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The Free Boundary Problem of a Predator-Prey Model with Fear Effect and Stage Structure

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Abstract: In this paper, fear effect and stage structure are introduced in a free boundary problem of a prey-predator model. This system simulates the spread of an invasive or newly introduced predator species, taking into account the presence of both immature and mature stages of prey that are affected by fear of the predator. The predator's predation behavior on adult prey induces fear in the prey, which in turn causes the prey to seek out safer habitats. While this short-term survival strategy may be effective, it ultimately leads to a decrease in the prey's long-term survival fitness, including reduced reproductive ability. Consequently, the overall population of prey is expected to decline over the long term. The existence and uniqueness of the solution is given, and the comparison principle is used to discuss the long-term behavior of the solution by constructing a sequence of upper and lower solutions. We obtain a spreading–vanishing dichotomy for this model, in other words, when the predator can only spread in a limited area, the predator will eventually become extinct, the population density of the two stages of prays will tend to two positive constants, and when the predator can spread to infinity, the predator ultimately survives, and their population density, defined as (u, v, w) will eventually tend to (u^*, v^*, w^*) which we defined blow.

Keywords: Fear Effect, Stage Structure, Free Boundary Problem, Asymptotic Property

1. Introduction

In recent years, the introduction of free boundary into prey-predator models [1-3] can better describe the dynamics of invading predator and prey, and is helpful to predict and prevent the invading predator. Due to the increasingly in-depth discussion of prey-predator models, more and more elements have been introduced into the model. By introducing the fear effect [4-7], it is concluded that the fear of the prey for the predator changes its growth and development rules, which becomes the main reason for the extinction of the prey. The species itself has multiple stages, and the predatory structure at different stages may not be the same [8-10].

This paper investigates a prey-predator model with a fear effect and stage structure by studying its free boundary problem. By combining free boundary, fear effect, and stage structure, we provide a comprehensive analysis of this model.

$$\begin{cases} u_t - d_1 u_{xx} = \dfrac{bv}{1+kw} - r_1 u - \alpha u, & t>0, x\in R, \\ v_t - d_2 v_{xx} = \alpha u - \lambda v^2 - r_2 v - \beta v w, & t>0, x\in R, \\ w_t - d_3 w_{xx} = \beta v w - \eta w^2 - r_3 w, & t>0, g(t)<x<h(t), \\ w=0, & t>0, x\le g(t), x\ge h(t), \\ g'(t) = -\mu w_x(t,g(t)), h'(t) = -\mu w_x(t,h(t)), & t\ge 0, \\ w(0,x) = w_0(x), & x\in[-h_0,h_0], \\ g(0) = -h_0, h(0) = h_0, u(0,x) = u_0(x), v(0,x) = v_0(x), & x\in R, \end{cases} \tag{1}$$

where u, v are the population densities of immature and mature prey, respectively, w is the density of the predator, b is the birth rate of the immature prey, $1/(1+kw)$ is the fear factor, r_1, r_2, r_3 are mortality rates, α is the probability of the prey transforming from immature to mature, λ, η are adult prey and predator environmental factors respectively, β is the predatory coefficient between two species, $x = g(t), x = h(t)$ are the invading front and evolves according to the Stefan condition, and the initial functions satisfy

$$u_0, v_0 \in C_b(R), u_0 > 0, v_0 > 0, x \in R, \quad w_0 \in W_p^2([-h_0, h_0]), w_0 > 0, x \in (-h_0, h_0), w_0(\pm h_0) = 0.$$

2. Existence and Uniqueness

Theorem 2.1 establishes the existence and uniqueness of a local solution, which can be extended to infinity and estimated from Theorem 2.2. The proofs of Theorem 2.1 and Theorem 2.2 are similar to them of Theorem 1.1 in [11], Theorem 2.2 and Theorem 3.1 in [12]. For convenience, we first give some symbols.

$$D_T^{g,h} = \{0 < t \le T, g(t) < x < h(t)\},$$

$$\mathfrak{C}_T = C_b([0,T] \times R) \cap C_{\mathrm{loc}}^{2+\alpha, 1+\frac{\alpha}{2}}((0,T] \times R).$$

Theorem 2.1 For any given $\alpha \in (0,1)$ and $p > \frac{3}{1-\alpha}$, there exists a $T > 0$ such that the problem (1) exists an unique solution (u, v, w, g, h) satisfying

$$(u, v, w, g, h) \in (\mathfrak{C}_T)^2 \times \left[W_p^{1,2}(D_T^{g,h}) \cap C^{\frac{1+\alpha}{2}, 1+\alpha}(\bar{D}_T^{g,h}) \right] \times [C^{1+\frac{\alpha}{2}}([0,T])]^2,$$

Moreover

$$u, v > 0 \ in \ [0,T] \times R, \quad w > 0 \ in \ D_T^{g,h}, \quad g'(t) < 0, h'(t) > 0 \ in \ [0,T].$$

Theorem 2.2 For any given $\alpha \in (0,1)$, the problem (1) exists an unique global solution (u, v, w, g, h) satisfying

$$0 < u(t,x), v(t,x) < M_1, (t,x) \in [0,T] \times R, \quad 0 < w(t,x) < M_1, (t,x) \in D_\infty^{g,h}, \quad g'(t) < 0, h'(t) > 0, t > 0.$$

$$g'(t) < 0, h'(t) > 0, t > 0, \quad \|w\|_{C^1([g(t), h(t)])} + \|g, h\|_{C^{1+\alpha/2}([1,\infty))} \le C, \ t \ge 1. \tag{2}$$

3. Long Time Behavior

This section is devoted to the analysis of the long-term behavior of the solution. For convenience, we first give some symbols.

$$\rho = \frac{\beta\alpha b}{\eta\lambda(r_1+\alpha)}, \quad \theta = \frac{\beta r_2}{\lambda\eta} + \frac{r_3}{\eta}, \quad f(z) = \frac{\rho - \theta - k\theta z}{1+kz} - \frac{\beta^2 z}{\eta\lambda}, \quad A = k(r_1+\alpha)(\lambda\eta + \beta^2),$$

$$B = (d_1+\alpha)(\lambda\eta + \lambda k r_3 + r_2 k\beta + \beta^2), \quad C = -\alpha b\beta + (r_2\beta + \lambda r_3)(r_1+\alpha),$$

If $C < 0$, then (1) has an unique positive constant equilibrium.

$$u^* = \frac{b\beta w^* - br_3}{\lambda(1+kw^*)(r_1+\alpha)}, v^* = \frac{\beta w^* - r_3}{\lambda}, w^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A}.$$

Theorem 3.1 Suppose that (u, v, w, g, h) is the solution of problem (1). If $h_\infty - g_\infty < \infty$, then

$$\lim_{t\to\infty} \|w(t,\cdot)\|_{C([g(t),h(t)])} = 0,$$

and

$$\lim_{t\to\infty} \|u(t,\cdot)\|_R = \frac{b}{\lambda(r_1+\alpha)}\left(\frac{b\alpha}{r_1+\alpha} - r_2\right), \quad \lim_{t\to\infty} \|v(t,\cdot)\|_R = \frac{1}{\lambda}\left(\frac{b\alpha}{r_1+\alpha} - r_2\right)$$

uniformly in any bounded subset of $(-\infty, \infty)$.

Proof. Theorem 2.2 implies that $0 < w < M_1$ for $t > 0, g(t) < x < h(t)$. And w satisfies

$$\begin{cases} w_t - d_3 w_{xx} \geq -(r_3 + \eta M1)w, & t > 0, g(t) < x < h(t), \\ w(t, g(t)) = w(t, h(t)) = 0, & t > 0, \\ g'(t) \leq -\mu w_x(t, g(t)), h'(t) \geq -\mu w_x(t, h(t)), & t > 0, \\ g(0) = -h_0, h(0) = h_0, w(0, x) = w_0(x), & -h_0 \leq x \leq h_0. \end{cases}$$

We can easily obtain $\lim_{t\to\infty} h'(t) = 0$ by using $h_\infty < \infty$ and (2). According to Proposition 2 in [13] and Theorem 2.2, we can see that

$$\lim_{t\to\infty} max_{g(t)\leq x\leq h(t)} w(t,x) = 0$$

For any $\varepsilon > 0$, there exists $T > 0$ such that $w(t,x) < \varepsilon$ for $t \geq T$. We consider Cauchy problem

$$\begin{cases} \underline{u}_t - d_1 \underline{u}_{xx} = \dfrac{b\underline{v}}{1+k\varepsilon} - r_1 \underline{u} - \alpha \underline{u}, & t > 0, x \in R, \\ \underline{v}_t - d_2 \underline{v}_{xx} = \alpha \underline{u} - \lambda \underline{v}^2 - r_2 \underline{v} - \beta\varepsilon \underline{v} & t > 0, x \in R, \end{cases}$$

Initial conditions are $\underline{u}(0,x) = u_0(x), \underline{v}(0,x) = u_0(x)$. By applying the comparison principle, we can conclude that $u(t,x) \geq \underline{u}(t,x), v(t,x) \geq \underline{v}(t,x)$. Note that Theorem 1.1 in [14], the solution of the above Cauchy problem satisfies

$$\lim_{t\to\infty} \underline{u}(t,x) = \frac{b}{\lambda(r_1+\alpha)}\left(\frac{b\alpha}{(r_1+\alpha)(1+k\varepsilon)} - r_2 - \beta\varepsilon\right),$$
$$\lim_{t\to\infty} \underline{v}(t,x) = \frac{1}{\lambda}\left(\frac{b\alpha}{(r_1+\alpha)(1+k\varepsilon)} - r_2 - \beta\varepsilon\right)$$

uniformly in any bounded subset of $(-\infty, \infty)$. Due to the arbitrariness of ε, we have

$$\liminf_{t\to\infty} u(t,x) \geq \frac{b}{\lambda(r_1+\alpha)}\left(\frac{b\alpha}{(r_1+\alpha)} - r_2\right),$$
$$\liminf_{t\to\infty} v(t,x) \geq \frac{1}{\lambda}\left(\frac{b\alpha}{(r_1+\alpha)} - r_2\right) \tag{3}$$

uniformly in any bounded subset of $(-\infty, \infty)$.

Next, we are going to consider the ODE problem

$$\begin{cases} \overline{u}_t = b\overline{v} - r_1\overline{u} - \alpha\overline{u}, \quad t > 0; \quad \overline{u}(0) = \|u_0\|_\infty, \\ \overline{v}_t = \alpha\overline{u} - \lambda\overline{v}^2 - r_2\overline{v}, \ t > 0; \quad \overline{v}(0) = \|v_0\|_\infty. \end{cases} \tag{4}$$

Applying the comparison principle, we have $u(t,x) \leq \overline{u}(t), v(t,x) \leq \overline{v}(t)$ for $t \in [0,\infty), x \in R$, where $(\overline{u}(t), \overline{v}(t))$ is the solution of (4). According to theorem 2.1 in [15], it is found that the positive constant equilibrium of (4) is globally asymptotically stable, so

$$\limsup_{t\to\infty} u(t,x) \leq \frac{b}{\lambda(r_1+\alpha)}\left(\frac{b\alpha}{(r_1+\alpha)} - r_2\right),$$
$$\limsup_{t\to\infty} v(t,x) \leq \frac{1}{\lambda}\left(\frac{b\alpha}{(r_1+\alpha)} - r_2\right) \tag{5}$$

Combining (3) and (5), we can see that

$$\lim_{t\to\infty} \|u(t,\cdot)\|_R = \frac{b}{\lambda(r_1+\alpha)}\left(\frac{b\alpha}{r_1+\alpha} - r_2\right), \quad \lim_{t\to\infty} \|v(t,\cdot)\|_R = \frac{1}{\lambda}\left(\frac{b\alpha}{r_1+\alpha} - r_2\right)$$

uniformly in any bounded subset of $(-\infty, \infty)$.

Theorem 3.2 Assume that (u, v, w, g, h) is the solution of problem (1) and the conditions $\rho > \theta, k\rho + \beta^2/\eta\lambda < 1$ hold. If $g_\infty = -\infty, h_\infty = \infty$, then

$$\lim_{t\to\infty} u(t,x) = u^*, \lim_{t\to\infty} v(t,x) = v^*, \lim_{t\to\infty} w(t,x) = w^* \tag{6}$$

Proof. Step 1 Denote $\overline{u}_1 = \dfrac{1}{\lambda}\dfrac{b}{r_1+\alpha}\left(\dfrac{\alpha b}{r_1+\alpha} - r_2\right)$, $\overline{v}_1 = \dfrac{1}{\lambda}\left(\dfrac{\alpha b}{r_1+\alpha} - r_2\right)$. By using (5), we have

$$\limsup_{t\to\infty} u(t,x) \leq \overline{u}_1, \ \limsup_{t\to\infty} v(t,x) \leq \overline{v}_1.$$

For any $\varepsilon > 0$, there exists $T > 0$ such that $v(t,x) \leq \overline{v}_1 + \varepsilon$ for $t > T, x \in R$. The problem

$$\begin{cases} \overline{w}_t = \beta(\overline{v}_1 + \varepsilon)\overline{w} - \eta\overline{w}^2 - r_3\overline{w}, \quad t > T \\ \overline{w}(T) = M_1. \end{cases}$$

exists an unique positive equilibrium $\overline{w}_1^\varepsilon$, which is globally asymptotically stable. According to the comparison principle, we easily see that $w(t,x) \leq \overline{w}(t)$ for $t > T$, therefore $\lim_{t\to\infty} w(t,x) \leq \overline{w}_1^\varepsilon$. Due to the arbitrariness of ε, we have

$$\limsup_{t\to\infty} w(t,x) \leq \frac{\beta\overline{v}_1 - r_3}{\eta} = \rho - \theta := \overline{w}_1. \tag{7}$$

for all $x \in R$.

Step 2 Note that (7), for any $\varepsilon > 0$, there exists $T > 0$ such that $w(t,x) \leq \overline{w}_1 + \varepsilon$ for $t \geq T$. By virtue of a similar analysis of (3), we can conclude that

$$\liminf_{t\to\infty} u(t,x) \ge \frac{b}{\lambda[1+k(\overline{w}_1+\varepsilon)](r_1+\alpha)}\left(\frac{b\alpha}{[1+k(\overline{w}_1+\varepsilon)](r_1+\alpha)} - r_2 - \beta(\overline{w}_1+\varepsilon)\right),$$

$$\liminf_{t\to\infty} v(t,x) \ge \frac{1}{\lambda}\left(\frac{b\alpha}{[1+k(\overline{w}_1+\varepsilon)](r_1+\alpha)} - r_2 - \beta(\overline{w}_1+\varepsilon)\right)$$

uniformly in any bounded subset of $(-\infty,\infty)$. Due to the arbitrariness of ε, we have

$$\liminf_{t\to\infty} u(t,x) \ge \frac{b}{\lambda(1+k\overline{w}_1)(r_1+\alpha)}\left(\frac{b\alpha}{[1+k\overline{w}_1](r_1+\alpha)} - r_2 - \beta\overline{w}_1\right) := \underline{u}_1,$$

$$\liminf_{t\to\infty} v(t,x) \ge \frac{1}{\lambda}\left(\frac{b\alpha}{(1+k\overline{w}_1)(r_1+\alpha)} - r_2 - \beta\overline{w}_1\right) := \underline{v}_1 \tag{8}$$

uniformly in any bounded subset of $(-\infty,\infty)$, and $\underline{u}_1 \le \overline{u}_1, \underline{v}_1 \le \overline{v}_1$.

For any given small enough positive constants ε,δ and large enough positive constant L, l_δ is defined by Proposition 8.1 in [12] with $d=d_1$, $a=\beta(\underline{v}_1-\varepsilon)-r_3$, $\theta=\eta$. Note that (8), there exists $T>0$ such that $v \ge \underline{v}_1-\varepsilon$ for $(t,x)\in[T,\infty)\times[-l_\delta,l_\delta]$. Therefore w satisfies

$$\begin{cases} w_t - d_2 w_{xx} \ge \beta(\underline{v}_1-\varepsilon)w - \eta w^2 - r_3 w, & t\ge T, x\in[-l_\delta,l_\delta],\\ w(t,\pm l_\lambda) \ge 0, & t \ge T. \end{cases}$$

Applying Proposition 8.1 in [11], we can see that $\liminf\limits_{t\to\infty} w(t,x) \ge [\beta(\underline{v}_1-\varepsilon)-r_3]/\eta$ uniformly in $[-L,L]$. Due to the arbitrariness of L,ε,δ, we have

$$\liminf_{t\to\infty} w(t,x) \ge \frac{\beta\underline{v}_1 - r_3}{\eta} = f(\overline{w}_1) := \underline{w}_1$$

uniformly in any bounded subset of $(-\infty,\infty)$. We can obtain $\frac{1-k\theta}{1+k(\rho-\theta)} > \frac{\beta^2}{\eta\lambda}$ on the conditions $\rho>\theta$ and $\frac{\beta^2}{\eta\lambda}+k\rho<1$, moreover $0<\underline{w}_1\le\overline{w}_1$.

Step 3 For any given large enough positive constant L and small enough positive constant ε, there exists $T>0$ such that $w(t,x)\ge\underline{w}_1-\varepsilon$ for $(t,x)\in[T,\infty)\times[-L,L]$. Then (u,v) satisfies

$$\begin{cases} u_t - d_1 u_{xx} \le \dfrac{bv}{1+k(\underline{w}_1-\varepsilon)} - r_1 u - \alpha u, & t\ge T, x\in(-L,L),\\ v_t - d_2 v_{xx} \le \alpha u - \lambda v^2 - r_2 v - \beta v(\underline{w}_1-\varepsilon), & t\ge T, x\in(-L,L), \end{cases}$$

and $u(t,\pm L)\le M_1, v(t,\pm L)\le M_1$ for $t\ge T$. We can obtain the desired result by following the same arguments as presented in Proposition 8.1 of [12].

$$\limsup_{t\to\infty} u(t,x) \le \frac{b}{\lambda[1+k(\underline{w}_1-\varepsilon)](r_1+\alpha)}\left(\frac{b\alpha}{[1+k(\underline{w}_1-\varepsilon)](r_1+\alpha)} - r_2 - \beta(\underline{w}_1-\varepsilon)\right),$$

$$\limsup_{t\to\infty} v(t,x) \le \frac{1}{\lambda}\left(\frac{b\alpha}{[1+k(\underline{w}_1-\varepsilon)](r_1+\alpha)} - r_2 - \beta(\underline{w}_1-\varepsilon)\right)$$

uniformly in any bounded subset of $(-\infty,\infty)$. Due to the arbitrariness of ε, we have

$$\limsup_{t\to\infty} u(t,x) \le \frac{b}{\lambda(1+k\underline{w}_1)(r_1+\alpha)}\left(\frac{b\alpha}{(1+k\underline{w}_1)(r_1+\alpha)} - r_2 - \beta\underline{w}_1\right) := \overline{u}_2,$$

$$\limsup_{t\to\infty} v(t,x) \le \frac{1}{\lambda}\left(\frac{b\alpha}{(1+k\underline{w}_1)(r_1+\alpha)} - r_2 - \beta\underline{w}_1\right) := \overline{v}_2$$

uniformly in any bounded subset of $(-\infty,\infty)$, and $\underline{u}_1 \le \overline{u}_2 \le \overline{u}_1, \underline{v}_1 \le \overline{v}_2 \le \overline{v}_1$.

For any given small enough positive constants ε,δ and large enough positive constant L, l_δ is defined by Proposition 8.1 in [12] with $d=d_1$, $a=\beta(\overline{v}_2+\varepsilon)-r_3$, $\theta=\eta$. There exists $T>0$ such that $v\le \overline{v}_2+\varepsilon$ for $(t,x)\in[T,\infty)\times[-l_\delta,l_\delta]$. Therefore w satisfies

$$\begin{cases} w_t - d_2 w_{xx} \le \beta(\overline{v}_2-\varepsilon)w - \eta w^2 - r_3 w, & t\ge T, x\in[-l_\delta,l_\delta], \\ w(t,\pm l_\lambda) \le M_1, & t\ge T. \end{cases}$$

Applying Proposition 8.1 in [12], we see that $\limsup_{t\to\infty} w(t,x) \le [\beta(\overline{v}_2+\varepsilon)-r_3]/\eta$ uniformly on $[-L,L]$. Due to the arbitrariness of L,ε,δ, we have

$$\limsup_{t\to\infty} w(t,x) \le \frac{\beta\overline{v}_2 - r_3}{\eta} = f(\underline{w}_1) := \overline{w}_2$$

uniformly in any bounded subset of $(-\infty,\infty)$. Moreover $\underline{w}_1 \le \overline{w}_2 \le \overline{w}_1$.

Step 4 Repeating the procedure above, we can find six sequences $\{\overline{u}_n\},\{\overline{v}_n\}$, $\{\overline{w}_n\}$, $\{\underline{u}_n\},\{\underline{v}_n\}$, $\{\underline{w}_n\}$ satisfying

$$\underline{u}_1 \le \underline{u}_2 \le \cdots \le \underline{u}_n \le \cdots \le \overline{u}_n \le \cdots \le \overline{u}_2 \le \overline{u}_1, \quad \underline{v}_1 \le \underline{v}_2 \le \cdots \le \underline{v}_n \le \cdots \le \overline{v}_n \le \cdots \le \overline{v}_2 \le \overline{v}_1,$$
$$\underline{w}_1 \le \underline{w}_2 \le \cdots \le \underline{w}_n \le \cdots \le \overline{w}_n \le \cdots \le \overline{w}_2 \le \overline{w}_1,$$

moreover

$$\underline{u}_n \le \liminf_{t\to\infty} u(t,x) \le \limsup_{t\to\infty} u(t,x) \le \overline{u}_n, \quad \underline{v}_n \le \liminf_{t\to\infty} v(t,x) \le \limsup_{t\to\infty} v(t,x) \le \overline{v}_n,$$

$$\underline{w}_n \le \liminf_{t\to\infty} w(t,x) \le \limsup_{t\to\infty} w(t,x) \le \overline{w}_n$$

uniformly in any bounded subset of $(-\infty,\infty)$. It is easy to know that all six sequences converge, and the limits are recorded as $\overline{u}_\infty,\overline{v}_\infty$, $\overline{w}_\infty$, $\underline{u}_\infty,\underline{v}_\infty$, $\underline{w}_\infty$. We next prove (3.4). It can be obtained from calculation that

$$\overline{u}_1 = \frac{1}{\lambda}\frac{b}{r_1+\alpha}\left(\frac{b\alpha}{r_1+\alpha}-r_2\right), \ \overline{v}_1 = \frac{1}{\lambda}\left(\frac{b\alpha}{r_1+\alpha}-r_2\right),$$
$$\overline{w}_1 = \frac{\beta\overline{v}_1 - r_3}{\eta}.$$

$$\underline{u}_n = \frac{b}{(r_1+\alpha)(1+k\overline{w}_n)}\underline{v}_n,$$
$$\underline{v}_n = \frac{1}{\lambda}\left(\frac{b\alpha}{(r_1+\alpha)(1+k\overline{w}_n)} - r_2 - \beta\overline{w}_n\right),$$
$$\underline{w}_n = f(\overline{w}_n), \quad n=1,2,\cdots$$

$$\overline{u}_n = \frac{b}{(r_1+\alpha)(1+k\underline{w}_{n-1})}\overline{v}_n,$$
$$\overline{v}_n = \frac{1}{\lambda}\left(\frac{b\alpha}{(r_1+\alpha)(1+k\underline{w}_{n-1})} - r_2 - \beta\underline{w}_{n-1}\right),$$
$$\overline{w}_n = f(\underline{w}_{n-1}), \quad n=2,3,\cdots$$

Taking limit, we can see that

$$\underline{u}_\infty = \frac{b}{(r_1+\alpha)(1+k\overline{w}_\infty)}\underline{v}_\infty,$$
$$\underline{v}_\infty = \frac{1}{\lambda}\left(\frac{b\alpha}{(r_1+\alpha)(1+k\overline{w}_\infty)} - r_2 - \beta\overline{w}_\infty\right), \quad \underline{w}_\infty = f(\overline{w}_\infty),$$

$$\overline{u}_\infty = \frac{b}{(r_1+\alpha)(1+k\underline{w}_{n-1})}\overline{v}_\infty,$$
$$\overline{v}_\infty = \frac{1}{\lambda}\left(\frac{b\alpha}{(r_1+\alpha)(1+k\underline{w}_\infty)} - r_2 - \beta\underline{w}_\infty\right), \quad \overline{w}_\infty = f(\underline{w}_\infty),$$

Note that $|f'(z)| < k\rho + \frac{\beta^2}{\eta\lambda} < 1$ for $z>0$, we obtain $\underline{w}_\infty = \overline{w}_\infty = w^*$, therefore

$$\underline{v}_\infty = \overline{v}_\infty = v^*, \ \underline{u}_\infty = \overline{u}_\infty = u^*.$$

(6) holds.

4. Conclusion

This paper investigates a free boundary problem in a prey-predator stage-structured model with a fear effect. Our main results include the existence and uniqueness of solutions (Theorems 2.1-2.2), the spreading-vanishing dichotomy, and the long-term behavior of solutions (Theorems 3.1-3.2).

Biologically, incorporating stage structure and fear effects into the model increases its realism. Our results suggest that the introduction of stage structure and fear effect can provide more flexible control over the vanishing and spreading of species. To facilitate species spreading, one can increase the birth rate b of the immature or reduce the fear rate k.

Although our work provides some insights into the prey-predator stage-structured model with fear effect, there are still some unresolved issues. For instance, it would be interesting to establish the sharp criteria for spreading and vanishing and to determine the spreading speed when spreading occurs. These problems are left for future work.

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