To solve the problem, an approximate-analytical solution is proposed here, which allows obtaining an approximate-analytical solution of problems of differential equations.

3. Obtaining an Approximate Analytical Solution

To obtain an approximate-analytical solution of the problem, we use the method of moving nodes. Consider three approaches to obtain a solution to the problem. The method for obtaining an approximate analytical solution is very simple, however, it represents a rougher representation of the solution to the problem. Very useful for engineering calculations.

3.1. Obtaining an Approximate Solution When Approximating in Both Variables

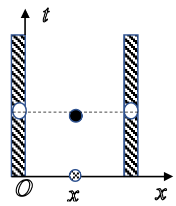


Figure 1. *Solution area.*

Consider problem (1) - (3) and choose an arbitrary point (x,t) inside the region (Figure 1) where the solution is sought. We approximate the derivative with respect to t by a finite forward difference, and approximate the second derivative with respect to x on a three-point template with a non-uniform step. Then instead of equation (1), we obtain an approximate equation:

$$\frac{U(x,t)-U(x,0)}{t}=\sigma\frac{b}{2}\left[\frac{U(b,t)-U(x,t)}{b-x}-\frac{U(x,t)-U(0,t)}{x}\right]+f(x,t), \tag{6}$$

Here $U(x,t).$ is an approximate solution $u(x,t).$. Using the boundary and initial conditions, we have,

$$U(x,0).=u(x,0)=\varphi(x)\ ,$$

$$U(0,t).=u(0,t)=u_0(t),$$
$$U(b,t).=u(b,t)=u_b(t),$$

Solving equations (4) taking into account the boundary and initial conditions, we obtain

$$U(x,t).=\frac{x(b-x)}{2\sigma t+x(b-x)}\varphi(x)+\frac{2\sigma t[u_b(t)x+u_0(t)(b-x)]}{b(2\sigma t+x(b-x))}+\frac{x(b-x)t}{x(b-x)+2\sigma t}\cdot f(x,t). \tag{7}$$

(7) is an approximate analytical solution of problem (1)-(3).

It is easy to see that the approximate solution satisfies conditions (2) and (3).

(7) the formula gives an approximate solution if the boundary values of the desired function are given, i.e. under conditions (3). Let us now consider if other boundary conditions. Let the boundary conditions be given, for example, in the form

$$u(0,t) = u_0(t), \quad \frac{\partial u(b,t)}{\partial x} = \psi_b(t). \tag{8}$$

In this case, we cannot obtain a solution for (7), due to the fact that $u_b(t)$ unknown. Consider obtaining on the basis of the second boundary condition (8).

In (7) we pass to the limit at $x \to b$, then we obtain

$$\frac{U(b,t)-U(x,0)}{t} = \sigma \frac{b}{2}\left[\frac{dU(b,t)}{dx} - \frac{u_b(t)-u_0(t)}{b}\right] + f(b,t),$$

From here we determine

$$\frac{dU(b,t)}{dx} = \frac{b^3+2\sigma t}{2b\sigma t}u_b(t) - \frac{u_0(t)}{b} - \frac{b}{2\sigma}f(b,t) - \frac{b}{2\sigma t}\varphi(x)$$

Using the second condition in (8), we find

$$u_b(t) = \frac{2b\sigma t}{b^3+2\sigma t}u_0(t) + \frac{b^3 t}{b^3+2\sigma t}f(b,t) + \frac{b^2}{b^3+2\sigma t}\varphi(b) + \frac{2b^2\sigma t}{b^3+2\sigma t}\psi(t) \tag{9}$$

Using expression (9) according to formula (7), we determine the unknown in a given area. By doing the same, we can find an approximate solution with different boundary conditions.

3.2. Obtaining an Approximate Solution of the Problem When Approximating with Respect to the Variable t

This approach is close to the method of straight lines, which is widely used in numerical analysis [11-16].

In equation (1), we approximate only with respect to the variable t. Then we have

$$\frac{U(x,t)-U(x,0)}{t} = \sigma\frac{d^2U}{dx^2} + f(x,t) \tag{10}$$

From here,

$$\sigma t\frac{d^2U}{dx^2} - U(x,t) = -t\cdot f(x,t) - u(x,0) \tag{11}$$

We solve equation (11) as an ordinary differential equation, while considering t as a parameter. The exact solution of the homogeneous equation (11) has the form (under the assumption σ. does not depend on x)

$$U^* = C_1\exp\left(-\sqrt{p}x\right) + C_2\exp\left(\sqrt{p}x\right).$$

Here $p = \frac{1}{\sigma t}$. Let $\bar{U} = Q(x,t)$. be a particular solution of equation (11). Then the solution of Eq. (11), taking into account the boundary conditions, has the form

$$U(x,t). = C_1\exp\left(\sqrt{p}x\right) + C_2\exp\left(-\sqrt{p}x\right) + Q(x,t). \tag{12}$$

The coefficients C_1 and C_2 are determined based on the boundary condition (2):

$$C_1 = \frac{\exp\left(-\sqrt{pb}\right)\left(\varphi(0)-Q(0,t)\right) - \left(\varphi(b)-Q(b,t)\right)}{\exp\left(-\sqrt{pb}\right)-\exp\left(\sqrt{pb}\right)}, \quad C_2 = \frac{\left(\varphi(b)-Q(b,t)\right) - \exp\left(\sqrt{pb}\right)\left(\varphi(0)-Q(0,t)\right)}{\exp\left(-\sqrt{pb}\right)-\exp\left(\sqrt{pb}\right)}$$

If the boundary conditions satisfy conditions (8), the coefficients in (12) have the form

$$C_1 = \frac{-\sqrt{p}\exp\left(-\sqrt{pb}\right)\left(u_0(t)-Q(0,t)\right) - \psi_b(t) + \frac{\partial Q(b,t)}{\partial x}}{-\sqrt{p}\left(\exp\left(-\sqrt{pb}\right)+\exp\left(\sqrt{pb}\right)\right)}, \quad C_2 = \frac{-\sqrt{p}\exp\left(-\sqrt{pb}\right)\left(u_0(t)-Q(0,t)\right) + \psi_b(t) - \frac{\partial Q(b,t)}{\partial x}}{-\sqrt{p}\left(\exp\left(-\sqrt{pb}\right)+\exp\left(\sqrt{pb}\right)\right)}$$

3.3. Obtaining an Approximate Solution of the Problem When Approximating with Respect to the Variable x

We approximate equation (1) as follows.

$$\frac{dU(x,t)}{dt}=\sigma\frac{b}{2}\left[\frac{U(b,t)-U(x,t)}{b-x}-\frac{U(x,t)-U(0,t)}{x}\right]+f(x,t) \tag{13}$$

From here

$$\frac{dU(x,t)}{dt}+p(x)\cdot U=q(x,t) \tag{14}$$

It's supposed to be here

$$p(x)=\frac{2\sigma}{(b-x)x},\ q(x,t)=\frac{2\sigma}{b}\cdot\frac{u_b(t)x+u_0(t)(b-x)}{x(b-x)}.+f(x,t)$$

We solve equation (13) as a linear ordinary differential equation. Considering x as a parameter and t as an independent variable, taking into account the initial condition, we obtain the solution

$$U(x,t)=\exp\left(-\int_0^t p(x)dt\right)\left(\int_0^t q(x,t)\exp\left(\int_0^t p(x)dt\right)dt+\varphi(x)\right) \tag{15}$$

If σ does not depend on t

$$\exp\left(-\int_0^t p(x)dt\right)=\exp\left(-\frac{2\sigma t}{x(b-x)}\right).$$

Then the solution looks like

$$U(x,t)=\exp\left(-\frac{2\sigma t}{x(b-x)}\right)\left[\varphi(x)+\int_0^t q(x,t)\exp\left(\frac{2\sigma t}{x(b-x)}\right)dt\right] \tag{16}$$

If the boundary conditions do not depend on t

$$U(x,t)=\varphi(x)\exp(-p\cdot t)+\frac{2\sigma}{b}\left(xu_b+(b-x)u_0\right)\left(1-\exp(-p\cdot t)\right) \tag{17}$$

Let us consider how the solution changes under condition (8). The unknown boundary condition appears in equation (14) within the coefficient $u_b(t)$. Let us write equation (13) for $x\to b$,

$$\frac{dU(b,t)}{dt}=\sigma\frac{b}{2}\left[\frac{dU(b,t)}{dx}-\frac{U(b,t)-U(0,t)}{b}\right]+f(b,t) \tag{18}$$

From here we get

$$\frac{dU(b,t)}{dt}+\frac{2\sigma}{b^2}U(b,t)=\frac{2\sigma}{b}\psi(t)+\frac{2\sigma}{b^2}u_0(t)+f(b,t) \tag{19}$$

We have obtained an ordinary differential equation, solving which we determine the right boundary value of the unknown function.

If the right boundary condition does not depend on time from (19) we determine

$$U(b,t)=b\psi(t)+u_0(t)+\frac{b^2}{2\sigma}f(b,t) \tag{20}$$

4. Examples

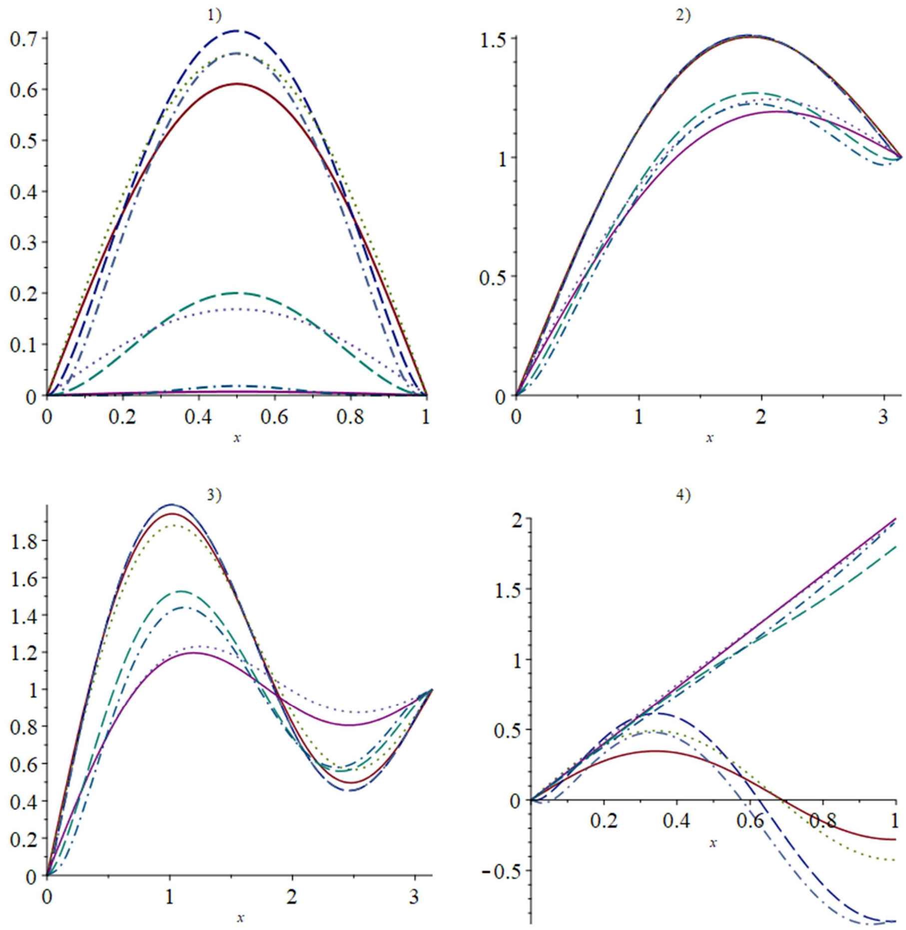
Let's look at some examples.

The initial data are given in table 1.

Table 1. *Initial data.*

№	b	$\varphi(x)$	$u_0(t)$	$u_b(t)$	$\psi(t)$	$f(x,t)$
1	1	$\sin(\pi x)$	0	0	-	0
2	π	$\sin(x)+\frac{x}{\pi}$	0	1	-	0
3	π	$\sin(x)+\sin(2x)$	0	1	-	0
4	1	$\sin(\frac{3\pi x}{2})$	0	-	t	x

The exact solutions of the above problems, respectively, have the form:



1) $\exp\left(-\pi^2 t\right)\sin(\pi x)$, 2) $\exp(-t)\sin(x)+\frac{x}{\pi}$, 3) $\exp(-t)\sin(x)+\exp(-2t)\sin(2x)+\frac{x}{\pi}$, 4) $xt+\exp\left(-\left(\frac{3\pi}{2}\right)^2 t\right)\sin\left(\frac{3\pi x}{2}\right)$

Figure 2. *Comparison of results. The numbering of the graphs in each figure above corresponds to the numbers of the problems. 1) t=0.05 and t=0.5; 2) t=0.05 and t=0.5; 3) t=0.05 and t=0.5; 4) t=0.05 and t=2.*

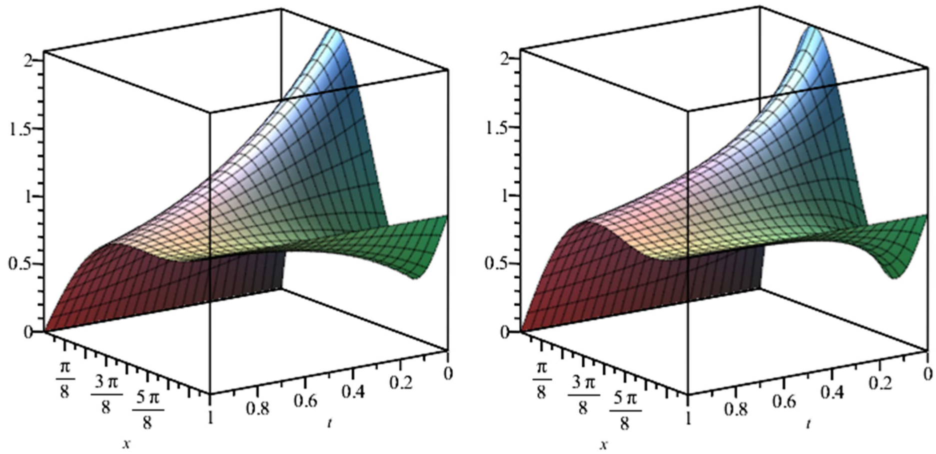
Figure 2 compares the approximate analytical solution with the exact ones. In all graphs, the solid lines are the exact solution, the dotted lines are the standard scheme, the dotted lines are the discrete scheme for t; dotted-dotted - discrete circuit in x. In graphs 1), 2) and 3), the upper lines correspond to the solutions at t=0.05, and the lower ones are obtained at t=0.5. In graph 4) the lower lines refer to t=0.05 and the upper lines are obtained at t=2.

It can be seen from the graphs that the proposed methods for obtaining an approximately analytical solution reasonably reflect the behavior of the solution.

Figure 3 shows the exact and approximate solution of problem 3.

Figure 3 shows the closeness of the approximate and exact solutions.

Table 2 shows the root-mean-square error of the considered problems.



a) Exact solution b) Approximate Solution

Figure 3. *Graphical representation of the exact and approximate solution of problem 3. The approximate solution corresponds to the case with discrete in variable t.*

Table 2. *Comparison by rms errors.*

№ Problems	**Standard scheme**	**Scheme discrete in x**	**Scheme discrete in t**
1	0.113332	0.104339	0.023974
2	0.067477	0.051968	0.048134
3	0.1915033	0.070304	0.139150
4	0.268632	0.227815	0.077888

From Table 2, It can be seen that partial approximations give better results compared to the standard scheme.

5. Conclusion

The proposed approach for obtaining an approximate analytical solution for a parabolic equation is a good tool. Obtaining by this method the solution of initial-boundary value problems for a parabolic equation can be successfully applied to engineering calculations, which describes the qualitative behavior of the solution of the process under study. Obtaining an approximate-analytical solution according to the standard scheme is more universal, compared to other approaches that use the method of lines in combination with the method of a moving node, but is inferior in accuracy.

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